Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

ANALYSIS OF A DOUBLE NONLINEAR PARABOLIC EQUATION WITH A SOURCE IN AN INHOMOGENEOUS MEDIUM

MERSAID ARIPOV AND MAKHMUD BOBOKANDOV

Abstract. This paper studies the properties of solutions for a double nonlinear parabolic equation with variable density, not in divergence form with a source. The problem is formulated as a partial differential equation with a nonlinear term. The main results are the existence of weak solutions in suitable function spaces; regularity and positivity of solutions; asymptotic behaviour of solutions as time goes to infinity; comparison principles; and maximum principles for solutions. The proofs are based on the energy method, comparison methods, and asymptotic techniques.

1. Introduction

We study a double nonlinear, non-divergent parabolic equation with a source in an inhomogeneous medium in $Q = \left\{(x,t) \,|\, x \in \mathbb{R}^N,\, t > 0\right\}$

$$\rho_1(x)\, u_t = u^q \nabla\left(\rho_2(x)\, u^{m-1} \left|\nabla u^k\right|^{p-2} \nabla u\right) + \rho_3(x)\, u^\beta; \quad (x,t) \in Q, \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^N, \tag{1.2}$$

where $q < 1$, $m \geq 1$, $k > 0$, $p \geq 2$, $\beta > q + m + k(p-2) > 1$, $N \geq 1$, $n_2 \neq p + n_1$, $0 \leq n_1 \leq n_3 < \frac{\beta - q - m - k(p-2)}{q + m + k(p-2) - 1} N$ - are given numerical parameters, $\rho_i(x) = |x|^{n_i}$, $i = 1, 2, 3$, and $u_0(x)$ is a non-negative bounded and continuous function in $\mathbb{R}^N$. Here we mean by a solution, a function $u(x,t)$ is nonnegative and continuous in $Q \setminus (0,0)$, satisfying (1.1)-(1.2) in the distribution sense.

The equation (1.1) included many known equations, such as the porous medium equation, p-Laplacian equation, heat equation, Leibenson equation, Boussinesq equation in filtration of liquid and gas, and so on [37]. To simulate a broad variety of physical processes, equation (1.1) is important. For example, curve shortening flow, resistive diffusion phenomena in force-free magnetic fields, diffusive processes found in biological species, and the spread of infectious diseases are among the many applications of equation (1.1) (see references [36, 6, 30]).

The problem (1.1) in the particular value of the numerical parameters is intensively studied by many authors [1, 5, 29].

2010 *Mathematics Subject Classification.* 35K55, 35D30, 35B51, 35B44.
Key words and phrases. Self-similar solution, global solvability, blow-up, Life span.

Zhaoyin Xiang, Chunlai Mu, and Xuegang Hu [38] investigated the properties of spatial localization, existence and non-existence of global solutions for problem (1.1)-(1.2) where the density function has the form

$$|x|^{-k}, (1+|x|)^{-k}.$$

Qualitative properties of solutions of a doubly nonlinear reaction-diffusion, self-similar profiles of solutions, global existence, and blow-up solutions studied in [25, 31, 41]. Asymptotic behaviour of solutions of the nonlinear diffusion equation with absorption at a critical exponent considered in the works [13, 17]. In [24, 22] the Cauchy problem for the following two equations with variable coefficients is studied:

$$\rho(x)\frac{\partial u}{\partial t} = div\left(u^{m-1}|\nabla u|^{p-2}\nabla u\right) + \rho_0(x)u^{\beta}, x \in \mathbb{R}^N, t > 0,$$

where $p > 1, m+p > 3, \beta > m+p-2, \rho(x) = \begin{cases} |x|^{-n}, \\ (1+|x|)^{-n} \end{cases}, \rho_0(x) = \rho(x)$ or $\rho_0(x) = 1$. Any nontrivial Cauchy problem solution blows up in a finite time, it has been shown, under particular parameter constraints. Furthermore, the authors found a sharp approximation of the solution that is universal near the blow-up point.

R. Gianni, A. Tedeev and V. Vespri studied the asymptotic behaviour of non-negative solutions of the following equation [18]

$$\begin{cases} \frac{\partial u}{\partial t} = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}\left(a_{ij}(x,t)u^{m-1}|Du|^{p-2}\frac{\partial u}{\partial x_i}\right), \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^N \end{cases}$$

where $a_{ij}(x,t) = a_{ji}(x,t), \ i,j = \overline{1,N}$ are measurable functions. They showed when the initial datum has a finite mass, the asymptotic expansion of the solution for a large time, uniformly in whole space, is established.

Galaktionov V.A. and Vazquez J.L. investigated [16] the nonlinear case of the Laplace equation with critical exponents

$$\begin{aligned} &u_t = \Delta\left(u^{\sigma+1}\right) - u^{\beta}; \quad (x,t) \in \mathbb{R}^N \times (0,\infty), \\ &u(x,0) = u_0(x) \geq 0, x \in \mathbb{R}^N \end{aligned} \tag{1.3}$$

where $N \geq 1, \sigma > 0, \beta = \sigma + 1 + 2/N$. The authors show that, the long-term asymptotic behaviour of the solution for the critical exponent in the following form

$$\begin{aligned} &u(t,x) = ((T+t)\ln(T+t))^{-k} F(\xi,a), \xi = |x|(T+t)^{-\frac{k}{N}}(\ln(T+t))^{\frac{k\sigma}{2}}, \\ &F(\xi,a) = C_0\left(a^2 - \xi^2\right)_+^{\frac{1}{\sigma}}, k = \frac{1}{\sigma + 2/N}, C_0 = \left[\frac{k\sigma}{2N(\sigma+1)}\right]^{\frac{1}{\sigma}}, T > 1. \end{aligned} \tag{1.4}$$

By creating sub- and super-solutions, they showed that solution (1.4) is the long-time asymptotic of the solution to problems (1.3). The sub- and super-solutions with variables are as follows:

$$\begin{aligned} &((T+t)\ln(T+t))^{-k} F(\xi,a_-) \leq u(t,x) \leq ((T+t)\ln(T+t))^{-k} F(\xi,a_+), \\ &0 < a_- < a_+. \end{aligned}$$

H. Brezis and A. Friedman [10] showed that the equation (1.1)-(1.2) has no (singular [21]) solution if the $\beta \geq 1+\frac{2}{N}$ when the equality (1.6) is equal to zero in the case $m=k=1, p=2, q=n_i=0, (i=1,2,3)$.
Furthermore, H. Brezis, L.A. Peletier, and D. Terman [11] found a different type of singular solution or very singular solution and it has a stronger singularity than the fundamental solutions at $(0,0)$, i.e., such that

$$\lim_{t\to 0}\int_{|x|<r} u(x,t)=\infty.$$

for every $r>0$.
Moreover, F. Nicolosi et al. [26], and K.M. Hui [19] showed removable singularities with point-wise and Schauder estimates under suitable conditions.
The source term $\rho_3(x)\,u^{\beta}$ can lead to a finite time blow-up, where solutions become unbounded in finite time. Analyzing the blow-up behaviour and the conditions under which it occurs is difficult, especially in inhomogeneous conditions. Additionally, studying the asymptotic behaviour for large time requires specialized techniques, as standard tools may not apply due to the nonlinearity and inhomogeneity of the equation (1.1). Furthermore, the equation (1.1) is doubly singular and therefore does not have a classical solution in general cases [2]. As a result, we need to define a weak solution to address this issue.

Definition 1.1. A non-negative function $u(x,t)$ defined in Q is called a weak solution of Cauchy problem (1.1)-(1.2), if for every bounded open set Ω with smooth boundary $\partial\Omega$, $u\in L^{\infty}_{loc}(\Omega\times(0,T))\cap C\left((0,T),\,L^{2}_{|x|^{n_1},loc}(\Omega)\right)\cap L^{\beta}_{|x|^{n_3},loc}(\Omega\times(0,T))$, $u^{m-1}\left|\nabla u^k\right|^p\in L^1_{loc}(\Omega\times(0,T))$ and the following integral identity fulfils

$$\int_\Omega \rho_1(x)\,u(x,t)\,\eta(x,t)\,dx-\int_\Omega \rho_1(x)\,u(x,t_0)\,\eta(x,t_0)\,dx=\int_0^t\int_\Omega \rho_3(x)\,u^{\beta}\eta dxd\tau$$
$$+\int_0^t\int_\Omega\left(\rho_1(x)\,u\cdot\frac{\partial\eta(x,\tau)}{\partial\tau}-\rho_2(x)\,u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u\cdot\nabla\left(u^q\eta\right)\right)dxd\tau \quad (1.5)$$

for all $0\le t\le T$ and for any test function $\eta\in C^1_0(\Omega\times(0,T))$. Moreover,

$$\lim_{t\to 0}\int_\Omega u(x,t)\,\zeta(x)\,dx=\int_\Omega u(x,0)\,\zeta(x)\,dx \quad (1.6)$$

for any $\zeta(x)\in C^1_0(\Omega)$ (see, [10], Chapter 5, page 77 [37]).

