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DEGENERATE FOURIER TRANSFORM ASSOCIATED WITH THE STURM-LIOUVILLE OPERATOR

A. V. GORSHKOV

Abstract. In this paper we present degenerate Fourier transform, which is associated with the Sturm-Liouville operator, and define a complete orthonormal system when the spectrum consists of a continuous part, and no bigger than a countable set of eigenvalues. We prove Parseval's identity and inversion formula for degenerate Fourier transform and give some examples.

1. Introduction

Most of the known Fourier transforms associated with the equations of mathematical physics have a trivial kernel, and an inversion formula as well as the Parseval's identity are fulfilled. In other words, the system of the eigenfunctions involved in the definition of the integral transform is complete.

However, in some cases, the differential operator, in addition to the continuous part of the spectrum that defines this transform, may contain a set of eigenfunctions $\{e_k\}$, and the Parseval's identity takes the form

$$\|f\|^2 = \|F[f]\|^2 + \sum_k (f, e_k)^2, \tag{1.1}$$

where $\{e_k\}$ are the elements from $\ker F$. In that case F becomes the *degenerate transform.*

For example, the sine-Fourier transform

$$\hat{f}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(\lambda s) f(s) ds$$

is based on the eigenfunctions of $A = d^2/dx^2$ in $L_2(0,\infty)$ with the Dirichlet condition $f(0) = 0$. The spectrum of the operator is continuous and fills the entire negative half-axis: $\sigma_c = (-\infty, 0]$. This transform is not degenerate, and the inversion formula has the form

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin(\lambda x) \left(\int_0^\infty \sin(\lambda s) f(s) ds \right) d\lambda.$$

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Functions $\varphi(x,\lambda) = \sin(\lambda x)$ are the *generalized eigenfunctions* (since they don't belong to the main space $L_2(0,\infty)$, where the operator is defined):

$$\partial_{xx}\varphi(x,\lambda) = -\lambda^2\varphi(x,\lambda).$$

Similarly, the same operator with the Neumann condition generates a cosine Fourier transform. However, d^2/dx^2 with a Robin boundary condition

$$f'(0) + af(0) = 0 \tag{1.2}$$

with $a > 0$, in addition to its generalized eigenfunctions contains an ordinary eigenfunction e^{-ax} with an eigenvalue of a^2. Its spectrum consists of a continuous part $\sigma_c = (-\infty, 0]$ and an eigenvalue $\lambda = a^2$. The Fourier transform generated by this operator will already have a nontrivial kernel (see [5]).

Functions

$$\varphi(x,\lambda) = \sqrt{\frac{2}{\pi}}\frac{\lambda\cos(\lambda x) - \sin(\lambda x)}{\sqrt{\lambda^2+1}}$$

satisfy Robin boundary condition (1.2) with $a = 1$. Together with an eigenfunction

$$e_0(x) = \frac{e^{-x}}{\sqrt{2}}$$

they form a complete set of orthogonal functions for Laplace operator. The inversion formula has the following form:

$$f(x) = \frac{2}{\pi}\int_0^\infty \varphi(x,\lambda)\left(\int_0^\infty \varphi(s,\lambda)f(s)ds\right)d\lambda + e_0(e_0,f).$$

Integral transform

$$F[f] = \int_0^\infty \frac{\lambda\cos(\lambda s) - \sin(\lambda s)}{\sqrt{\lambda^2+1}}f(s)ds$$

vanishes on function e_0, which means that $\varphi(x,\lambda)$ is orthogonal to e_0 and the Parseval's identity holds:

$$\|f\|^2_{L_2(0,\infty)} = \|F[f]\|^2_{L_2(0,\infty)} + (e_0,f)^2.$$

However, the mixed boundary condition with $a > 0$ refers to *non-physical*, and, as a rule, is not considered in differential equations.

Another example of a family of operators which are connected with the degenerate transforms are

$$\Delta_k = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{k^2}{r^2},\ k \in \mathbb{R}.$$

For integers k, these operators are the Fourier coefficients of the Laplacian when it is decomposed into a Fourier series by an angular variable in polar coordinates.

These operators with the mixed boundary condition

$$r_0\frac{\partial w(t,r)}{\partial r}\Big|_{r=r_0} \pm kw(t,r_0) = 0 \tag{1.3}$$

in addition to the continuous spectrum $\sigma_c = (-\infty, 0)$ also have an eigenvalue $\lambda = 0$ from the kernel of Δ_k (here $r_0 > 0$ is a fixed number). The kernel $\ker(\Delta_k)$ consists of the eigenfunctions $1/r^{\pm k}$ (for certain values of k). They also produce a degenerate transform, which is a generalization of the integral Weber transform, the properties of which were investigated by the author in [1].

