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BEREZIN NORM AND BEREZIN RADIUS INEQUALITIES OF PRODUCTS AND SUMS WITH SELBERG OPERATOR

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Abstract. We prove new inequalities related to the Berezin norm and Berezin radius of some products and sums with the Selberg operator on a reproducing kernel Hilbert space.

1. Introduction

By a reproducing kernel Hilbert space (briefly, RKHS) we mean a Hilbert space $\mathcal{H} = \mathcal{H}(X)$ of complex-valued functions on some set X such that evaluation at any point of X is a continuous functional on $\mathcal{H}$. The Riesz representation theorem ensures that the Hilbert function space $\mathcal{H}$ has a reproducing kernel, that is, for each $\lambda \in X$ there exists a function $k_\lambda(z) \in \mathcal{H}$ such that $\langle f, k_\lambda \rangle = f(\lambda)$ for each $f \in \mathcal{H}$ and $\lambda \in X$. This function is called reproducing kernel of the space $\mathcal{H}$. We denote by $\widehat{k}_\lambda := \frac{k_\lambda}{||k_\lambda||_{\mathcal{H}}}$ the normalized reproducing kernel of $\mathcal{H}$. The prototypical RKHSs are the Hardy space $H^2(\mathbb{D})$, the Bergman space $L_a^2(\mathbb{D})$, the Dirichlet space $\mathcal{D}^2(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc, and the Fock space $\mathcal{F}(\mathbb{C})$. A detailed presentation of the theory of reproducing kernels and RKHSs is given, for instance in Aronzajn [1]. Reproducing kernels play important role in many branches of pure and applied mathematics including frame theory, wavelets, signals, fractals theories (see for instance, Jorgensen's book [27] and its references).

For every bounded linear operator P on $\mathcal{H}$ (i.e., for $P \in \mathcal{B}(\mathcal{H})$ its Berezin symbol $\widetilde{P}$ is defined by (see, Berezin [6] and Nordgren and Rosenthal [29])

$$\widetilde{P}(\lambda) := \langle P\widehat{k}_\lambda, \widehat{k}_\lambda \rangle, \ \lambda \in X.$$

This is s scalar valued function which is bounded on X since by Cauchy-Schwarz inequality $\left|\widetilde{P}(\lambda)\right| \leq \left\|P\widehat{k}_\lambda\right\| \leq \|P\|$ for all $\lambda \in X$. The Berezin symbol of an operator provides important information about the operator. For instance, it is well-known that on the RKHSs of analytic functions (including the Hardy, Bergmam, Dirichlet and Fock spaces), the Berezin symbol uniquely determines the operator, i.e., $P_1 = P_2$ if and only if $\widetilde{P}_1 = \widetilde{P}_2$ (see, for example, Engliš [12] and Zhu [34]).

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The Berezin set of operator $P \in \mathcal{B}(\mathcal{H})$ is defined by

$$\mathrm{Ber}(P) := \mathrm{Range}(\widetilde{P})$$

and Berezin radius of P is the number defined by

$$\mathrm{ber}(P) := \sup\left\{|\mu| : \mu \in \mathrm{Ber}(P)\right\}.$$

(see Karaev [22, 23]). The Berezin norms of $P \in \mathcal{B}(\mathcal{H})$ is defined by

$$\|P\|_{B,1} := \sup_{\lambda \in X} \left\| P\widehat{k}_\lambda \right\|_{\mathcal{H}} \text{ and } \|P\|_{B,2} := \sup_{\lambda,\mu \in X} \left| \left\langle P\widehat{k}_\lambda, \widehat{k}_\mu \right\rangle \right|.$$

Clearly, $\|P\|_{B,1} \le \|P\|_{B,2}$. Here, if $P \ge 0$ then $\|P\|_{B,2} = \mathrm{ber}\,(P)$, but the equality does not hold for general self adjoint operators, see [9].

Let $W(P)$ and $w(P)$ denote the numerical range and numerical radius of P, respectively:

$$W(P) := \{\langle Pf, f\rangle : f \in \mathcal{H} \text{ and } \|f\|_{\mathcal{H}} = 1\}$$

$$w(P) := \sup\{|\langle Pf, f\rangle| : f \in \mathcal{H} \text{ and } \|f\|_{\mathcal{H}} = 1\}.$$

It is obvious that $\mathrm{Ber}(P) \subseteq W(P)$, $\mathrm{ber}(P) \le \|P\|_{B,i} \le \|P\|$ for $i = 1, 2$ and $\mathrm{ber}(P) \le w(P)$. Also, it is well-known that (see Halmos [26])

$$\frac{1}{2}\|P\| \le w(P).$$

So, the study of the new numerical characteristics $\mathrm{Ber}(P)$, $\mathrm{ber}(P)$ and $\|P\|_{B,i}$ is important firstly for the deep study of the numerical range and the numerical radius of operators on the RKHSs.

Recall that a function $\theta \in H^\infty(\mathbb{D})$ (the Hardy space of bounded analytic functions f on the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with infinite sup-norm $\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < +\infty$) is called inner if its boundary function is unimodular on the unit circle $\partial\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| = 1\}$. For example, $f(z) = z^n$ is an inner function for each $n \ge 1$.

Let θ be an inner function. We set

$$N_\theta := T_\theta\left(I - T_\theta T_{\overline{\theta}}\right) = T_\theta P_\theta,$$

where T_θ is an analytic Toeplitz operator, $T_\theta f = \theta f$, $T_{\overline{\theta}}$ is a co-analytic Toeplitz operator on the Hardy space $H^2 = H^2(\mathbb{D})$ defined by $T_{\overline{\theta}} f = P_+(\overline{\varphi} f)$, $f \in H^2$, here $P_+ : L^2(\partial\mathbb{D}) \to H^2$ is the Riesz projector, and $P_\theta : H^2 \to \left(\theta H^2\right)^\perp$ is an orthogonal projection. It is elementary that $\|N_\theta\| = 1$, $N_\theta^2 = \Theta$, i.e., N_θ is a square zero operator. It is also easy to verify that

$$\widetilde{N_\theta}(\lambda) = \theta(\lambda)\left(1 - |\theta(\lambda)|^2\right), \ \lambda \in \mathbb{D}.$$

We set $f(x) := x\left(1 - x^2\right)$, $0 \le x < 1$. Clearly $f'(x) = 1 - 3x^2$, which shows that if $f'(x) = 0$ then $x = 1/\sqrt{3}$. Since $f''(x) = -6x$, and $f''\left(1/\sqrt{3}\right) < 0$, we deduce that $\sup_{0<x<1} f(x) = f\left(1/\sqrt{3}\right) = 1/\sqrt{3}\left(1 - \frac{1}{3}\right) = \frac{2}{3\sqrt{3}}$. Since $\theta(\mathbb{D})$ is always an open dense connected subset of $\mathbb{D}$, this implies that

$$\sup_{\mathbb{D}} |\theta(\lambda)|\left(1 - |\theta(\lambda)|^2\right) = \frac{2}{3\sqrt{3}},$$

that is $\operatorname{ber}(N_\theta) = \frac{2}{3\sqrt{3}}$. Now it is clear that $\operatorname{ber}(N_\theta) < \frac{\|N_\theta\|}{2} = \frac{1}{2}$. This example shows that the inequality $\frac{1}{2}\|P\| \leq \operatorname{ber}(P)$ does not hold in general for general bounded linear operators on reproducing kernel Hilbert spaces.