Let us denote

$$T*=\sup\left\{T>0;\ \sup_{t\in[0,T)}\|u(x,t)\|_{\infty}<\infty\right\},$$

then $T*$ is called "the life span" of the solution $u(x,t)$. If $T*=\infty$, then the solution $u(x,t)$ is global in time mean. On the other side, if $T*<\infty$, the solution $u(x,t)$ is called "blow-up" in finite time $T*$.
Due to the non-standard growth conditions, obtaining existence and uniqueness results for weak solutions is challenging. The solutions often have limited regularity and may only be defined in the weak or distributional sense, which requires careful functional analysis and compactness methods [35, 37, 2].
The Cauchy problem for equation (1.1) has been extensively studied in the literature, specifically in reference [8, 39, 40], where it has been studied for the

particular case $q = n_2 = 0$, $k = 1$. Also, they found the second critical exponent value. This paper aims to present the primary findings by using the techniques proposed by [34] and utilizing the methodologies outlined in the aforementioned literature. We express our sincere gratitude to the authors for their invaluable contributions to this field of research. Their work has enabled us to delve deeper into this subject matter and apply their established theorems and methods to shed light on new insights.
The main objective of our study is to investigate the behaviour of the solution $u(x,t)$ to problems (1.1)-(1.2), when the initial data $u_0(x)$ has slow decay in proximity to $x = \infty$. For instance, consider the following case:

$$u_0(x) \cong M|x|^{-a} \quad \text{with } a > 0 \text{ and } M > 0, \tag{1.7}$$

we are investigating whether solutions to equations (1.1)-(1.2) exist globally or not by analyzing them in relation to the variables denoted by M and a.
Throughout the paper, we denote by $C_b\left(\mathbb{R}^N\right)$ the space of all bounded continuous functions in $\mathbb{R}^N$. For $a \geq 0$, we define

$$F_a = \left\{ \rho(x) \in C_b\left(\mathbb{R}^N\right) \mid \rho(x) \geq 0 \ \text{ and } \lim_{|x|\to\infty} \inf |x|^a \rho(x) > 0 \right\},$$

$$F^a = \left\{ \rho(x) \in C_b\left(\mathbb{R}^N\right) \mid \rho(x) \geq 0 \ \text{ and } \lim_{|x|\to\infty} \sup |x|^a \rho(x) < \infty \right\}.$$

In addition, let

$$\beta > \beta_c = q + m + k(p-2) + \frac{(1-q)(p-n_2+n_3)}{N+n_1},\ a_c = \frac{p+n_3-n_2}{\beta - q - m - k(p-2)}.$$

2. Global existence of the solution

Within this section, we will be discussing the condition for global existence as well as the behaviour of the global solution over a large time.

Theorem 2.1. *Suppose that* $u_0(x) = \lambda\varphi(x)$, $\lambda > 0$ *and* $\varphi(x) \in F^a$ *for some* $a \in (a_c,\ N+n_1)$, *then there exists* $\lambda_0 = \lambda_0(\varphi) > 0$ *such that the solution* $u(x,t)$ *of the problem* (1.1)-(1.2) *exists globally for all* $\lambda < \lambda_0$. *Furthermore, the solution has the following estimate:*

$$\|u(x,t)\|_\infty \leq Ct^{-\frac{a}{a(q+m+k(p-2)-1)+p+n_1-n_2}},\ t > 0 \tag{2.1}$$

Theorem 2.2. *Let* $a \in (a_c,\ N+n_1), \varphi(x) \in F^a$ *and* $u_0(x) = \lambda\varphi(x)$, $\lambda > 0$ *fulfill the following identity,*

$$\lim_{|x|\to\infty} |x|^a \varphi(x) = M > 0, \tag{2.2}$$

then there exists $\lambda_0 = \lambda_0(\varphi) > 0$ *such that for* $\lambda < \lambda_0$, *the solution* $u(x,t)$ *fulfils*

$$t^{\frac{a}{a(q+m+k(p-2)-1)+p+n_1-n_2}} \left|u(x,t) - U_{\lambda M,a}(x,t)\right| \to 0, \quad \text{as } t \to \infty \tag{2.3}$$

uniformly in compact set of $\mathbb{R}^N$, *where* $U_{\lambda M,a}(x,t)$ *is the solution to the following Cauchy problem*

$$\begin{cases} \rho_1(x)\, u_t = u^q \nabla\left(\rho_2(x)\, u^{m-1}\left|\nabla u^k\right|^{p-2} \nabla u\right),\ (x,t) \in Q, \\ u(x,0) = \lambda M|x|^{-a},\ x \in \mathbb{R}^N. \end{cases} \tag{2.4}$$

Proof. We introduce the radially symmetric self-similar solution $U_{M,a}(x,t)$, to the following Cauchy problem to prove theorems 2.1-2.2

$$\begin{cases} \rho_1(x)\, u_t = u^q \nabla\left(\rho_2(x)\, u^{m-1}\left|\nabla u^k\right|^{p-2}\nabla u\right), \ (x,t)\in Q, \\ u(x,0) = u_0(x) = M|x|^{-a}, \ x\in\mathbb{R}^N. \end{cases} \tag{2.5}$$

It is widely acknowledged that, under specific suitable conditions, the local time existence of a solution to equation (2.5) has been firmly established in research articles [22, 23]. Additionally, the uniqueness of the solution to (2.5) can be proven by using the same method described in Chapter 5 of the research article [28]. Due to the symmetric properties of (2.5), the solution $U_{M,a}(x,t)$, can be expressed in the following form:

$$\begin{aligned} &U_{M,a}(x,t) = t^{-a\alpha} f_M(r), \text{ with } r = |x|\, t^{-\alpha}, \\ &\alpha = \frac{1}{a\,(q+m+k\,(p-2)-1)+p+n_1-n_2} \end{aligned} \tag{2.6}$$

where the positive function f_M is the solution of the following problem

$$\begin{cases} \left(f_M^{m-1}\left|\left(f_M^k\right)'\right|^{p-2} f_M'\right)' + \frac{N+n_2-1}{r} f_M^{m-1}\left|\left(f_M^k\right)'\right|^{p-2}(f_M)' + \alpha r^{n_1-n_2+1} f_M^{-q} f_M' \\ + a\alpha r^{n_1-n_2} f_M^{1-q} = 0, \ r>0, \\ f_M(r)\geq 0, \ r\geq 0, \ f_M'(0) = 0, \ \lim\limits_{r\to+\infty} r^a f_M(r) = M. \end{cases} \tag{2.7}$$

We will use the following ordinary differential equation to show the existence of the solution $f_M(r)$ to (2.7). In addition, we will derive the non-increasing property of $f_M(r)$.

Initially, given a fixed $\eta>0$, we consider the following Cauchy problem

$$\begin{cases} \left(g^{m-1}\left|\left(g^k\right)'\right|^{p-2} g'\right)' + \frac{N+n_2-1}{r} g^{m-1}\left|\left(g^k\right)'\right|^{p-2} g' + \alpha r^{n_1-n_2+1} g^{-q} g' \\ + a\alpha r^{n_1-n_2} g^{1-q} = 0, \ r>0, \\ g(0) = \eta, \ g'(0) = 0. \end{cases} \tag{2.8}$$

Using the standard approach for solving Cauchy problem to Ordinary Differential Equations and following the techniques described in references [34] and [14], we may deduce that the solution $g(r)$ of the problem (2.8) is positive, and $g(r) \overset{r\to\infty}{\longrightarrow} 0$, furthermore,

$$\lim_{r\to+\infty} r^a g(r) = M,$$

for some $M = M(\eta) > 0$.
Next, we aim to prove that there is a one-to-one correspondence between $M\in(0,+\infty)$and $\eta\in(0,+\infty)$. Indeed, this can be seen from the following relation,

$$g_\eta(r) = \eta g_1(\eta^\sigma r), \ \sigma = -\frac{q+m+k\,(p-2)-1}{p+n_1-n_2} < 0, \tag{2.9}$$

where $g_1(r)$ is the solution of (2.8) for $\eta = 1$ and $a \neq 1/\sigma$. Hence,

$$M(\eta) = \eta^{1-a\sigma} M(1), \quad \text{with } M(1) = \lim_{r\to+\infty} r^a g_1(r). \tag{2.10}$$

Consequently, we can deduce that for each $M > 0$, there exists a positive, bounded, and global solution $f_M(r)$ satisfying (2.7).
Finally, we will especially prove that the solution $g(r)$ will be non-increasing. It means that $f_M(r)$is also non-increasing. We divide the proof into several lemmas.

Lemma 2.1. *Let $g(r)$ be the solution* (2.8), *then*

$$\lim_{r\to+\infty} \frac{N+n_2}{r} g^{m-1}(r) \left|\left(g^k(r)\right)'\right|^{p-2} g'(r) = 0. \tag{2.11}$$

Proof. We multiply the equation (2.8) by a smooth sequence of test function $\chi_n(x)$ such that $\lim\limits_{n\to\infty} \chi_n(x) = 1$ if $r \in [0,\varepsilon]$ and $\lim\limits_{n\to\infty} \chi_n(x) = 0$, otherwise, for some $\varepsilon > 0$. After integrating, letting $n \to \infty$, we have

$$\begin{aligned} &\left(g^{m-1}\left|\left(g^k\right)'\right|^{p-2} g'\right)(\varepsilon) + \int_0^\varepsilon \frac{N+n_2-1}{r} g^{m-1}\left|\left(g^k\right)'\right|^{p-2} g' dr \\ &+ \alpha \int_0^\varepsilon r^{n_1-n_2+1} g^{-q} g' dr + a\alpha \int_0^\varepsilon r^{n_1-n_2} g^{1-q} dr = 0. \end{aligned} \tag{2.12}$$

Dividing by ε and taking $\varepsilon \to 0$ in (2.12), we obtain,

$$\lim_{r\to+\infty} \left[\frac{g^{m-1}\left|\left(g^k\right)'\right|^{p-2} g'(\varepsilon)}{\varepsilon} + \frac{N+n_2-1}{\varepsilon} g^{m-1}\left|\left(g^k\right)'\right|^{p-2} g'(\varepsilon) \right] = 0, \tag{2.13}$$

which implies that (2.11) holds. We have finished proving Lemma 2.1.