Weber transform $W_{k,k\pm1}[\cdot]$ for $k \in \mathbb{R}$, $r_0 > 0$ is defined as

$$W_{k,k\pm1}[f](\lambda) = \int_{r_0}^{\infty} \frac{J_k(\lambda s)Y_{k\pm1}(\lambda r_0) - Y_k(\lambda s)J_{k\pm1}(\lambda r_0)}{\sqrt{J^2_{k\pm1}(\lambda r_0) + Y^2_{k\pm1}(\lambda r_0)}} f(s)s\,\mathrm{d}s, \tag{1.4}$$

where J_k, Y_k are the Bessel functions of the first and second kind.

The inverse transform is given by the following formula:

$$W^{-1}_{k,k\pm1}[\hat{f}](r) = \int_{0}^{\infty} \frac{J_k(\lambda r)Y_{k\pm1}(\lambda r_0) - Y_k(\lambda s)J_{k\pm1}(\lambda r_0)}{\sqrt{J^2_{k\pm1}(\lambda r_0) + Y^2_{k\pm1}(\lambda r_0)}} \hat{f}(\lambda)\lambda\,\mathrm{d}\lambda\,. \tag{1.5}$$

Operator Δ_k combined with the Robin boundary condition (1.3) has the eigenfunctions $e^{\pm}_{0,k}$ from the kernel of W:

$$e^{+}_{0,k} = \begin{cases} c^{+}_k r^k, & k < -1, \\ 0, & k \geq -1, \end{cases} \qquad e^{-}_{0,k} = \begin{cases} c^{-}_k / r^k, & k > 1, \\ 0, & k \leq 1, \end{cases}$$

with normalization coefficients $c^{\pm}_k$.

In [1] there was proved the inversion formula for functions $f(\cdot)$, $f(r)\sqrt{r} \in L_1(r_0,\infty) \cap L_2(r_0,\infty)$, $r_0 > 0$:

$$f = W^{-1}_{k,k\pm1}\left[W_{k,k\pm1}[f]\right] + (f, e^{\pm}_{0,k})e^{\pm}_{0,k}.$$

And these transforms, precisely as a degenerate one, had found an application in mathematical physics. With its help, the author obtained a solution to the Stokes problem in the exterior of the circle (see [2]). All this confirms the importance of such a degenerate transforms not only for the theory of functions, but also as having practical importance.

2. Completeness of a system of eigenfunctions

Let the Sturm-Liouville operator $Af(x) = f'' - q(x)f$, defined on the semiaxis $x \in (0,\infty)$, have a spectrum $\sigma(A)$, which consists of a continuous part $E \subset \mathbb{R}$, and no bigger than a countable set of eigenvalues $\{\lambda_k\}$ of the finite multiplicity with eigenfunctions $\{e_k\}$. As is known[5][4], if $q(x)$ is continuous on $\mathbb{R}_+$, then there exist the generalized eigenfunctions $\varphi(x,\lambda)$ specifying the Fourier transform

$$F[f] = \int_0^{\infty} \varphi(x,\lambda)f(x)dx, \;\; A\varphi(x,\lambda) = \lambda\varphi(x,\lambda).$$

At the same time, there is a spectral function $\rho(\lambda)$ such that $F[f]$ belongs to the space L_2 with the weight $\rho(\lambda)$: $F[f] \in L_2\left(E, \rho(\lambda)\right)$.

Definition 2.1. Let's call the functions $\varphi(x,\lambda)$ to be *orthonormal*, denoting as $\langle \varphi(\cdot,\lambda), \varphi(\cdot,\zeta)\rangle = \delta(\lambda - \zeta)$ if

$$\hat{f}(\lambda) = \int_0^{\infty} \varphi(x,\lambda)\left(\int_E \varphi(x,\zeta)\hat{f}(\zeta)d\rho(\zeta)\right)dx \tag{2.1}$$

for any function $\hat{f}(\lambda)$ of the form $\hat{f}(\lambda) = F[f(\cdot)](\lambda)$, $f(x) \in L_2(\mathbb{R}_+)$.