On the other hand, for $\theta(z) = z$ we consider the operator $N_z = S(I - SS^*)$ on H^2, $N_z x = S(I - SS^*)x = \widehat{x}(0)z$ for any $x \in H^2$. Then we have that

$$\langle N_z x, x\rangle = \left\langle \widehat{x}(0)z, \sum_{k=0}^{\infty} \widehat{x}(k)z^k \right\rangle = \widehat{x}(0)\overline{\widehat{x}(1)}$$

for any $x \in H^2$ with $\|x\|_2 = 1$. This shows that

$$W(N_z) = \left\{ \widehat{x}(0)\widehat{x}(1) : \left[\sum_{k=0}^{\infty} |\widehat{x}(k)|^2\right]^{1/2} = 1 \right\}. \tag{1.1}$$

Since N_z is a one dimensional square zero operator (nilpotent operator) on the Hardy space H^2, it is clear from (1.1) that $w(N_z) = \frac{1}{2}$ and $W(N_\theta) = \overline{\mathbb{D}}_{1/2}$ (see Karaev and Iskenderov [24]). On the other hand, we know that $\operatorname{ber}(N_z) = \frac{2}{3\sqrt{3}}$ which strictly less than $\frac{w(N_z)}{2} = \frac{1}{4}$. This example shows that in general the inequality $\frac{1}{2}w(P) \leq \operatorname{ber}(P)$ is not true.

Thus, the following questions are natural.

Question 1. Under which conditions the inequality $\frac{1}{2}\|P\| \leq \operatorname{ber}(P)$ holds?

Question 2. Is it true that $\frac{1}{2}\|P\|_{B,1} \leq \operatorname{ber}(P)$?

Question 3. Under which conditions the inequality $\frac{1}{2}w(P) \leq \operatorname{ber}(P)$ holds?

There are a large literature devoted to the investigation of the above mentioned numerical characteristics of operators and to their relationship, see, for instance, [4, 5, 7, 8, 10, 11, 14, 15, 16, 17, 18, 19, 20, 30, 31, 32, 33].

Given $\mathcal{Z} = \{z_i : i = 1, 2, ..., n\} \subset \mathcal{H}$, we define the so-called Selberg operator associated to $\mathcal{Z}$ as follows:

$$S_{\mathcal{Z}} = \sum_{i=1}^{n} \frac{z_i \otimes z_j}{\sum_{j=1}^{n} |\langle z_i, z_j\rangle|} \in \mathcal{B}(\mathcal{H}),$$

where $T := x \otimes y$ is the rank one operator defined by $T(z) = \langle z, y\rangle x$ with $x, y, z \in \mathcal{H}$.

It can be shown that $0 \leq S_{\mathcal{Z}} \leq I$, i.e., $S_{\mathcal{Z}}$ is a positive contraction (see, for instance, [2]).

Also, it is shown in [2] that $w(I - S_{\mathcal{Z}}) = \|I - S_{\mathcal{Z}}\| \leq 1$.

We recall that Selberg determined the following important inequality (see [28]) for given nonzero vectors $\mathcal{Z} = \{z_i : i = 1, ..., n\}$,

$$\sum_{i=1}^{n} \frac{|\langle x, z_i\rangle|^2}{\sum_{j=1}^{n} |\langle z_i, z_j\rangle|} \leq \|x\|^2, \tag{1.2}$$

which holds for all $x \in \mathcal{H}$. This inequality is known as the Selberg inequality. The inequality in (1.2) holds if and only if $x = \sum_{i=1}^{n} a_i z_i$ for some complex numbers $a_1, ..., a_n$ such that for any $i \neq j$, $\langle z_i, z_j\rangle = 0$ or $|a_i| = |a_j|$ with $\langle a_i z_i, a_j z_j\rangle \geq 0$ (see [13, Theorem 1]). It is relevant to notice that, from inequality (1.2), one can derive other well-known inequalities including Cauchy-Schwarz inequality $(\langle x, y\rangle \leq \|x\|\|y\|)$, Buzano inequality $(|\langle x, z\rangle\langle z, y\rangle| \leq \frac{1}{2}(|\langle x, y\rangle| + \|x\|\|y\|\|z\|)^2)$, Bessel inequality $\left(\sum_{i=1}^{n}|\langle x,l_i\rangle|^2 \leq \|x\|^2\right)$ and Bombieri inequality $(\sum_{i=1}^{n}|\langle x,z_i\rangle|^2 \leq \|x\|^2 \max_{1\leq i\leq n}\sum_{j=1}^{n}|\langle z_i,z_j\rangle|)$.

In the present paper, which is motivated by the paper [2], we focus on establishing appropriate bounds for the Berezin norm and the Berezin radius of the product of three bounded linear operators on a RKHS, one of them being a Selberg operator which shed some light to the above stating questions 1, 2 and 3 (Section 2). In Section 3, we prove some new Berezin radius inequalities for sums with the Selberg operator.

2. Berezin norm and Berezin radius inequalities for the products of operators

In the present section, we obtain upper estimates for both the Berezin norm and the Berezin radius of the product of three operators, one of which is the Selberg operator. For the proofs of our results in this section, we will use the following lemma found in [3].

Lemma 2.1. *Let $\mathcal{H}=\mathcal{H}(X)$ be a RKHS over some suitable set X. For any $x,y\in X$, the following inequalities hold:*

$$|\langle S_{\mathcal{Z}}x,y\rangle| \leq \left|\langle S_{\mathcal{Z}}x,y\rangle - \frac{1}{2}\langle x,y\rangle\right| + \frac{1}{2}|\langle x,y\rangle| \leq \frac{1}{2}\left(|\langle x,y\rangle| + \|x\|\,\|y\|\right) \tag{2.1}$$

and

$$\left|\left\langle \left(S_{\mathcal{Z}} - \frac{1}{2}I\right)x,y\right\rangle\right| \leq \frac{1}{2}\|x\|\,\|y\|. \tag{2.2}$$

Now we can present our results.