Lemma 2.2. *If there exists $r_0 \in [0,+\infty)$ such that $g(r_0) = 0$, then $g(r) = 0$ for all $r \geq r_0$.*

Proof. We will prove this through contradiction. Suppose that Lemma 2.2 does not hold, it is clear that there is $\varepsilon > 0$ such that

$$g(r) > 0 \text{ and } g'(r) > 0 \text{ in } (r_0, r_0+\varepsilon). \tag{2.14}$$

Multiplying equation (2.8) by r^{1-N-n_2}, and integrating over (r_0, r) with $r \in (r_0, r_0+\varepsilon)$,we obtain

$$\begin{aligned} &r^{N+n_2-1} g^{m-1}\left|\left(g^k\right)'\right|^{p-2} g'(r) + \frac{\alpha r^{N+n_1}}{1-q} g^{1-q}(r) \\ &= \frac{\alpha(N+n_1)}{1-q} \int_{r_0}^r r^{N+n_1-1} g^{1-q} dr - \alpha a \int_{r_0}^r r^{N+n_1-1} g^{1-q} dr. \end{aligned} \tag{2.15}$$

It follows from (2.14)-(2.15) that

$$\frac{\alpha r^{N+n_1}}{1-q} g^{1-q}(r) \leq \alpha\left(\frac{N+n_1}{1-q} - a\right) g^{1-q}(r)\left(r^{N+n_1} - r_0^{N+n_1}\right),$$

or equivalently

$$1 \leq (N+n_1-a(1-q))\left(1-\left(\frac{r_0}{r}\right)^{N+n_1}\right). \tag{2.16}$$

Let $r \to r_0$ in (2.16), we obtain the inequality $1 \leq 0$, which is a contradiction. We have finished proving Lemma 2.2.

Lemma 2.3. *The solution* $g(r)$ *of* (2.8) *is monotone non-increasing in* $[0,+\infty)$.

Proof. We use the contradiction argument one more time. Assume that for some $r_0>0, g'(r)>0$, by Lemmas 2.1-2.2, there exists $r_1\in(0,r_0)$ such that

$$g(r_1)>0, \quad \text{and} \left(g^{m-1}\left|\left(g^k\right)'\right|^{p-2}g'\right)(r_1)>0. \tag{2.17}$$

Using the similar argument in Lemma 2.1, we deduce that

$$\lim_{r\to r_1}\frac{N+n_2}{r-r_1}\left(g^{m-1}\left|\left(g^k\right)'\right|^{p-2}g'\right)(r)=0, \tag{2.18}$$

which is a contradiction with (2.17). We have finished proving Lemma 2.3.
Next, we apply the monotone properties of $f_M(r)$ to infer the condition on the global existence of the solution to (1.1)-(1.2).
Proof of Theorem 2.1. We present a proof of Theorem 2.1 using a series of steps. By doing these steps systematically, we can prove the theorem 2.2.
Step 1. Since $\varphi(x)\in F^a$ with $a_c<a<N+n_1$, there exists a constant $K>0$ such that

$$\varphi(x)\le K(1+|x|)^{-a}, \ \forall x\in\mathbb{R}^N.$$

Taking $M>K$ and the self-similar solution $U_{M,a}(x,t)$ of (2.5) defined as (2.6), since $\lim\limits_{r\to+\infty} r^a f_M(r)=M>K$, there exists a positive constant R_0 such that

$$r^a f_M(r)>K \quad \text{for } r\ge R_0.$$

Set $\ell=f_M(R_0)=\min\{f_M(r)\mid r\in[0,R_0]\}>0$, it is easy to verify that $\varphi(x)\le U_{M,a}(x,t_0)$ for all $x\in\mathbb{R}^N$, where $t_0\in(0,1)$ and $t_0^{-a\alpha}\ell>\|\varphi\|_\infty$.
Let $\lambda>0$, then $w(x,t)=\lambda U_{M,a}\left(x,\lambda^{q+m+k(p-2)-1}t+t_0\right)^{-a\alpha}$ is the solution of the following problem

$$\begin{cases}\rho_1(x)\,w_t=w^q\nabla\left(\rho_2(x)\,w^{m-1}\left|\nabla w^k\right|^{p-2}\nabla w\right), \ (x,t)\in Q,\\ w(x,0)=\lambda U_{M,a}(x,t_0)\ge\lambda\varphi(x), \ x\in\mathbb{R}^N.\end{cases} \tag{2.19}$$

Taking $\eta=f_M(0)$ and noting that $f_M(r)$ is non-increasing, we have

$$\|w(x,t)\|_\infty=\eta\lambda\left(\lambda^{q+m+k(p-2)-1}t+t_0\right)^{-a\alpha}.$$

Extracting that

$$|x|^{\frac{n_3-n_1}{\beta-1}}U_{M,a}(x,t)=t^{-\alpha\left(a-\frac{n_3-n_1}{\beta-1}\right)}\left(\frac{|x|}{t^\alpha}\right)^{\frac{n_3-n_1}{\beta-1}}f_M\left(\frac{|x|}{t^\alpha}\right), \tag{2.20}$$

Then we could infer from $\lim\limits_{r\to+\infty}r^a f_M(r)=M$ and $\frac{n_3-n_1}{\beta-1}<a_c<a$ that

$$\|w(x,t)\|_{\infty,n_1,n_3}\equiv\sup_{x\in\mathbb{R}^N}|x|^{\frac{n_3-n_1}{\beta-1}}|w(x,t)|\le C\lambda\left(\lambda^{q+m+k(p-2)-1}t+t_0\right)^{-\alpha\left(a-\frac{n_3-n_1}{\beta-1}\right)}, \tag{2.21}$$

where C is a positive constant.

Step 2. Set $Z(x,t) = I(t)\, w(x, J(t))$, where $I(t)$ and $J(t)$ are solutions of the following problem

$$\begin{cases} I'(t) = C^{\beta-1}\lambda^{\beta-1}\left[\lambda^{q+m+k(p-2)-1} J(t) + t_0\right]^{-\frac{a(\beta-1)+n_1-n_3}{a(q+m+k(p-2)-1)+p+n_1-n_2}} I^{\beta}(t), \\ J'(t) = I^{q+m+k(p-2)-1}(t), \quad , t \in (0,\infty), \\ I(0) = 1, \; J(0) = 0. \end{cases} \tag{2.22}$$

By performing a direct calculation, we can obtain from equation (2.21) that $Z(x,t)$ fulfils

$$\begin{cases} \rho_1(x) Z_t \geq Z^q \nabla\left(\rho_2(x) Z^{m-1}\left|\nabla Z^k\right|^{p-2}\nabla Z\right) + \rho_3(x) Z^{\beta}, \; (x,t) \in Q, \\ Z(x,0) = w(x,0) = \lambda U_{M,a}(x,t_0) \geq \lambda\varphi(x), \; x \in \mathbb{R}^N. \end{cases} \tag{2.23}$$

Step 3. We will prove that there exists a positive constant $\lambda_0 = \lambda_0(\varphi)$ such that the problem (2.22) has a global solution $(I(t), J(t))$ with $I(t)$ bounded in $[0, +\infty)$ if $\lambda \in [0, \lambda_0)$. According to the standard theory of ODE, the local existence and uniqueness of solution $(I(t), J(t))$ of (2.22) holds (see Theorem II.3.2 in [32] and [9]). By (2.22), we have $I'(t) > 0, I(t) > 1$ for $t > 0$; moreover, the solution is continuous as long as the solution exists and $I'(t)$ is finite.
From (2.22), while $I(t)$ exists in $[0,t]$, then $J(t)$ is uniquely defined by

$$J(t) = \int_0^t I^{q+m+k(p-2)-1}(s)\, ds.$$

Since $q + m + k(p-2) > 1$ and $I(t)$ is increasing, we can deduce that

$$J(s) = \int_0^s I^{q+m+k(p-2)-1}(\tau)\, d\tau \geq I^{q+m+k(p-2)-1}(0)\, s = s \;\; for\ all\ s \in [0,t]. \tag{2.24}$$

By (2.22), (2.24) and $a > a_c = \frac{p+n_3-n_2}{\beta-q-m-k(p-2)}$, implies that

$$1 - I^{1-\beta}(t) = (\beta-1)(C\lambda)^{\beta-1}\int_0^t [\lambda^{q+m+k(p-2)-1} J(s) + t_0]^{-\frac{a(\beta-1)+n_1-n_3}{a(q+m+k(p-2)-1)+p+n_1-n_2}} ds$$

$$\leq (\beta-1)\, C^{\beta-1}\lambda^{\beta-1}\int_0^t \left[\lambda^{q+m+k(p-2)-1} s + t_0\right]^{-\frac{a(\beta-1)+n_1-n_3}{a(q+m+k(p-2)-1)+p+n_1-n_2}} ds$$

$$\leq \frac{(\beta-1)\, C^{\beta-1}\lambda^{\beta-q-m-k(p-2)}}{\frac{a(\beta-1)+n_1-n_3}{a(q+m+k(p-2)-1)+p+n_1-n_2} - 1}\, t_0^{1-\frac{a(\beta-1)+n_1-n_3}{a(q+m+k(p-2)-1)+p+n_1-n_2}}. \tag{2.25}$$

Let $\lambda_0 = \lambda_0(\varphi)$ be a positive constant defined by

$$\frac{(\beta-1)\, C^{\beta-1}\lambda^{\beta-q-m-k(p-2)}}{\frac{a(\beta-1)+n_1-n_3}{a(q+m+k(p-2)-1)+p+n_1-n_2} - 1}\, t_0^{1-\frac{a(\beta-1)+n_1-n_3}{a(q+m+k(p-2)-1)+p+n_1-n_2}} = \frac{1}{2}, \tag{2.26}$$

then it follows from (2.25), $\beta > \beta_c > q + m + k(p-2) + \frac{(1-q)(p-n_2-n_3)}{N+n_1} > 1$ that $1 \leq I(t) \leq 2^{\frac{1}{\beta-1}}$ for any $\lambda \in (0, \lambda_0]$, as long as $I(t)$ exists globally.
On the other side, by (2.22) and (2.24), we have

$$t \leq J(t) \leq 2^{\frac{q+m+k(p-2)-1}{\beta-1}}\, t \;\; for\ all\ t \geq 0. \tag{2.27}$$