Equality in this definition follows from the formal applying of the δ-function $\delta(\lambda-\zeta)$ to $\hat{f}(\lambda)$. The formula (2.1) is the inversion formula $\hat{f}=F[F^*[\hat{f}]]$ for the inverse Fourier transform F^*, and it will be valid only for functions $\hat{f}$ from the image of F. And the inversion formula for the direct transform $f=F^*[F[f]]$ will be valid for $f\in\ker^{\perp}F$.

Definition 2.2. We will say that $\{\{\varphi(\cdot,\lambda)\},\{e_k\}\}$ is a complete orthonormal system of eigenfunctions of the operator A if $\langle\varphi(\cdot,\lambda),\varphi(\cdot,\zeta)\rangle=\delta(\lambda-\zeta)$, $\langle\varphi(\cdot,\lambda),e_k\rangle_{L_2(\mathbb{R}_+)}=0$, $(e_k,e_j)_{L_2(\mathbb{R}_+)}=\delta_{k,j}$ for any $\lambda,\zeta\in E$, any eigenfunctions e_k,e_j, and the Parseval's identity (1.1) holds.

The condition $\langle\varphi(\cdot,\lambda),e_k\rangle_{L_2(\mathbb{R}_+)}=0$ means that F is the degenerate transform, i.e. $F[e_k]=0$.

Theorem 2.1. *Let the Sturm-Liouville operator A be a generator of a strongly continuous semigroup e^{tA} in $L_2(0,\infty)$; its spectrum is real, bounded from above, consists of a continuous part and no bigger than a countable set of eigenvalues $\{e_k\}$, and the resolvent satisfies the estimate $\|R(A,\lambda)\|\le C/\lambda$ with some $C>0$. Then the system of its eigenfunctions forms a complete orthonormal system and the following inversion formula holds:*

$$f(x)=\int_E\varphi(x,\lambda)\left(\int_{\mathbb{R}_+}\varphi(s,\lambda)f(s)ds\right)d\rho(\lambda)+\sum_k(f,e_k)e_k. \tag{2.2}$$

Proof. Consider the boundary value problem in $L_2(\mathbb{R}_+)$:

$$\partial_t y(t,x)-Ay=0, y(0,x)=f(x).$$

Using the estimate on the resolvent $R(A,\lambda)$ from the conditions of the theorem, the solution $y(t,\cdot)=e^{tA}f(\cdot)$ of this equation can be given by the formula

$$y(t,\cdot)=\frac{1}{2\pi i}\int_\gamma e^{\lambda t}R(A,\lambda)f(\cdot)d\lambda,$$

where the contour γ covers the real spectrum of the operator A. In the case of an unbounded spectrum γ is the boundary of the sector $S_{a,\theta}=\{\lambda\in\mathbb{C},\ |\arg(\lambda-a)|>\theta\}$ with some $a>0$, $\theta\in(\frac{\pi}{2},\pi)$.

The resolvent has a gap along the set E, and the points λ_k are its poles. Integral

$$P_k=\frac{1}{2\pi i}\int_{|\lambda-\lambda_k|=\varepsilon}R(A,\lambda)d\lambda$$

for a sufficiently small ε is a finite-dimensional projector onto the proper subspace E_k (see [3]). And then the solution takes the following form:

$$y(t,\cdot)=\frac{1}{2\pi i}\int_E e^{\lambda t}\left(R(A,\lambda-i\cdot 0)-R(A,\lambda+i\cdot 0)\right)f(\cdot)d\lambda+\sum_k e^{\lambda_k t}(f,e_k)e_k.$$

The integral in the last equality defines a family of projectors on the generalized proper subspace and can be expressed in terms of the eigenfunctions $\varphi(x,\lambda)$, giving the final formula for $y(t,x)$:

$$y(t,x)=\int_E e^{\lambda t}\varphi(x,\lambda)\left(\int_{\mathbb{R}_+}\varphi(s,\lambda)f(s)ds\right)d\rho(\lambda)+\sum_k e^{\lambda_k t}(f,e_k)e_k. \tag{2.3}$$

Let's prove the inversion formula (2.2). Then it will follow the completeness of a system of eigenfunctions. Since A is the generator of the strongly continuous semigroup e^{tA}, then $y(t,\cdot) \to f(x)$ strongly at $t \to 0$. If the set of the continuous spectrum E is bounded, then passing to the limit at $t \to 0$ is allowed under the sign of the integral in (2.3) and the inversion formula is proved. If E is not bounded, then we must justify the limit transition at $t \to 0$. It is enough to prove the validity of a limit transition only in the integral part of the equality (2.3). Let's prove weak convergence at $t \to 0$:

$$y(t,x) \rightharpoonup \int_E \varphi(x,\lambda) \left(\int_{\mathbb{R}_+} \varphi(s,\lambda) f(s) ds \right) d\rho(\lambda) + \sum_k (f, e_k) e_k.$$