Proposition 2.1. *Let $P,R\in\mathcal{B}(\mathcal{H})$ be two operators on the RKHS $\mathcal{H}=\mathcal{H}(X)$. Then we have:*

$$\|RS_{\mathcal{Z}}P\|_{B,2} \leq \frac{1}{2}\left(\|RP\|_{B,2} + \|P\|_{B,1}\|R^*\|_{B,1}\right) \tag{2.3}$$

and

$$\left\|R\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)P\right\|_{B,2} \leq \frac{1}{2}\|P\|_{B,1}\|R^*\|_{B,1}, \tag{2.4}$$

where $S_{\mathcal{Z}}$ is, as before, the Selberg operator on $\mathcal{H}$.

Also, we have the following Berezin radius inequalities:

$$\operatorname{ber}(RS_{\mathcal{Z}}P) \leq \frac{1}{2}\left[\operatorname{ber}(RP) + \frac{1}{2}\left\||P|^2+|R^*|^2\right\|_{B,1}\right] \tag{2.5}$$

and

$$\operatorname{ber}\left(R\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)P\right) \leq \frac{1}{4}\left\||P|^2+|R^*|^2\right\|_{B,1}. \tag{2.6}$$

Proof. If we replace x by $P\widehat{k}_\lambda$ and y by $R^*\widehat{k}_\mu$, it follows from (2.1) in Lemma 2.1 that

$$\left|\left\langle RS_{\mathcal{Z}}P\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right| \leq \frac{1}{2}\left[\left|\left\langle RP\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right| + \left\|P\widehat{k}_\lambda\right\|\left\|R^*\widehat{k}_\mu\right\|\right] \tag{2.7}$$

and

$$\left|\left\langle R\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)P\widehat{k}_{\lambda},\widehat{k}_{\mu}\right\rangle\right|\leq\frac{1}{2}\left\|P\widehat{k}_{\lambda}\right\|\left\|R^{*}\widehat{k}_{\lambda}\right\| \tag{2.8}$$

for every $\lambda,\mu\in X$. Therefore by taking the supremum over all λ and μ in (2.7) and (2.8), we have

$$\|RS_{\mathcal{Z}}P\|_{B,2}\leq\frac{1}{2}\left[\|RP\|_{B,2}+\|P\|_{B,1}\|R^{*}\|_{B,1}\right]$$

and

$$\left\|R\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)P\right\|_{B,2}\leq\frac{1}{2}\|P\|_{B,1}\|R^{*}\|_{B,1},$$

which prove (2.3) and (2.4).

From (2.7), for $\mu=\lambda$, we obtain that

$$\begin{aligned}\left|\left\langle RS_{\mathcal{Z}}P\widehat{k}_{\lambda},\widehat{k}_{\lambda}\right\rangle\right|&\leq\frac{1}{2}\left(\left|\left\langle RP\widehat{k}_{\lambda},\widehat{k}_{\lambda}\right\rangle\right|+\left\|P\widehat{k}_{\lambda}\right\|\left\|R^{*}\widehat{k}_{\lambda}\right\|\right)\\&\leq\frac{1}{2}\left(\left|\widetilde{RP}(\lambda)\right|+\frac{1}{2}\left(\left\|P\widehat{k}_{\lambda}\right\|^{2}+\left\|R^{*}\widehat{k}_{\lambda}\right\|^{2}\right)\right)\\&=\frac{1}{2}\left(\left|\widetilde{RP}(\lambda)\right|+\frac{1}{2}\left(\left\langle P^{*}P\widehat{k}_{\lambda},\widehat{k}_{\lambda}\right\rangle+\left\langle RR^{*}\widehat{k}_{\lambda},\widehat{k}_{\lambda}\right\rangle\right)\right)\\&=\frac{1}{2}\left(\left|\widetilde{RP}(\lambda)\right|+\frac{1}{2}\left(\widetilde{|P|^{2}}(\lambda)+\widetilde{|R^{*}|^{2}}(\lambda)\right)\right)\end{aligned}$$

for all $\lambda\in X$. By taking the supremum over all λ, we get

$$\operatorname{ber}(RS_{\mathcal{Z}}P)\leq\frac{1}{2}\left[\operatorname{ber}(RP)+\frac{1}{2}\operatorname{ber}\left(|P|^{2}+|R^{*}|^{2}\right)\right],$$

as desired to prove.

Now from (2.8) for $\mu=\lambda$ we have that

$$\left|\left\langle R\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)P\widehat{k}_{\lambda},\widehat{k}_{\lambda}\right\rangle\right|\leq\frac{1}{2}\left\|P\widehat{k}_{\lambda}\right\|\left\|R^{*}\widehat{k}_{\lambda}\right\|\leq\frac{1}{4}\left(\left\|P\widehat{k}_{\lambda}\right\|^{2}+\left\|R^{*}\widehat{k}_{\lambda}\right\|^{2}\right),$$

whence

$$\left|R\widetilde{\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)}P(\lambda)\right|\leq\frac{1}{4}\widetilde{\left(|P|^{2}+|R^{*}|^{2}\right)}(\lambda)$$

for all $\lambda\in X$, which implies that

$$\operatorname{ber}\left(R\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)P\right)\leq\frac{1}{4}\operatorname{ber}\left(|P|^{2}+|R^{*}|^{2}\right),$$

as desired. This completes the proof. □

If P is isometry and R is unitary in Proposition 2.1, then we have the following bounds:

1) $\|RS_{\mathcal{Z}}P\|_{B,2}\leq\frac{1}{2}\left(\|RP\|_{B,2}+1\right)$
2) $\left\|R\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)P\right\|_{B,2}\leq\frac{1}{2}$
3) $\operatorname{ber}(RS_{\mathcal{Z}}P)\leq\frac{1}{2}\left[\operatorname{ber}(RP)+1\right]$
4) $\operatorname{ber}\left(R\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)P\right)\leq\frac{1}{2}$.