Moreover, $J(t)$ is also global.
Step 4. For any $\lambda \in (0, \lambda_0]$, where $\lambda_0 = \lambda_0(\varphi)$ is defined as (2.26), the solution $u(x,t)$ of (1.1)-(1.2) with initial value $u_0(x) = \lambda\varphi(x)$ exists globally, and $u(x,t) \le Z(x,t)$ in Q. Therefore, there exists a positive constant C such that

$$\|u(.,t)\|_\infty \le \|Z(.,t)\|_\infty \le 2^{\frac{1}{\beta-1}}\eta\lambda(\lambda^{q+m+k(p-2)-1}J(t)+t_0)^{-a\alpha} \le Ct^{-a\alpha}, \ \forall\, t>0. \tag{2.28}$$

The proof of Theorem 2.1 is complete.
Proof of Theorem 2.2. Let

$$u_n(x,t) = n^a u\left(nx,\, n^{1/\alpha}t\right),\ n>1,$$

then it is easy to see that $u_n(x,t)$ is the solution to the following Cauchy problem

$$\begin{cases} |x|^{n_1} u_{nt} = u_n^q \nabla\left(|x|^{n_2} u_n^{m-1}\left|\nabla u_n^k\right|^{p-2}\nabla u_n\right) \\ +n^{p-n_2+n_3-(\beta-q-m-k(p-2))a}|x|^{n_3}u_n^\beta, \quad (x,t)\in Q, \\ u_n(x,0) = \lambda n^a\varphi(nx), \qquad\qquad x\in\mathbb{R}^N. \end{cases} \tag{2.29}$$

It follows from (2.28) that

$$\|u_n(x,t)\|_\infty = n^a\left\|u\left(nx,\, n^{1/\alpha}t\right)\right\|_\infty \le Cn^a\left(n^{1/\alpha}t\right)^{-a\alpha} = Ct^{-a\alpha}. \tag{2.30}$$

Therefore, $\{u_n(x,t)\}$ is uniformly bounded in $\mathbb{R}^N\times[\delta,\infty)$ for any $\delta>0$. Hence, we can apply the idea in Chapter 9 in [15] and [33], to conclude that the family $\{u_n(x,t)\}$ is relatively compact in $L^\infty_{loc}(Q)$. Then using the Ascoli-Arzela theorem and a diagonal sequence method in δ, we see that for any sequence $n_j\to\infty$, there exists a subsequence $n'_j\to\infty$ and a function $w(x,t)\in C(Q)$ such that

$$u_{n'_j}(x,t)\to w(x,t)\ \ as\ n'_j\to\infty,$$

local uniformly in Q. We will prove that

$$w(x,t) = U_{\lambda M,a}(x,t).$$

According to the definition of weak solution, $u_n(x,t)$ being a weak solution to equation (2.29) implies that the integral identity

$$\int_{\mathbb{R}^N}|x|^{n_1}u_n(x,t)\eta(x,t)\,dx - \int_{\mathbb{R}^N}|x|^{n_1}u_n(x,0)\eta(x,0)\,dx = \int_0^t\int_{\mathbb{R}^N}\Big(|x|^{n_1}u_n\eta_\tau - |x|^{n_2}u_n^{m-1}\left|\nabla u_n^k\right|^{p-2}\nabla u_n\cdot\nabla\left(u_n^q\eta\right) + n^{p-n_2+n_3-(\beta-q-m-k(p-2))a}|x|^{n_3}u_n^\beta\eta\Big)dxd\tau \tag{2.31}$$

is satisfied for any non-negative $\eta\in C_0^\infty\left(\mathbb{R}^N\times[0,\infty)\right)$.
Under the assumption $\lim\limits_{|x|\to\infty}|x|^a\varphi(x) = M>0$, implies that

$$\int_{\mathbb{R}^N}|x|^{n_1}u_n(x,0)\eta(x,0)\,dx = \int_{\mathbb{R}^N}\lambda n^a|x|^{n_1}\varphi(nx)\eta(x,0)\,dx$$
$$\to \lambda M\int_{\mathbb{R}^N}|x|^{n_1}|x|^{-a}\eta(x,0)\,dx,\ \text{as } n=n'_j\to\infty. \tag{2.32}$$

On the other hand, we can obtain

$$\int_0^t \int_{\mathbb{R}^N} n^{p-n_2+n_3-(\beta-q-m-k(p-2))a} |x|^{n_3} u_n^{\beta} \eta dx d\tau$$
$$= \int_0^{n^{1/\alpha}t} \int_{\mathbb{R}^N} n^{a+n_3-n_1} |x|^{n_3} u^{\beta}(nx,\tau)\, \eta\left(x, n^{-1/\alpha}\tau\right) dx d\tau. \tag{2.33}$$

It follows from the proof Theorem 2.1 that $Z(x,t)$ is the super solution to (1.1)-(1.2); thus, we have

$$n^{a+n_3-n_1} |x|^{n_3} u^{\beta}(nx,t) \le n^{a+n_3-n_1} |x|^{n_3} Z^{\beta}(nx,t) \le$$
$$\le |nx|^a Z(nx,t) \left(n^{n_3-n_1} |x|^{n_3-a} Z^{\beta-1}(nx,t)\right). \tag{2.34}$$

It is easy to see that $|nx|^a Z(nx,t)$ is bounded by some constant independent of $n, x,$ and t; therefore, we deduce

$$n^{n_3-n_1} |x|^{n_3-a} Z^{\beta-1}(nx,t) = |x|^{-(a-n_1)} \left[|nx|^{\frac{n_3-n_1}{\beta-1}} Z(nx,t)\right]^{\beta-1}$$
$$\le C\lambda^{\beta-1} |x|^{-(a-n_1)} \Bigg[\left(\frac{|nx|}{\left(\lambda^{q+m+k(p-2)-1}J(t)+t_0\right)^{\alpha}}\right)^{\frac{n_3-n_1}{\beta-1}} \cdot$$
$$\left(\lambda^{q+m+k(p-2)-1}J(t)+t_0\right)^{-\alpha\left(a-\frac{n_3-n_1}{\beta-1}\right)} f_M\left(\frac{|nx|}{\left(\lambda^{q+m+k(p-2)-1}J(t)+t_0\right)^{\alpha}}\right)\Bigg]^{\beta-1}$$
$$\le C\lambda^{\beta-1} |x|^{-(a-n_1)} \left(\lambda^{q+m+k(p-2)-1}J(t)+t_0\right)^{-\alpha(a(\beta-1)-(n_3-n_1))}$$
$$\left[\left(\frac{|nx|}{\left(\lambda^{q+m+k(p-2)-1}J(t)+t_0\right)^{\alpha}}\right)^{\frac{n_3-n_1}{\beta-1}} f_M\left(\frac{|nx|}{\left(\lambda^{q+m+k(p-2)-1}J(t)+t_0\right)^{\alpha}}\right)\right]^{\beta-1}. \tag{2.35}$$

Using the assumption $\lim_{r\to+\infty} |x|^a f_M(r) = M$ and $a > \frac{n_3-n_1}{\beta-1}$, then for some $x \neq 0$ we have

$$\left(\frac{|nx|}{\left(\lambda^{q+m+k(p-2)-1}J(t)+t_0\right)^{\alpha}}\right)^{\frac{n_3-n_1}{\beta-1}} f_M\left(\frac{|nx|}{\left(\lambda^{q+m+k(p-2)-1}J(t)+t_0\right)^{\alpha}}\right) \to 0 \text{ as } n\to\infty. \tag{2.36}$$

Using the Lebesgue dominated convergence theorem and aggregating (2.33)-(2.36), we get

$$\int_0^t \int_{\mathbb{R}^N} n^{p-n_2+n_3-(\beta-q-m-k(p-2))a} |x|^{n_3} u_n^{\beta} \eta dx d\tau \to 0 \text{ as } n = n_j' \to \infty. \tag{2.37}$$

Moreover, letting $n = n_j' \to \infty$ in (2.31), we obtain

$$\int_{\mathbb{R}^N} |x|^{n_1} w(x,t)\,\eta(x,t)\,dx - \lambda M \int_{\mathbb{R}^N} |x|^{n_1} |x|^{-a} \eta(x,0)\,dx$$
$$= \int_0^t \int_{\mathbb{R}^N} \left(|x|^{n_1} w\eta_\tau - |x|^{n_2} w^{m-1} \left|\nabla w^k\right|^{p-2} \nabla w\cdot\nabla\left(w^q\eta\right)\right) dx d\tau, \tag{2.38}$$

which implies that $w(x,t)$ is the weak solution to the following problem

$$\begin{cases} \rho_1(x)\, w_t = u^q \nabla\left(\rho_2(x)\, w^{m-1} \left|\nabla w^k\right|^{p-2} \nabla w\right), \ (x,t)\in Q, \\ w(x,0) = \lambda M |x|^{-a}, \ x\in\mathbb{R}^N. \end{cases} \tag{2.39}$$

By the uniqueness of the weak solution of (2.39), we obtain $w\left(x,t\right)=U_{\lambda M,a}\left(x,t\right).$ Consequently, we have proven that

$$u_n\left(x,t\right)\to U_{\lambda M,a}\left(x,t\right)\ \text{ as } n\to\infty, \tag{2.40}$$

uniformly in a compact set of Q. Let $t=1$ in (2.40), we deduce

$$u_n\left(x,1\right)\to U_{\lambda M,a}\left(x,1\right)\ \text{ as } n\to\infty, \tag{2.41}$$

that is

$$n^a u\left(nx,n^{1/\alpha}\right)\to f_{\lambda M}\left(|x|\right)\ \text{ as } n\to\infty, \tag{2.42}$$

uniformly in compact set of $\mathbb{R}^N$. We denote $y=nx$ and $s=n^{1/\alpha}$ in (2.42), we obtain

$$s^{a\alpha}u\left(y,s\right)\to f_{\lambda M}\left(|y|\,s^{-\alpha}\right)\ \text{ as } s\to\infty, \tag{2.43}$$

which is

$$s^{a\alpha}\left|u\left(y,s\right)-U_{\lambda M,a}\left(y,s\right)\right|\to 0 \text{ as } s\to\infty, \tag{2.44}$$

uniformly in a compact set of $\mathbb{R}^N$. The proof of Theorem 2.2 is concluded.