Take an arbitrary $g \in C_0^\infty(r_0, \infty)$. Then $\hat{g}(\lambda) = F[g(\cdot)](\lambda)$ will decrease rapidly by λ as it claimed in the following proposition:

Proposition 2.1. *For an arbitrary* $g \in C_0^\infty(r_0,\infty)$, $k > 0$ *holds* $\lambda^k \hat{g}(\lambda) \in L_2(E, \rho(\lambda))$.

Proof. Since $A\varphi = \lambda\varphi$, then

$$\hat{g}(\lambda) = \int_0^\infty \varphi(x,\lambda) g(x) dx = \frac{1}{\lambda^k} \int_0^\infty A^k[\varphi(x,\lambda)] g(x) dx$$
$$= \frac{1}{\lambda^k} \int_0^\infty \varphi(x,\lambda) A^k[g(x)] dx.$$

Since $A^k[g(x)] \in L_2(\mathbb{R}_+)$, then $\int_0^\infty \varphi(x,\lambda) A^k[g(x)] dx \in L_2(E, \rho(\lambda))$ and the proposition is proved. □

Denote

$$F^*[\hat{g}(\cdot)](x) = \int_E \varphi(x,\lambda) \hat{g}(\lambda) d\rho(\lambda).$$

Then

$$\left(F^* \left[e^{\lambda t} F[f] \right], g(\cdot) \right)_{L_2(\mathbb{R}_+)} = \left(e^{\lambda t} F[f], F[g] \right)_{L_2(E,\rho(\lambda))}$$
$$= \left(e^{\lambda t} \hat{f}(\lambda), \hat{g}(\lambda) \right)_{L_2(E,\rho(\lambda))},$$

where $\hat{f}(\lambda) = F[f](\lambda)$, $\hat{g}(\lambda) = F[g](\lambda)$.

The residuals of integrals

$$\int_{-\infty}^{L} \left| e^{\lambda t} \hat{f}(\lambda) \hat{g}(\lambda) \right| d\rho(\lambda) = \int_{-\infty}^{L} \left| e^{\lambda t} \frac{\hat{f}(\lambda)}{\lambda} \hat{g}(\lambda) \lambda \right| d\rho(\lambda) \le \frac{1}{|L|} \|\hat{f}(\lambda)\| \|\hat{g}(\lambda)\lambda\|$$

converge to zero uniformly over t as $L \to -\infty$. Consequently, we have proved the validity of the transition at $t \to 0$:

$$\left(e^{\lambda t} \hat{f}(\lambda), \hat{g}(\lambda) \right)_{L_2(E,\rho(\lambda))} \to \left(\hat{f}(\lambda), \hat{g}(\lambda) \right)_{L_2(E,\rho(\lambda))}.$$

From the uniqueness of the weak limit, taking into account $y(t,\cdot) \to f(\cdot)$, the inversion formula is valid almost everywhere

$$f(\cdot) = F^*[F[f]] + \sum_k (f, e_k) e_k, \tag{2.4}$$

and the formula (2.2) is proved.

Let's prove that $e_k \in \ker(F)$:

$$\int_0^\infty \varphi(x,\lambda)e_k(x)dx = \frac{1}{\lambda}\int_0^\infty A[\varphi(x,\lambda)]e_k(x)dx = \frac{1}{\lambda}\int_0^\infty \varphi(x,\lambda)Ae_k(x)dx$$
$$= \frac{\lambda_k}{\lambda}\int_0^\infty \varphi(x,\lambda)e_k(x)dx,$$

which implies $F[e_k] = 0$.

The orthogonality condition (2.1) follows from the inversion formula (2.4) if we apply the transform F to the latter:

$$\hat{f} = F\left[F^*\hat{f}\right].$$

The Parseval's identity (1.1) follows from the inversion formula (2.2) in virtue of the orthonormality of $\{\{\varphi(\cdot,\lambda)\},\{e_k\}\}$. The theorem is proved. □

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A. V. Gorshkov

Lomonosov Moscow State University, Faculty of Mechanics and Mathematics, GSP-1, Leninskie Gory, Moscow, 119991, Russian Federation.

E-mail address: alexey.gorshkov.msu@gmail.com

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