The following corollaries can be established as direct applications based on Proposition 2.1 :

Corollary 2.1. *Let $S_{\mathcal{Z}}$ be the Selberg operator defined above and $P, R \in \mathcal{B}(\mathcal{H})$. Then*

$$\begin{aligned}\operatorname{ber}(RS_{\mathcal{Z}}P) &\leq \frac{1}{2}\left[\operatorname{ber}(RP)+\frac{1}{2}\left\|\frac{\|R\|_{B,1}}{\|P\|_{B,1}}|P|^2+\frac{\|P\|_{B,1}}{\|R\|_{B,1}}|R^*|^2\right\|_{B,1}\right] \\ &\leq \frac{1}{2}\left\|\frac{\|R\|_{B,1}}{\|P\|_{B,1}}|P|^2+\frac{\|P\|_{B,1}}{\|R\|_{B,1}}|R^*|^2\right\|_{B,1} \leq \frac{1}{2}\left(\|RP\|_{B,1}+\|P\|_{B,1}\|R\|_{B,1}\right).\end{aligned}$$

Proof. Replacing P by $\frac{P}{\|P\|_{B,1}}$ and $\frac{R}{\|R\|_{B,1}}$ in (2.5), respectively, we obtain the first inequality. On the other hand, we have:

$$\operatorname{ber}(RP) \leq \frac{1}{2}\left\|\frac{\|R\|_{B,1}}{\|P\|_{B,1}}|P|^2+\frac{\|P\|_{B,1}}{\|R\|_{B,1}}|R^*|^2\right\|_{B,1}.$$

Hence,

$$\begin{aligned}&\frac{1}{2}\left[\operatorname{ber}(RP)+\frac{1}{2}\left\|\frac{\|R\|_{B,1}}{\|P\|_{B,1}}|P|^2+\frac{\|P\|_{B,1}}{\|R\|_{B,1}}|R^*|^2\right\|_{B,1}\right] \\ &\leq \frac{1}{2}\left\|\frac{\|R\|_{B,1}}{\|P\|_{B,1}}|P|^2+\frac{\|P\|_{B,1}}{\|R\|_{B,1}}|R^*|^2\right\|_{B,1}.\end{aligned} \tag{2.9}$$

By considering that $\frac{\|R\|_{B,1}}{\|P\|_{B,1}}|P|^2$ and $\frac{\|P\|_{B,1}}{\|R\|_{B,1}}|R^*|^2$ are positive operators, it can be easily seen that

$$\frac{1}{2}\left\|\frac{\|R\|_{B,1}}{\|P\|_{B,1}}|P|^2+\frac{\|P\|_{B,1}}{\|R\|_{B,1}}|R^*|^2\right\|_{B,1} \leq \frac{1}{2}\left(\|RP\|_{B,1}+\|P\|_{B,1}\|R\|_{B,1}\right). \tag{2.10}$$

Thus, by combining (2.9) and (2.10), we obtain the desired result. □

The following result generalize the inequalities (2.3) and (2.4) presented in Proposition 2.1.

Theorem 2.1. *Let $S_{\mathcal{Z}}$ be the Selberg operator defined above with $r \geq 1$ and $P, R \in \mathcal{B}(\mathcal{H})$. Then for $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$,*

$$\operatorname{ber}(RS_{\mathcal{Z}}P) \leq \frac{1}{2^{\frac{1}{r}}}\left(\operatorname{ber}^r(RP)+\left\|\frac{1}{p}|P|^{rp}+\frac{1}{q}|R^*|^{rq}\right\|_{B,1}\right)^{\frac{1}{r}}, \tag{2.11}$$

provided that $rp \geq 2, rq \geq 2$; and for $s > 0$,

$$\operatorname{ber}\left(R\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)P\right) \leq \frac{1}{2}\left\|\frac{1}{p}|P|^{sp}+\frac{1}{q}|R^*|^{sq}\right\|_{B,1}^{\frac{1}{s}} \tag{2.12}$$

for $sp \geq 2$ and $sq \geq 2$.

Proof. It follows from the proof of Proposition 2.1 that

$$\left|\left\langle RS_{\mathcal{Z}}P\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \leq \frac{1}{2}\left(\left|\left\langle RP\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|+\left\|P\widehat{k}_\lambda\right\|\left\|R^*\widehat{k}_\lambda\right\|\right) \tag{2.13}$$

If we take the power $r \geq 1$ in (2.13), we have, by the convexity of power functions, that

$$\left|\left\langle RS_{\mathcal{Z}}P\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^r \leq \left(\frac{\left|\left\langle RP\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left\|P\widehat{k}_\lambda\right\| \left\|R^*\widehat{k}_\lambda\right\|}{2}\right)^r,$$

for all $\lambda \in X$, so, we infer that

$$\left|\left\langle RS_{\mathcal{Z}}P\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^r \leq \frac{\left|\left\langle RP\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^r + \left\|P\widehat{k}_\lambda\right\|^r \left\|R^*\widehat{k}_\lambda\right\|^r}{2} \tag{2.14}$$

for all $\lambda \in X$. From Young's inequality

$$ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q, a, b \geq 0, p, q > 1 \text{ with } \frac{1}{p} + \frac{1}{q} = 1,$$

we have

$$\begin{aligned}\left\|P\widehat{k}_\lambda\right\|^r \left\|R^*\widehat{k}_\lambda\right\|^r &\leq \frac{1}{p}\left\|P\widehat{k}_\lambda\right\|^{rp} + \frac{1}{q}\left\|R^*\widehat{k}_\lambda\right\|^{rq} \\ &= \frac{1}{p}\left\|P\widehat{k}_\lambda\right\|^{\frac{2rp}{2}} + \frac{1}{q}\left\|R^*\widehat{k}_\lambda\right\|^{\frac{2rq}{2}} \\ &= \frac{1}{p}\left\langle |P|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{rp}{2}} + \frac{1}{q}\left\langle |R^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{rq}{2}}\end{aligned}$$

for all $\lambda \in X$. On the other hand, by McCarthy's inequality [27], $\langle Px, x\rangle^s \leq \langle P^s x, x\rangle$, $s \geq 1$ for $x \in \mathcal{H}$, $\|x\| = 1$, we have that

$$\frac{1}{p}\left\langle |P|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{rp}{2}} + \frac{1}{q}\left\langle |R^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{rq}{2}} \leq \frac{1}{p}\left\langle |P|^{rp}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle |R^*|^{rq}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle$$

for all $\lambda \in X$. Hence, we deduce that

$$\frac{1}{p}\left\langle |P|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{rp}{2}} + \frac{1}{q}\left\langle |R^*|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\frac{rq}{2}} \leq \left\langle\left(\frac{1}{p}|P|^{rp} + \frac{1}{q}|R^*|^{rq}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \tag{2.15}$$