3. Blow-up

We introduce a notation $v=u^{1-q}$ and put this into problem (1.1)-(1.2)

$$L\left(v\right)\equiv v^{\frac{q}{1-q}}\left[\rho_1\left(x\right)v_t-\nabla\left(\rho_2\left(x\right)v^{m_2-1}\left|\nabla v^{k_2}\right|^{p-2}\nabla v\right)-\left(1-q\right)\rho_3\left(x\right)v^{\beta_2}\right]=0, \tag{3.1}$$

$$v|_{t=0}=v_0\left(x\right)=\left[u_0\left(x\right)\right]^{1-q}, \tag{3.2}$$

where $m_2=\frac{m}{1-q}$, $k_2=\frac{k}{1-q}$, $\beta_2=\frac{\beta-q}{1-q}$.
We note that if $q(q-1)>0$, then the equation (3.1) has a nontrivial solution and $v=0$ a trivial solution; otherwise, only a nontrivial solution exists. Therefore, we consider only nontrivial solutions.

Theorem 3.1. *Let* $0<a<a_c$, $u_0\left(x\right)\in F_a$ *and* $\beta<\beta_c$, *then the solution* $v\left(x,t\right)$ *of the problem* (3.1)-(3.2) *blows up in a finite time.*

Proof. To obtain conditions for blow-up related to the problem (1.1)-(1.2), we will employ the energy method. To accomplish this, it is necessary to select a suitable test function as follows:

$$\psi_c\left(x\right)=Ac^Ne^{-c|x|}\text{ with }A=\left(\int_{\mathbb{R}^N}e^{-|x|}dx\right)^{-1},\ \nabla\psi_c=-c\frac{x}{|x|}\psi_c,\ c>0.$$

Suppose that $v\left(x,t\right)$ is the solution of the Cauchy problem (3.1)-(3.2) and T is the blow-up time. Let

$$E\left(t\right)=\frac{1}{s}\int_{\mathbb{R}^N}\rho_1\left(x\right)v^s\left(x,t\right)\psi_c\left(x\right)dx,\ t\in\left[0,T\right), \tag{3.3}$$

where $0 < s < \min\left\{\frac{1}{p} - q, \frac{\beta-q-m-k(p-2)}{n_3}\big(N - \big(1 - \frac{1}{q+m+k(p-2)}\big)n_3\big)\right\}$, hence, we get

$$\begin{aligned}&E'(t) = \int_{\mathbb{R}^N} \rho_1(x)\, v_t v^{s-1}\psi_c(x)\,dx = \int_{\mathbb{R}^N} \nabla\big(\rho_2(x)\, v^{m_2-1}\left|\nabla v^{k_2}\right|^{p-2}\nabla v\big)v^{s-1}\cdot\\ &\psi_c(x)\,dx + (1-q)\int_{\mathbb{R}^N}\rho_3(x)\, v^{\beta_2+s-1}\psi_c(x)\,dx \ge (1-s)\,k_2^{p-2}\int_{\mathbb{R}^N}\rho_2(x)\cdot\\ &v^{s+m_2+k_2(p-2)-p}\left|\nabla v\right|^p\psi_c(x)\,dx - ck_2^{p-2}\int_{\mathbb{R}^N}\rho_2(x)\, v^{s+m_2+k_2(p-2)-p+1}\left|\nabla v\right|^{p-1}\cdot\\ &\psi_c(x)\,dx + (1-q)\int_{\mathbb{R}^N}\rho_3(x)\, v^{\beta_2+s-1}\psi_c(x)\,dx.\end{aligned} \tag{3.4}$$

By making use of Young's inequality, we can obtain the following result

$$\begin{aligned}&c\int_{\mathbb{R}^N}\rho_2(x)\, v^{s+m_2+k_2(p-2)-p+1}\left|\nabla v\right|^{p-1}\psi_c(x)\,dx \le \frac{p-1}{p}\cdot\\ &\int_{\mathbb{R}^N}\rho_2(x)v^{s+m_2+k_2(p-2)-p}\left|\nabla v\right|^p\psi_c(x)dx + \frac{c^p}{p}\int_{\mathbb{R}^N}\rho_2(x)v^{s+m_2+k_2(p-2)}\psi_c(x)dx\end{aligned} \tag{3.5}$$

Given that $s < 1/p - q$, we can infer from (3.4)- (3.5) that

$$E'(t) \ge (1-q)\int_{\mathbb{R}^N}\rho_3(x)v^{\beta_2+s-1}\psi_c(x)dx - \frac{c^p k_2^{p-2}}{p}\int_{\mathbb{R}^N}\rho_2(x)v^{s+m_2+k_2(p-2)-1}\psi_c(x)dx \tag{3.6}$$

By $s < \frac{\beta-q-m-k(p-2)}{n_3}\left(N - \left(1 - \frac{1}{q+m+k(p-2)}\right)n_3\right)$, $\int_{\mathbb{R}^N}\psi_c(x)\,dx = 1$ and using Holder's inequality, we obtain

$$\begin{aligned}&\textstyle\int_{\mathbb{R}^N}\rho_2 v^{s+m_2+k_2(p-2)-1}\psi_c(x)\,dx = \int_{\mathbb{R}^N}\rho_3^{1/p'}(x)\, v^{(\beta_2+s-1)/p'}\cdot\psi_c^{1/p'}\cdot\\ &\left(\rho_2\rho_3^{-1/p'}\psi_c^{1/q'}\right)dx \le \left[\int_{\mathbb{R}^N}\rho_3(x)\, v^{\beta_2+s-1}\psi_c dx\right]^{\frac{1}{p'}}\left[\int_{\mathbb{R}^N}\rho_2^{q'}\rho_3^{-q'/p'}\psi_c dx\right]^{\frac{1}{q'}} \le\\ &C_1 c^{n_3/p'-n_2}\left[\int_{\mathbb{R}^N}\rho_3(x)\, v^{\beta_2+s-1}\psi_c dx\right]^{\frac{1}{p'}},\end{aligned} \tag{3.7}$$

where $p' = \frac{\beta_2+s-1}{s+m_2+k_2(p-2)-1}$, $q' = \frac{\beta_2+s-1}{\beta_2-m_2-k_2(p-2)}$,
$C_1 = \left[A\int_{\mathbb{R}^N}|x|^{q'n_2-\frac{q'}{p'}n_3}e^{-|x|}dx\right]^{\frac{1}{q'}} > 0.$
Using (3.6)-(3.7), we can derive the following

$$\begin{aligned}&E'(t) \ge \left[\int_{\mathbb{R}^N}\rho_3(x)\, v^{\beta_2+s-1}\psi_c dx\right]^{\frac{1}{p'}}\left[(1-q)\left[\int_{\mathbb{R}^N}\rho_3(x)\, v^{\beta_2+s-1}\psi_c(x)\,dx\right]^{\frac{1}{q'}}\right.\\ &\left.-\frac{C_1 k_2^{p-2}c^{p+n_3/p'-n_2}}{p}\right].\end{aligned} \tag{3.8}$$

By using Holder's inequality once more, we can derive the following

$$\int_{\mathbb{R}^N} \rho_1(x) v^s \psi_c dx \leq \left[\int_{\mathbb{R}^N} \rho_3(x) v^{\beta_2+s-1} \psi_c dx\right]^{\frac{1}{p''}} \left[\int_{\mathbb{R}^N} \rho_1^{q''} \rho_3^{-q''/p''} \psi_c dx\right]^{\frac{1}{q''}} \leq$$

$$C_2 c^{n_3/p''-n_1} \left[\int_{\mathbb{R}^N} \rho_3(x) v^{\beta_2+s-1} \psi_c dx\right]^{\frac{1}{p''}},$$

where $p'' = \frac{\beta_2+s-1}{s}$, $q'' = \frac{\beta_2+s-1}{\beta_2-1}$, $C_2 = \left[A \int_{\mathbb{R}^N} |x|^{q'' n_1 - \frac{q''}{p''} n_3} e^{-|x|} dx\right]^{\frac{1}{q''}} > 0$,
Using formula (3.3) and the last inequality, we can conclude that

$$\int_{\mathbb{R}^N} \rho_3(x) v^{\beta_2+s-1} \psi_c dx \geq \left(\frac{s}{C_2}\right)^{p''} c^{p'' n_1 - n_3} [E(t)]^{p''}. \tag{3.9}$$

According to expressions (3.8) - (3.9), it can be inferred that

$$E'(t) \geq (1-q) \left(\frac{s}{C_2}\right)^{p''} c^{p'' n_1 - n_3} [E(t)]^{p''/p'} \left[[E(t)]^{p''/q'} - \right.$$
$$\left. \frac{C_1 k_2^{p-2} c^{p+n_3-n_2-p'' n_1/q'}}{p(1-q)} \left(\frac{C_2}{s}\right)^{p''/q'} \right] \tag{3.10}$$