for all $\lambda \in X$. By using (2.14) and (2.15), we obtain

$$\left|\left\langle RS_{\mathcal{Z}}P\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^r \leq \frac{1}{2}\left[\left|\left\langle RP\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^r + \left\langle\left(\frac{1}{p}|P|^{rp} + \frac{1}{q}|R^*|^{rq}\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right]$$

for all $\lambda \in X$, and by taking the supremum over all $\lambda \in X$, we have

$$\operatorname{ber}^r(RS_{\mathcal{Z}}P) \leq \frac{1}{2}\left[\operatorname{ber}^r(RP) + \left\|\frac{1}{p}|P|^{rp} + \frac{1}{q}|R^*|^{rq}\right\|_{B,1}\right]$$

which gives (2.11). As it was proved (see the proof of Proposition 2.1),

$$\operatorname{ber}\left(R\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)P\right) \leq \frac{1}{4}\operatorname{ber}\left(|P|^2 + |R^*|^2\right).$$

Then we have,

$$\left|\left\langle R\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)P\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^s \leq \frac{1}{2^s}\left\|P\widehat{k}_\lambda\right\|^s \left\|R^*\widehat{k}_\lambda\right\|^s \tag{2.16}$$

for all $\lambda \in X$. By Young's inequality and McCarthy's inequality for $\frac{sp}{2} \geq 1, \frac{sq}{2} \geq 1$ we also have that

$$\begin{aligned} \left\| P\widehat{k}_\lambda \right\|^s \left\| R^*\widehat{k}_\lambda \right\|^s &\leq \frac{1}{p} \left\| P\widehat{k}_\lambda \right\|^{sp} + \frac{1}{q} \left\| R^*\widehat{k}_\lambda \right\|^{sq} \\ &= \frac{1}{p} \left\| P\widehat{k}_\lambda \right\|^{\frac{2sp}{2}} + \frac{1}{q} \left\| R^*\widehat{k}_\lambda \right\|^{\frac{2sq}{2}} \\ &= \frac{1}{p} \left\langle |P|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\frac{sp}{2}} + \frac{1}{q} \left\langle |R^*|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\frac{sq}{2}} \\ &\leq \frac{1}{p} \left\langle |P|^{sp} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle + \frac{1}{q} \left\langle |R^*|^{sq} \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \end{aligned}$$

for all $\lambda \in X$. Thus, we get

$$\left\| P\widehat{k}_\lambda \right\|^s \left\| R^*\widehat{k}_\lambda \right\|^s \leq \left\langle \left(\frac{1}{p} |P|^{sp} + \frac{1}{q} |R^*|^{sq} \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \tag{2.17}$$

for all $\lambda \in X$. By making use of (2.16) and (2.17), we have that

$$\left| \left\langle R \left(S_{\mathcal{Z}} - \frac{1}{2} I \right) P\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^s \leq \frac{1}{2^s} \left\langle \left(\frac{1}{p} |P|^{sp} + \frac{1}{q} |R^*|^{sq} \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle,$$

for all $\lambda \in X$, and finally by taking the supremum over all $\lambda \in X$, we obtain the inequality (2.12)

$$\operatorname{ber} \left(R \left(S_{\mathcal{Z}} - \frac{1}{2} I \right) P \right) \leq \frac{1}{2} \left\| \frac{1}{p} |P|^{sp} + \frac{1}{q} |R^*|^{sq} \right\|_{B,1}^{\frac{1}{s}},$$

as desired. □

If we put $\mathcal{H} = H^2(\mathbb{D})$, $R = S^*$ and $P = S$, where $Sf(z) = zf(z)$ is the shift operator on H^2, then we have the following bounds in Theorem 2.1:

1) $\operatorname{ber}(S^* S_{\mathcal{Z}} S) \leq 1$

2) $\operatorname{ber}\left(S^* \left(S_{\mathcal{Z}} - \frac{1}{2} I\right) S\right) \leq 2^{\frac{1}{s}-1}$ for $sp \geq 2$ and $sq \geq 2$.

Corollary 2.2. *If $r \geq 1$ and $P \in \mathcal{B}(\mathcal{H})$, then, for $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\operatorname{ber}(PS_{\mathcal{Z}}P) \leq \frac{1}{2^{\frac{1}{r}}} \left(\operatorname{ber}^r\left(P^2\right) + \left\| \frac{1}{p} |P|^{rp} + \frac{1}{q} |P^*|^{rq} \right\|_{B,1} \right)^{\frac{1}{r}},$$

provided that $rp \geq 2$, $rq \geq 2$; and for $s > 0$,

$$\operatorname{ber} \left(P \left(S_{\mathcal{Z}} - \frac{1}{2} I \right) P \right) \leq \frac{1}{2} \left\| \frac{1}{p} |P|^{sp} + \frac{1}{q} |P^*|^{sq} \right\|_{B,1}^{\frac{1}{s}},$$

provided that $sp \geq 2$, $sq \geq 2$.

Using a convex combination of $|P|$ and $|R^*|$, we prove an upper bound for the Berezin number in the following theorem.

Theorem 2.2. *Let $S_{\mathcal{Z}}$ be the Selberg operator defined above and $P, R \in \mathcal{B}(\mathcal{H})$. Then for $\alpha \in [0,1]$,*

$$\operatorname{ber}^2\left(RS_{\mathcal{Z}}P\right) \leq \frac{1}{2}\left(\operatorname{ber}^2\left(RP\right) + \left\|(1-\alpha)\left|P\right|^2 + \alpha\left|R^*\right|^2\right\|_{B,2} \left\|P\right\|_{B,1}^{2\alpha} \left\|R\right\|_{B,1}^{2(1-\alpha)}\right) \tag{2.18}$$

and

$$\operatorname{ber}^2\left(R\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)P\right) \leq \frac{1}{4}\left\|(1-\alpha)\left|P\right|^2 + \alpha\left|R^*\right|^2\right\|_{B,2} \left\|P\right\|_{B,1}^{2\alpha} \left\|R\right\|_{B,1}^{2(1-\alpha)} \tag{2.19}$$

In particular, we obtain

$$\operatorname{ber}^2\left(RS_{\mathcal{Z}}P\right) \leq \frac{1}{2}\left(\operatorname{ber}^2\left(RP\right) + \frac{1}{2}\left\|\left|P\right|^2 + \left|R^*\right|^2\right\|_{B,1} \left\|P\right\|_{B,1} \left\|R\right\|_{B,1}\right)$$

and

$$\operatorname{ber}^2\left(R\left(S_{\mathcal{Z}} - \frac{1}{2}I\right)P\right) \leq \frac{1}{8}\left\|\left|P\right|^2 + \left|R^*\right|^2\right\|_{B,1} \left\|P\right\|_{B,1} \left\|R\right\|_{B,1}$$

Proof. We have from (2.14) for $r=2$ that

$$\begin{aligned}
&\left|\left\langle RS_{\mathcal{Z}}P\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \\
&\leq \frac{1}{2}\left(\left|\left\langle RP\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 + \left\|P\widehat{k}_\lambda\right\|^2 \left\|R^*\widehat{k}_\lambda\right\|^2\right) \\
&= \frac{1}{2}\left(\left|\left\langle RP\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 + \left\langle \left|P\right|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle \left|R^*\right|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right) \\
&= \frac{1}{2}\left(\left|\left\langle RP\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left\langle \left|P\right|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{1-\alpha} \left\langle \left|R^*\right|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\alpha} \left\langle \left|P\right|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{\alpha} \left\langle \left|R^*\right|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{1-\alpha}\right) \\
&\leq \frac{1}{2}\left(\left|\left\langle RP\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + (1-\alpha)\left\langle \left|P\right|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \alpha\left\langle \left|R^*\right|^2\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\|P\widehat{k}_\lambda\right\|^{2\alpha} \left\|R^*\widehat{k}_\lambda\right\|^{2(1-\alpha)}\right) \\
&= \frac{1}{2}\left(\left|\left\langle RP\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| + \left\langle \left[(1-\alpha)\left|P\right|^2 + \alpha\left|R^*\right|^2\right]\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\|P\widehat{k}_\lambda\right\|^{2\alpha} \left\|R^*\widehat{k}_\lambda\right\|^{2(1-\alpha)}\right),
\end{aligned}$$

for all $\lambda \in X$. Hence it follows that

$$\begin{aligned}
&\operatorname{ber}^2\left(RS_{\mathcal{Z}}P\right) \\
&= \sup_{\lambda\in X}\left|\left\langle RS_{\mathcal{Z}}P\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \\
&\leq \frac{1}{2}\sup_{\lambda\in X}\left(\left|\left\langle RP\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 + \left\langle \left[(1-\alpha)\left|P\right|^2 + \alpha\left|R^*\right|^2\right]\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\|P\widehat{k}_\lambda\right\|^{2\alpha} \left\|R^*\widehat{k}_\lambda\right\|^{2(1-\alpha)}\right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\operatorname{ber}^2\left(RS_{\mathcal{Z}}P\right) &\leq \frac{1}{2}\sup_{\lambda\in X}\left|\left\langle RP\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \\
&\quad + \frac{1}{2}\sup_{\lambda\in X}\left(\left\langle \left[(1-\alpha)\left|P\right|^2 + \alpha\left|R^*\right|^2\right]\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\|P\widehat{k}_\lambda\right\|^{2\alpha} \left\|R^*\widehat{k}_\lambda\right\|^{2(1-\alpha)}\right)
\end{aligned} \tag{2.20}$$

and by using that

$$\begin{aligned}
&\sup_{\lambda\in X}\left(\left\langle\left[(1-\alpha)\left|P\right|^2+\alpha\left|R^*\right|^2\right]\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\left\|P\widehat{k}_\lambda\right\|^{2\alpha}\left\|R^*\widehat{k}_\lambda\right\|^{2(1-\alpha)}\right)\\
&\le\sup_{\lambda\in X}\left\langle\left[(1-\alpha)\left|P\right|^2+\alpha\left|R^*\right|^2\right]\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\sup_{\lambda\in X}\left\|P\widehat{k}_\lambda\right\|^{2\alpha}\sup_{\lambda\in X}\left\|R^*\widehat{k}_\lambda\right\|^{2(1-\alpha)}\\
&\le\sup_{\lambda,\mu\in X}\left\langle\left[(1-\alpha)\left|P\right|^2+\alpha\left|R^*\right|^2\right]\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\\
&=\left\|(1-\alpha)\left|P\right|^2+\alpha\left|R^*\right|^2\right\|_{B,2}\left\|P\right\|_{B,1}^{2\alpha}\left\|R\right\|_{B,1}^{2(1-\alpha)},
\end{aligned}$$

by (2.20), we get the required result (2.18).

By (2.16), we obtain for $s=2$ and for all $\lambda\in X$ that

$$\begin{aligned}
\left|\left\langle R\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)P\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2&\le\frac{1}{4}\left\|P\widehat{k}_\lambda\right\|^2\left\|R^*\widehat{k}_\lambda\right\|^2\\
&=\frac{1}{4}\left\langle\left|P\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\left\langle\left|R^*\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\\
&\le\frac{1}{4}\left\langle\left[(1-\alpha)\left|P\right|^2+\alpha\left|R^*\right|^2\right]\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\left\|P\widehat{k}_\lambda\right\|^{2\alpha}\left\|R^*\widehat{k}_\lambda\right\|^{2(1-\alpha)},
\end{aligned}$$

which obviously implies (2.19). □

3. Berezin number inequalities for summations with Selberg operator

In the present section, we prove some new bounds related to the summations with the Selberg operator. The initial statement in this section offers an expanded interpretation of the inequality defined in Proposition 2.1, which is as follows:

$$\begin{aligned}
\left\|RS_{\mathcal{Z}}P-\frac{1}{2}RP\right\|_{B,2}^2&=\left\|R\left(S_{\mathcal{Z}}-\frac{1}{2}I\right)P\right\|_{B,2}^2\\
&\le\frac{1}{4}\left\|P\right\|_{B,1}^2\left\|R\right\|_{B,1}^2=\frac{1}{4}\left\|\left|P\right|^2\right\|_{B,1}\left\|\left|R^*\right|^2\right\|_{B,1}
\end{aligned}$$

for every $P,R\in\mathcal{B}(\mathcal{H})$.