Therefore, we can deduct from equation (3.10) that

$$E'(t) \geq \frac{1}{2} \left(\frac{s}{C_2}\right)^{p''} c^{p'' n_1 - n_3} [E(t)]^{p''}, \tag{3.11}$$

while

$$E(0) \geq \frac{C_2}{s} \left[\frac{2 C_1 k_2^{p-2} c^{p+n_3-n_2-p'' n_1/q'}}{p(1-q)}\right]^{q'/p''}. \tag{3.12}$$

Integrating (3.11), on the interval $[0,t]$, we have

$$E(t) \geq \left[(E(0))^{1-p''} - C_3 t\right]^{-\frac{1}{p''-1}}, \tag{3.13}$$

where $C_3 = \frac{p''-1}{2} \left(\frac{s}{C_2}\right)^{p''} c^{p'' n_1 - n_3} > 0$.
Therefore, from (3.12)-(3.13), we obtain that $v(x,t)$ blows up in finite time $T = \dfrac{(E(0))^{1-p''}}{C_3}$ and get an estimate on the blow-up time of the solution $v(x,t)$ as follows:

$$T \leq \frac{2^{1+q'(1-p'')/p''} c^{n_3-n_1+q'(1-p'')(p+n_3-n_2)/p''}}{p''-1} \frac{C_2}{s} \left[\frac{C_1 k_2^{p-2}}{p(1-q)}\right]^{q'(1-p'')/p''}.$$

Ultimately, we need to verify the blow-up condition which is stated in equation (3.12). Since $u_0(x) \in F_a$ for some $0 < a < a_c$, there exist two positive constants M and $R_0 > 1$ such that $u_0(x) \geq M|x|^{-a}$ for all $|x| \geq R_0$, and we have the following inequality:

$$E(0)=\frac{1}{s}\int_{\mathbb{R}^N}\rho_1(x)\,v_0^s(x)\,\psi_c(x)\,dx\geq\frac{AM^{s(1-q)}c^N}{s}\int_{|x|\geq R_0}|x|^{n_1-s(1-q)a}\,e^{-c|x|}dx$$

$$=\frac{AM^{s(1-q)}c^{s(1-q)a-n_1}}{s}\int_{|x|\geq cR_0}|x|^{n_1-s(1-q)a}\,e^{-|x|}dx. \qquad (3.14)$$

By the definition of A and $0<a<a_c$, we could choose $0<c\leq\frac{1}{R_0}$ so small that (3.12) holds. The proof of Theorem 3.1 is complete.

4. Life span of the solution

Within this section, we will give estimates of the life span T_λ^* of solution to (1.1)-(1.2) both from below and above.

Theorem 4.1. *Assume that* $u_0(x)=\lambda\varphi(x)$ *for some* $\lambda>0$.

- *Let* $\varphi(x)\in F_a$ *with* $a\in(0,a_c)$, *then there exist* $\lambda_1=\lambda_1(\varphi)>0$ *and* $C_0>0$ *such that*
$$T_\lambda^*\leq C_0\lambda^{-\frac{2\beta-2+(\beta-m-k(p-2))n_1+(q+m+k(p-2)-1)n_3}{p-n_2+n_3-(q+m+k(p-2))a}}\quad \textit{for all}\ \ \lambda<\lambda_1.$$
- *Let* $\varphi(x)\in F^a$ *with* $a\in\left(\frac{n_3-n_1}{\beta-1},a_c\right)$, *then there exist* $\lambda_1=\lambda_1(\varphi)>0$ *and* $c_0>0$ *such that*
$$T_\lambda^*\geq c_0\lambda^{-\frac{2\beta-2+(\beta-m-k(p-2))n_1+(q+m+k(p-2)-1)n_3}{p-n_2+n_3-(q+m+k(p-2))a}}\quad \textit{for all}\ \ \lambda<\lambda_1.$$

Proof. In order to obtain an upper estimate of T_λ^*, we introduce $u_l(x,t)$, as follows

$$u_l(x,t)=lu\left(l^{1/a_c}x,\,l^{\frac{2\beta-2+(\beta-m-k(p-2))n_1+(q+m+k(p-2)-1)n_3}{p-n_2+n_3-(q+m+k(p-2))a}}t\right), \qquad (4.1)$$

where $l>0$ and $u(x,t)$ is the solution to (1.1)-(1.2) with $u_0(x)=\lambda\varphi(x)$, then $u_l(x,t)$ solves the following problem

$$\rho_1(x)\,\partial_t u_l=u_l^q\nabla\left(\rho_2(x)\,u_l^{m-1}\left|\nabla u_l^k\right|^{p-2}\nabla u_l\right)+\rho_3(x)\,u_l^\beta, \qquad (x,t)\in Q, \quad (4.2)$$

$$u_l(x,0)=\lambda l\varphi\left(l^{1/a_c}x\right), \qquad x\in\mathbb{R}^N, \qquad (4.3)$$

We use the notation $v_l=u_l^{1-q}$, then (4.2)-(4.3) problem becomes to the next form

$$\rho_1(x)\,\partial_t v_l=\nabla\left(\rho_2(x)\,v_{l-1}^{m_2}\left|\nabla v_l^{k_2}\right|^{p-2}\nabla v_l\right)+(1-q)\,\rho_3(x)\,v_l^{\beta_2}, \qquad (4.4)$$

$$v_l|_{t=0}=\left[\lambda l\varphi\left(l^{1/a_c}x\right)\right]^{1-q}, \qquad x\in\mathbb{R}^N, \qquad (4.5)$$

where m_2,k_2,β_2 defined above.
Let T_l^*, be the life span of $v_l(x,t)$, it is easy to see that

$$T_\lambda^*=l^{\frac{2\beta-2+(\beta-m-k(p-2))n_1+(q+m+k(p-2)-1)n_3}{p-n_2+n_3-(q+m+k(p-2))a}(1-q)}T_l^{*1-q}. \qquad (4.6)$$

We define

$$E_c(t, v_l) = \frac{1}{s} \int_{\mathbb{R}^N} \rho_1(x) v_l^s(x,t) \psi_c(x)\, dx, \ t \in [0, T_\lambda^*), \tag{4.7}$$

where $\psi_c(x)$ defined above.
We deduce from the arguments in Blow-up section 3 that if

$$E_c(0, v_l(x,0)) \geq \frac{C_2}{s} \left[\frac{2C_1 k_2^{p-2} c^{p+n_3-n_2-p''n_1/q'}}{p(1-q)} \right]^{q'/p''}, \tag{4.8}$$

then

$$E_c'(t, v_l) \geq \frac{1}{2} \left(\frac{s}{C_2} \right)^{p''} c^{p''n_1 - n_3} \left[E_c(t, v_l) \right]^{p''}. \tag{4.9}$$

Integrating the above inequality over an interval $(0, t)$, yields

$$T_l^* \leq \frac{2}{p''-1} \left(\frac{C_2}{s} \right)^{p''} c^{n_3 - p''n_1} \left[E_c(0, v_l(x,0)) \right]^{1-p''}. \tag{4.10}$$

It needs to confirm the blow-up condition (4.8). Notice that $\varphi(x) \in F_a$ with $a \in (0, a_c)$ and choose

$$\lambda l = l^{a/a_c}, \tag{4.11}$$

then

$$E_c(0, v_l(x,0)) = \frac{(\lambda l)^{s(1-q)}}{s} \int_{\mathbb{R}^N} |x|^{n_1} \left[\varphi\left(l^{1/a_c} x \right) \right]^{s(1-q)} \psi_c(x)\, dx =$$

$$= \frac{A}{s} l^{as(1-q)/a_c} c^{-n_1} \int_{\mathbb{R}^N} |x|^{n_1} \left[\varphi\left(l^{1/a_c} c^{-1} x \right) \right]^{s(1-q)} e^{-|x|} dx.$$

Since $\lim_{x\to\infty} |x|^a \varphi(x) > M$, for some $M > 0$, there exists a positive constant l_0 such that

$$l^{as(1-q)/a_c} c^{-as(1-q)} |x|^{as(1-q)} \left[\varphi\left(l^{1/a_c} c^{-1} x \right) \right]^{s(1-q)} > M^{s(1-q)} \ \ for\ l > l_0. \tag{4.12}$$

Furthermore, we obtain

$$\liminf_{l\to\infty} E_c(0, v_l(x,0)) \geq \frac{AM^{s(1-q)}}{s} c^{as(1-q)-n_1} \int_{\mathbb{R}^N} |x|^{n_1 - as(1-q)} e^{-|x|} dx = C_0 c^{as(1-q)-n_1} \tag{4.13}$$

Since $a \in (0, a_c)$, it follows from (4.13) that (4.7) holds if c is small enough. Finally, we see that $l \to \infty$ as $\lambda \to 0$ and apply (4.5), (4.9), (4.10) to conclude the upper bound of life span T_λ^*.
To establish a lower bound of life span T_λ^*, we will construct a suitable super solution to (4.4). As demonstrated in the section on the global existence of the solution, for each $\varphi \in F^a$, there exist $M_0 > 0$ and t_0 such that

$$\varphi(x) \leq U_{M_0,a}(x, t_0). \tag{4.14}$$

Let z_l be the solution to the Cauchy problem

$$\rho_1(x) \partial_t z_l = z_l^q \nabla \left(\rho_2(x) z_l^{m-1} \left| \nabla z_l^k \right|^{p-2} \nabla z_l \right); \qquad (x,t) \in Q, \tag{4.15}$$

$$z_l(x, 0) = \lambda l \varphi\left(l^{1/a_c} x \right), \qquad x \in \mathbb{R}^N. \tag{4.16}$$

Using the similarity of $U_{M_0,a}(x,t_0)$, we have

$$\lambda l U_{M_0,a}\left(l^{1/a_c}x,t_0\right)=U_{M_0,a}\left(x,l^{-\frac{1}{a_c\alpha}}t\right), \tag{4.17}$$

where λ,α defined above. Using the comparison principle, we infer

$$z_l(x,t)\le U_{M_0,a}\left(x,t+l^{-\frac{1}{a_c\alpha}}t_0\right). \tag{4.18}$$