Theorem 3.1. *We suppose that $S_{\mathcal{Z}}$ is the Selberg operator defined above, $P_i,R_i\in\mathcal{B}(\mathcal{H})$, $i\in\{1,...,m\}$ and $Q_i\ge 0$, $i\in\{1,...,m\}$ with $\sum_{i=1}^m Q_i=1$. Then*

$$\left\|\sum_{i=1}^m Q_iR_iS_{\mathcal{Z}}P_i-\frac{1}{2}\sum_{i=1}^m Q_iR_iP_i\right\|_{B,2}^2\le\frac{1}{4}\left\|\sum_{i=1}^m Q_i\left|P_i\right|^2\right\|_{B,1}\left\|\sum_{i=1}^m Q_i\left|R_i^*\right|^2\right\|_{B,1}$$

and

$$\operatorname{ber}\left(\sum_{i=1}^m Q_iR_iS_{\mathcal{Z}}P_i-\frac{1}{2}\sum_{i=1}^m Q_iR_iP_i\right)\le\left\|\sum_{i=1}^m Q_i\frac{\left|P_i\right|^2+\left|R_i^*\right|^2}{4}\right\|_{B,1}.\tag{3.1}$$

Proof. We will use inequality (2.8) in the proof of Proposition 2.1. We have for any $\lambda,\mu\in X$ that

$$\sum_{i=1}^m Q_i\left|\left\langle\left(R_iS_{\mathcal{Z}}P_i-\frac{1}{2}R_iP_i\right)\widehat{k}_\lambda,\widehat{k}_\mu\right\rangle\right|\le\frac{1}{2}\sum_{i=1}^m Q_i\left\|P_i\widehat{k}_\lambda\right\|\left\|R^*\widehat{k}_\mu\right\|\tag{3.2}$$

The generalized triangle inequality implies that

$$\sum_{i=1}^{m} Q_i \left|\left\langle \left(R_i S_{\mathcal{Z}} P_i - \frac{1}{2} R_i P_i\right) \widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right| \geq \left|\sum_{i=1}^{m} Q_i \left\langle \left(R_i S_{\mathcal{Z}} P_i - \frac{1}{2} R_i P_i\right) \widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right|$$
$$= \left|\left\langle \left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i - \frac{1}{2} \sum_{i=1}^{m} Q_i R_i P_i\right) \widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right|$$

for all $\lambda, \mu \in X$.

Now applying Cauchy-Bunyakovski-Schwarz inequality, we have:

$$\sum_{i=1}^{m} Q_i \left\|P_i \widehat{k}_\lambda\right\| \left\|R_i^* \widehat{k}_\mu\right\| \leq \left(\sum_{i=1}^{m} Q_i \left\|P_i \widehat{k}_\lambda\right\|^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} Q_i \left\|R_i^* \widehat{k}_\mu\right\|^2\right)^{\frac{1}{2}}$$
$$= \left(\sum_{i=1}^{m} Q_i \left\langle |P_i|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle\right)^{\frac{1}{2}} \left(\sum_{i=1}^{m} Q_i \left\langle |R_i^*|^2 \widehat{k}_\mu, \widehat{k}_\mu \right\rangle\right)^{\frac{1}{2}}$$

for $\lambda, \mu \in X$, which implies that

$$\sum_{i=1}^{m} Q_i \left\|P_i \widehat{k}_\lambda\right\| \left\|R_i^* \widehat{k}_\mu\right\| \leq \left\langle \sum_{i=1}^{m} Q_i |P_i|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{m} Q_i |R_i^*|^2 \widehat{k}_\mu, \widehat{k}_\mu \right\rangle^{\frac{1}{2}} \tag{3.3}$$

for all $\lambda, \mu \in X$. Using (3.2) and (3.3), we obtain for all $\lambda, \mu \in X$ that

$$\left|\left\langle \left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i - \frac{1}{2} \sum_{i=1}^{m} Q_i R_i P_i\right) \widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right| \tag{3.4}$$
$$\leq \frac{1}{2} \left\langle \sum_{i=1}^{m} Q_i |P_i|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\frac{1}{2}} \left\langle \sum_{i=1}^{m} Q_i |R_i^*|^2 \widehat{k}_\mu, \widehat{k}_\mu \right\rangle^{\frac{1}{2}}.$$

After taking the supremum over all $\lambda, \mu \in X$, we have that

$$\left\|\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i - \frac{1}{2} \sum_{i=1}^{m} Q_i R_i P_i\right\|_{B,2}$$
$$= \sup_{\lambda,\mu \in X} \left|\left\langle \left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i - \frac{1}{2} \sum_{i=1}^{m} Q_i R_i P_i\right) \widehat{k}_\lambda, \widehat{k}_\mu \right\rangle\right|$$
$$\leq \frac{1}{2} \sup_{\lambda,\mu \in X} \left(\left\langle \sum_{i=1}^{m} Q_i |P_i|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\frac{1}{2}} . \left\langle \sum_{i=1}^{m} Q_i |R_i^*|^2 \widehat{k}_\mu, \widehat{k}_\mu \right\rangle^{\frac{1}{2}}\right)$$
$$= \frac{1}{2} \sup_{\lambda \in X} \left\langle \sum_{i=1}^{m} Q_i |P_i|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle^{\frac{1}{2}} . \sup_{\mu \in X} \left\langle \sum_{i=1}^{m} Q_i |R_i^*|^2 \widehat{k}_\mu, \widehat{k}_\mu \right\rangle^{\frac{1}{2}}$$
$$= \frac{1}{2} \left\|\sum_{i=1}^{m} Q_i |P_i|^2\right\|_{B,1}^{\frac{1}{2}} \left\|\sum_{i=1}^{m} Q_i |R_i^*|^2\right\|_{B,1}^{\frac{1}{2}}$$

which proves (2.20).

We have from (3.4) that

$$
\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i-\frac{1}{2}\sum_{i=1}^{m} Q_i R_i P_i\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right| \\
&\leq \frac{1}{2}\left\langle\sum_{i=1}^{m} Q_i\left|P_i\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^{\frac{1}{2}}\left\langle\sum_{i=1}^{m} Q_i\left|R_i^*\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^{\frac{1}{2}} \\
&\leq \frac{1}{4}\left(\left\langle\sum_{i=1}^{m} Q_i\left|P_i\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle+\left\langle\sum_{i=1}^{m} Q_i\left|R_i^*\right|^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right) \\
&= \frac{1}{4}\left\langle\left(\sum_{i=1}^{m} Q_i\left|P_i\right|^2+\sum_{i=1}^{m} Q_i\left|R_i^*\right|^2\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle \\
&= \left\langle\left(\sum_{i=1}^{m} Q_i\frac{\left|P_i\right|^2+\left|R_i^*\right|^2}{4}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle,
\end{aligned}
$$

and

$$
\begin{aligned}
&\sup_{\lambda\in X}\left|\left\langle\left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i-\frac{1}{2}\sum_{i=1}^{m} Q_i R_i P_i\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right| \\
&\leq \sup_{\lambda\in X}\left\langle\left(\sum_{i=1}^{m} Q_i\frac{\left|P_i\right|^2+\left|R_i^*\right|^2}{4}\right)\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle
\end{aligned}
$$

which is equivalent to

$$
\operatorname{ber}\left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i-\frac{1}{2}\sum_{i=1}^{m} Q_i R_i P_i\right)\leq\left\|\sum_{i=1}^{m} Q_i\frac{\left|P_i\right|^2+\left|R_i^*\right|^2}{4}\right\|_{B,1}.
$$