Applying the assumption $\frac{n_3-n_1}{\beta-1}<a$ and using the similar arguments for (2.21), we obtain

$$\|z_l(x,t)\|_{\infty,n_1,n_3}\equiv\sup_{x\in\mathbb{R}^N}|x|^{\frac{n_3-n_1}{\beta-1}}|z_l(x,t)|\le\sup_{x\in\mathbb{R}^N}|x|^{\frac{n_3-n_1}{\beta-1}}U_{M_0,a}\left(x,t+l^{-\frac{1}{a_c\alpha}}t_0\right)$$
$$\le C\lambda\left(t+l^{-\frac{1}{a_c\alpha}}t_0\right)^{-\alpha\left(a-\frac{n_3-n_1}{\beta-1}\right)}. \tag{4.19}$$

Let $h(t)$ be defined by

$$h(t)=\left[1-(\beta-1)\int_0^t\left(s+l^{-\frac{1}{a_c\alpha}}t_0\right)^{-\alpha((\beta-1)a-n_3+n_1)}ds\right]^{-\frac{1}{\beta-1}}, \tag{4.20}$$

then $h(t)$ fulfills the following problem

$$\begin{cases}h'(t)=C^{\beta-1}\left(t+l^{-\frac{1}{a_c\alpha}}t_0\right)^{-\alpha((\beta-1)a-n_3+n_1)}h^{\beta}(t),\\ h(0)=1.\end{cases} \tag{4.21}$$

Set

$$\overline{u_l}(x,t)=h(t)\,z_l(x,\tau(t)),\ \text{ with }\ \tau(t)=\int_0^t h^{q+m+k(p-2)-1}(s)\,ds. \tag{4.22}$$

Next, we will show that $\overline{u_l}(x,t)$ is a super solution to (4.4)-(4.5). Recall that

$$\tau(t)=\int_0^t h^{q+m+k(p-2)-1}(s)\,ds\ge\int_0^t h^{q+m+k(p-2)-1}(0)\,ds=t,$$

Based on equation (4.19), it can be inferred that

$$\begin{cases}\rho_1(x)\,\partial_t\overline{u_l}\ge\overline{u}_l^q\nabla\left(\rho_2(x)\,\overline{u}_l^{m-1}\left|\nabla\overline{u}_l^k\right|^{p-2}\nabla\overline{u}_l\right)+\rho_3(x)\,\overline{u}_l^{\beta}; & (x,t)\in Q,\\ \overline{u}_l(x,0)=z_l(x,0)=\lambda l\varphi\left(l^{1/a_c}x\right)=u_l(x,0), & x\in\mathbb{R}^N,\end{cases} \tag{4.23}$$

According to the principle of comparison, we obtain

$$T_l^*\ge T_h^*, \tag{4.24}$$

where T_h^* is the life span of $h(t)$. Since, $a\in(0,a_c)$, we obtain $\dfrac{a(\beta-1)+n_1-n_3}{a(q+m+k(p-2)-1)+p+n_1-n_2}<1$. Furthermore, from (4.19), we can deduce that T_h^* fulfills

$$\frac{(\beta-1)\,C^{\beta-1}\left(T_h^*+l^{-\frac{1}{a_c\alpha}}t_0\right)^{1-\frac{a(\beta-1)+n_1-n_3}{a(q+m+k(p-2)-1)+p+n_1-n_2}}}{1-\frac{a(\beta-1)+n_1-n_3}{a(q+m+k(p-2)-1)+p+n_1-n_2}}\ge 1 \tag{4.25}$$

Moreover, we obtain that $l \to \infty$ *as* $\lambda \to 0$ and apply (4.6), (4.11), (4.24), (4.25) to obtain the lower bound of life span T_λ^*. The proof of Theorem 4.1 has been concluded.

Corollary. If $u_0(x) = \lambda\varphi(x)$, $\lambda > 0$ and $\varphi \in F_a \cap F^a$ with $a \in \left(\frac{n_3-n_1}{\beta-1}, a_c\right)$, then we have the following an estimate of T_λ^*:

$$c_0\lambda^{-\frac{2\beta-2+(\beta-m-k(p-2))n_1+(q+m+k(p-2)-1)n_3}{p-n_2+n_3-(q+m+k(p-2))a}} \le T_\lambda^* \le$$
$$C_0\lambda^{-\frac{2\beta-2+(\beta-m-k(p-2))n_1+(q+m+k(p-2)-1)n_3}{p-n_2+n_3-(q+m+k(p-2))a}}, \quad \text{for all } \lambda < \lambda_1.$$

5. The self-similar analysis

In this section, we construct the Barenblatt-type self-similar solution. The problem (2.1)-(2.2) is an equivalent form of the problem (1.1)-(1.2). First, we rewrite the equation (2.1) as follows

$$r^{n_1}v_t = r^{1-N}\frac{\partial}{\partial r}\left(r^{N-1+n_2}v^{m_2-1}\left|\frac{\partial v^{k_2}}{\partial r}\right|^{p-2}\frac{\partial v}{\partial r}\right) + (1-q)\,r^{n_3}v^{\beta_2}, \tag{5.1}$$

where $r = |x|$.

Then, we look $v(t,r)$, as follows

$$v(t,r) = (T_1+t)^{-\alpha_1} f\left(\varphi(r)(T_1+t)^{\alpha_2}\right), \tag{5.2}$$

where $\varphi(r) = \begin{cases} \frac{r^{p_2}}{p_2}, \ p_2 = \frac{p-n_2+n_1}{p} \ne 0, \\ \ln r, \ p_2 = 0, \end{cases}$

$\alpha_1 = \frac{p+n_3/p_2}{\Delta_\alpha}$, $\alpha_2 = \frac{\beta_2-m_2-k_2(p-2)}{\Delta_\alpha}$, $\Delta_\alpha = p(\beta_2-1) + \frac{n_3}{p_2}(m_2+k_2(p-2)-1) \ne 0$.

We consider the case $p_2 \ne 0$, then the equation (5.1) transforms into the following form

$$\xi^{1-n}\frac{d}{d\xi}\left(\xi^{n-1}f^{m_2-1}\left|\frac{df^{k_2}}{d\xi}\right|^{p-2}\frac{df}{d\xi}\right) + \alpha_2\xi\frac{df}{d\xi} + \alpha_1 f + (1-q)\,p_2^{n_3/p_2}\xi^{n_3}f^{\beta_2} = 0, \tag{5.3}$$

where $n = \frac{N+n_1}{p_2}$, $\xi = \varphi(r)(T_1+t)^{\alpha_2}$.

Let us introduce new notations

$$U(x,t) = (T_1+t)^{-\alpha_1}\bar{f}(\xi), \ \ \bar{f}(\xi) = B\left(b - \xi^{\frac{p}{p-1}}\right)_+^{\gamma}, \tag{5.4}$$

where $\gamma = \frac{p-1}{m_2+k_2(p-2)-1}$, $B = \left[\frac{\gamma(p-1)}{p}\left(\frac{\alpha_2}{k_2^{p-2}}\right)^{\frac{1}{p-1}}\right]^{\gamma}$, $b = const \ge 0$, $(a)_+ = \max\{a, 0\}$.

Theorem 5.1. *Let us* $p_2 > 0$, $u(x,0) \le U(x,0)$, $x \in \mathbb{R}^N$, *and of the following inequalities satisfy*

$$\begin{aligned} n_3 &> \max\left((N+n_1)(\beta_2-m_2-k_2(p-2)) - pp_2, \ -\frac{pp_2(\beta_2-1)}{m_2+k_2(p-2)-1}\right), \\ n_3 &< \min\left((N+n_1)(\beta_2-m_2-k_2(p-2)) - pp_2, \ -\frac{pp_2(\beta_2-1)}{m_2+k_2(p-2)-1}\right), \end{aligned} \tag{5.5}$$

Then for solution of the problem (1.1)-(1.2) *an estimate*

$$u(x,t) \le U(x,t) \quad in \ Q,$$

hold.

Proof. We prove Theorem 5.1 by applying the comparing solution technique outlined in [2]. The function $U(x,t)$ is used to compare solutions. Substituting (5.4) into (2.1) yields the following:

$$L(U) = -(T_1+t)^{-\alpha_1}\left[\xi^{1-n}\frac{d}{d\xi}\left(\xi^{n-1}\bar{f}^{m_2-1}\left|\frac{d\bar{f}^{k_2}}{d\xi}\right|^{p-2}\frac{d\bar{f}}{d\xi}\right)+\alpha_2\xi\frac{d\bar{f}}{d\xi}+\alpha_1\bar{f}+\right.$$
$$\left.(1-q)\,p_2^{n_3/p_2}\xi^{n_3}\bar{f}^{\beta_2}\right]. \tag{5.6}$$

Now to prove the theorem we should show that

$$L(U)\le 0 \ \text{ in } D=\left\{(x,t)\mid |x|<\left(p_2 b^{(p-1)/p}(T+t)^{\alpha_2}\right)^{1/p_2},\ t>0\right\}.$$

For this purpose, we need to show that

$$\xi^{1-n}\frac{d}{d\xi}\left(\xi^{n-1}\bar{f}^{m_2-1}\left|\frac{d\bar{f}^{k_2}}{d\xi}\right|^{p-2}\frac{d\bar{f}}{d\xi}\right)+\alpha_2\xi\frac{d\bar{f}}{d\xi}+\alpha_1\bar{f}+(1-q)\,p_2^{n_3/p_2}\xi^{n_3}\bar{f}^{\beta_2}\ge 0. \tag{5.7}$$

It is easy to show that the inequality (5.7) can be rewritten as follows, for the function $\bar{f}(\xi)$

$$\bar{f}\left[\alpha_1-\alpha_2 n+(1-q)\,p_2^{n_3/p_2}\xi^{n_3}\bar{f}^{\beta_2-1}\right]\ge 0.$$

Since $\bar{f}$ and ξ have a non-negative property, we have

$$\alpha_1-\alpha_2 n\ge 0 \ \text{ in } \ \xi^{\frac{p}{p-1}}<b.$$

To fulfil the last inequality, it is necessary that

$$0\le\alpha_1-\alpha_2 n=\frac{pp_2+n_3+(N+n_1)(m_2+k_2(p-2))-\beta_2(N+n_1)}{pp_2\beta_2+n_3(m_2+k_2(p-2))-pp_2-n_3}.$$

Since one of the inequalities in (5.5) holds, the last inequality is satisfied.
The proof of the Theorem 5.1 is complete.