The evidence is now complete. $\square$

Theorem 3.2. *With the assumptions of Theorem 2.2, we have*

$$
\begin{aligned}
&\operatorname{ber}^2\left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i-\frac{1}{2}\sum_{i=1}^{m} Q_i R_i P_i\right) \\
&\leq \frac{1}{2}\left\|\frac{1}{Q}\left(\sum_{i=1}^{m} Q_i\left|P_i\right|^2\right)^{Q}+\frac{1}{q}\left(\sum_{i=1}^{m} Q_i\left|R_i^*\right|^2\right)^{q}\right\|_{B,1}
\end{aligned}
\tag{3.5}
$$

for $Q,q>1$ with $\frac{1}{Q}+\frac{1}{q}=1$ and

$$
\begin{aligned}
&\operatorname{ber}^2\left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i-\frac{1}{2}\sum_{i=1}^{m} Q_i R_i P_i\right) \\
&\leq \frac{1}{4}\left(\operatorname{ber}\left(\sum_{i=1}^{m} Q_i\left|R_i^*\right|^2\sum_{i=1}^{m} Q_i\left|P_i\right|^2\right)+\left\|\sum_{i=1}^{m} Q_i\left|P_i\right|^2\right\|_{B,1}\left\|\sum_{i=1}^{m} Q_i\left|R_i^*\right|^2\right\|_{B,1}\right).
\end{aligned}
\tag{3.6}
$$

Proof. By virtue of (3.4), taking $\mu = \lambda$, we have that

$$\begin{aligned} &\left|\left\langle \left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i - \frac{1}{2}\sum_{i=1}^{m} Q_i R_i P_i\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \\ &\leq \frac{1}{2}\left\langle \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle. \end{aligned} \tag{3.7}$$

By applying Young inequality we obtain

$$\begin{aligned} &\left\langle \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \left\langle \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \\ &\leq \frac{1}{Q}\left\langle \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^p + \frac{1}{q}\left\langle \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^q \end{aligned}$$

for all $\lambda \in X$ and $Q, q > 1$ with $\frac{1}{Q} + \frac{1}{q} = 1$. By the McCarthy inequality, we also have

$$\begin{aligned} &\frac{1}{Q}\left\langle \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^p + \frac{1}{q}\left\langle \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^q \\ &\leq \frac{1}{p}\left\langle \left(\sum_{i=1}^{m} Q_i \left|P_i\right|^2\right)^p \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + \frac{1}{q}\left\langle \left(\sum_{i=1}^{m} Q_i \left|R_i^*\right|^2\right)^q \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \end{aligned}$$

for all $\lambda \in X$, which yields that

$$\begin{aligned} &\frac{1}{p}\left\langle \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^p + \frac{1}{q}\left\langle \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^q \\ &\leq \left\langle \left[\frac{1}{p}\left(\sum_{i=1}^{m} Q_i \left|P_i\right|^2\right)^p + \frac{1}{q}\left(\sum_{i=1}^{m} Q_i \left|R_i^*\right|^2\right)^q\right] \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle, \end{aligned} \tag{3.8}$$

for all $\lambda \in X$. Hence, according to (3.7) and (3.8), we have

$$\begin{aligned} &\left|\left\langle \left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i - \frac{1}{2}\sum_{i=1}^{m} Q_i R_i P_i\right)\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^2 \\ &\leq \frac{1}{2}\left\langle \left[\frac{1}{p}\left(\sum_{i=1}^{m} Q_i \left|P_i\right|^2\right)^p + \frac{1}{q}\left(\sum_{i=1}^{m} Q_i \left|R_i^*\right|^2\right)^q\right] \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle \end{aligned}$$

for all $\lambda \in X$, this implies the desired inequality (3.5).

By using Buzano's inequality

$$\left|\langle u, e\rangle \langle e, v\rangle\right| \leq \frac{1}{2}\left(\left|\langle u, v\rangle\right| + \|u\| \|v\|\right),$$

where $\|e\|=1$, we have

$$\begin{aligned}
&\left\langle \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \left\langle \widehat{k}_\lambda, \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \widehat{k}_\lambda \right\rangle \\
&\leq \frac{1}{2}\left(\left| \left\langle \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda, \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \widehat{k}_\lambda \right\rangle \right| + \left\| \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda \right\| \left\| \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \widehat{k}_\lambda \right\| \right) \\
&= \frac{1}{2}\left(\left| \left\langle \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| + \left\| \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda \right\| \left\| \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \widehat{k}_\lambda \right\| \right)
\end{aligned}$$

for all $\lambda \in X$.

Now, by (3.6), we get for all $\lambda \in X$ that

$$\begin{aligned}
&\left| \left\langle \left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i - \frac{1}{2} \sum_{i=1}^{m} Q_i R_i P_i \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \\
&\leq \frac{1}{4}\left(\left| \left\langle \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| + \left\| \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda \right\| \left\| \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \widehat{k}_\lambda \right\| \right),
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{\lambda \in X} \left| \left\langle \left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i - \frac{1}{2} \sum_{i=1}^{m} Q_i R_i P_i \right) \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right|^2 \\
&\leq \frac{1}{4} \sup_{\lambda \in X} \left(\left| \left\langle \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| + \left\| \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \widehat{k}_\lambda \right\|_{B,1} \left\| \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \widehat{k}_\lambda \right\|_{B,1} \right).
\end{aligned}$$

Thus, we deduce that

$$\begin{aligned}
&\operatorname{ber}^2 \left(\sum_{i=1}^{m} Q_i R_i S_{\mathcal{Z}} P_i - \frac{1}{2} \sum_{i=1}^{m} Q_i R_i P_i \right) \\
&\leq \frac{1}{4}\left(\operatorname{ber}\left(\sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \right) + \left\| \sum_{i=1}^{m} Q_i \left|P_i\right|^2 \right\|_{B,1} \left\| \sum_{i=1}^{m} Q_i \left|R_i^*\right|^2 \right\|_{B,1} \right).
\end{aligned}$$

which implies the desired result (3.6). The theorem is proved. □

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