5.1. Asymptotic of compactly supported weak solution. Let us introduce a new notation

$$g(\xi)=B\left(b+\xi^{\frac{p}{p-1}}\right)^{\gamma},$$

where b, B, γ and ξ defined above.
We shall investigate nontrivial, non-negative solutions to the equation (5.3) that fulfil the following conditions:

$$f(0)=c_1>0,\ f(a_1)=0,\ a_1<\infty. \tag{5.8}$$
$$f(\infty)=f'(0)=0. \tag{5.9}$$

Theorem 5.2. *Let* $m_2+k_2(p-2)>1$, $\beta>\beta_c$, *and* $\gamma(1-\beta_2)<1$, *then a finite solution of the problem* (5.3), (5.8) *has an asymptotic representation*

$$f(\xi)=\bar{f}(\xi)(1+o(1)),\ \text{at}\ \xi\to b_-^{\frac{p-1}{p}}.$$

Theorem 5.3. *Let* $m_2+k_2(p-2)<1$, $\beta>\beta_c$, *and* $\gamma(1-\beta_2)<1$, *then the solution of the problem* (5.3), (5.9) *has an asymptotic representation*

$$f(\xi)=g(\xi)(1+o(1)),\ \text{at}\ \xi\to+\infty.$$

Proof. Theorems 5.2 and 5.3 could be proved in the same manner as it was done in [3].
Note that the singular(critical) cases $m_2+k_2(p-2)=1$, $p=n_2-n_1$, and $\Delta_\alpha=0$ were studied in the work [3, 4].

5.2. Fundamental solution. In $\beta_c = q+m+k(p-2)+\frac{(1-q)(p-n_2+n_3)}{N+n_1}$, value of b can be obtained if the initial condition (3.2) is the equivalent to the condition $v_0(x)=E_0\delta(x)$, where $\delta(x)$ - Dirac delta function $E_0=\int_{\mathbb{R}^N}\rho_2(x)v(x,t)\,dx$.
To determine the value of b, we use the method outlined in [7]. Let the constant b satisfy the Dirac condition, then we say that the function $v(t,x)$ is the generalized (weak) fundamental solution of the problem (3.1)-(3.2).
In particular, the source-less form of (3.1)-(3.2), in the case $q=n_1=0, k=1, p=2$ the authors of the work [35], in $q=n_2=0, k=1, p=2$ S. Kamin and P. Rosenau [20], and in $q=n_1=n_2=n_3=0, k=1, p=2$, V. Galaktionov, J. Vazquez [16] determined the value of b.
Consider the following integrals:

$$I_N(\gamma,\gamma_1)=\int_{\mathbb{R}^N}(b_0^{\gamma_1}-\xi^{\gamma_1})_+^{\gamma}\,d\xi, \tag{5.10}$$

$$J_N(\gamma,\gamma_1,\gamma_2)=\int_{\mathbb{R}^N}(c_0^{\gamma_1}-\xi^{\gamma_1})_+^{\gamma}\,\xi^{\gamma_2}d\xi. \tag{5.11}$$

In [3] the authors showed that $I_N(\gamma,\gamma_1)$ and $J_N(\gamma,\gamma_1,\gamma_2)$ have the following relation:

$$J_N(\gamma,\gamma_1,\gamma_2)=\left(\frac{1}{\gamma_2+1}\right)^N I_N\left(\gamma,\frac{\gamma_1}{\gamma_2+1}\right).$$

Therefore, it is enough to consider the integral (5.10). Introduce the following notations:

$$\theta_i=(\xi_1,\ldots,\xi_i),\quad \chi_i=\left(b_0^{\gamma_1}-\sum_{j=1}^{i}\xi_j^{\gamma_1}\right)^{\frac{1}{\gamma_1}},\ i=\overline{N-1,1}.$$

Then we can rewrite (5.10) as follows:

$$\begin{aligned}
&I_N(\gamma,\gamma_1)=\textstyle\int_{\mathbb{R}^N}(b_0^{\gamma_1}-\xi^{\gamma_1})_+^{\gamma}d\xi=\int_{R^{N-1}}d\theta_{N-1}\int_R(\chi_{N-1}^{\gamma_1}-\xi_N^{\gamma_1})_+^{\gamma}d\xi_N\\
&\textstyle\int_{R^{N-1}}d\theta_{N-1}\int_{-\chi_{N-1}}^{\chi_{N-1}}(\chi_{N-1}^{\gamma_1}-\xi_N^{\gamma_1})_+^{\gamma}d\xi_N=\left|\xi_N=\chi_{N-1}z_N^{\frac{1}{\gamma_1}}\right|\\
&=\textstyle\frac{2}{\gamma_1}\int_{R^{N-1}}\chi_{N-1}^{\gamma\gamma_1+1}d\theta_{N-1}\int_0^1(1-z_N)_+^{\gamma}z_N^{\frac{1}{\gamma_1}-1}dz_N=\frac{2}{\gamma_1}B\left(\gamma+1,\frac{1}{\gamma_1}\right)I_{N-1}\left(\gamma+\frac{1}{\gamma_1}\right)\\
&=\ldots=I_1\left(\gamma+\frac{N-1}{\gamma_1}\right)\textstyle\prod_{i=1}^{N-1}\left(\frac{2}{\gamma_1}B\left(\gamma+1+\frac{i-1}{\gamma_1},\frac{1}{\gamma_1}\right)\right)\\
&=b_0^{N+\gamma\gamma_1}\textstyle\prod_{i=1}^{N}\left[\frac{2\Gamma\left(\frac{1}{\gamma_1}\right)}{\gamma_1}\cdot\frac{\Gamma\left(\gamma+1+\frac{i-1}{\gamma_1}\right)}{\Gamma\left(\gamma+1+\frac{i}{\gamma_1}\right)}\right]=b_0^{N+\gamma\gamma_1}\frac{\Gamma(\gamma+1)}{\Gamma\left(\gamma+1+\frac{N}{\gamma_1}\right)}\left(\frac{2}{\gamma_1}\Gamma\left(\frac{1}{\gamma_1}\right)\right)^N.
\end{aligned} \tag{5.12}$$

Given that $b=b_0^{\gamma_1}, \gamma_1=\frac{p-n_2+n_1}{p-1}$, $\gamma_2=n_1$ and put that to (5.12), then we have the following equation:

$$\frac{E_0}{B}=J(\gamma,\gamma_1,\gamma_2)=\left(\frac{1}{\gamma_2+1}\right)^N I_N\left(\gamma,\frac{\gamma_1}{\gamma_2+1}\right)=\frac{b^{\frac{N(p-1)}{p-n_2+n_1}+\gamma}\Gamma(\gamma+1)}{\Gamma\left(\gamma+1+N\left(\frac{(p-1)(1+n_1)}{(p-n_2+n_1)}\right)\right)}$$
$$\left(\frac{2(p-1)}{p-n_2+n_1}\Gamma\left(\frac{(p-1)(1+n_1)}{(p-n_2+n_1)}\right)\right)^N.$$

Hence

$$b = b\left(E_0\right) = \left[\frac{E_0}{B} \frac{\Gamma\left(\gamma+1+N\left(\frac{(p-1)(1+n_1)}{(p-n_2+n_1)}\right)\right)}{\Gamma(\gamma+1)} \left(\frac{p-n_2+n_1}{2(p-1)\Gamma\left(\frac{(p-1)(1+n_1)}{(p-n_2+n_1)}\right)} \right)^N \right]^{\frac{1}{\frac{N(p-1)}{p-n_2+n_1}+\gamma}}.$$

6. Conclusion

In this paper, we have studied the global existence and blow-up phenomena of a double nonlinear parabolic equation with a source in an inhomogeneous medium. By using the energy method, we have proved that the solution of the problem blows up in a finite time. We have also shown the global existence of the solutions and provided lower and upper estimates of the life span. In addition, we have constructed a self-similar Barenblatt-type solution and obtained an upper estimate of the solution to the problem. Using the Beta function, we have calculated the exact value of the Barenblatt-type solution parameter b, when the initial energy is given.

References

[1] D. Andreucci, A. Tedeev, Sharp estimates and finite speed of propagation for a Neumann problem in domains narrowing at infinity, *Adv. Differ. Equ.* **5** (2000), 833–860.

[2] M. Aripov and S. Sadullaeva, *Computer Simulation of Nonlinear Diffusion Processes* [in Russian], Press of the National University of Uzbekistan, Tashkent, 2020.

[3] M. Aripov and M. Bobokandov, The Cauchy problem for a doubly nonlinear parabolic equation with variable density and nonlinear time-dependent absorption, *J. Math. Sci.* **277** (2023), no. 3, 355-365.

[4] M. Aripov and M. Bobokandov, Analysis of a double nonlinear diffusion equation in inhomogeneous medium, *J. Math. Sci.* 2024 (in press).

[5] M. Aripov, A. Matyakubov and M. Bobokandov, Cauchy problem for the heat dissipation equation in non-homogeneous medium, *AIP Conf. Proc.* **2781** (2023), no.1, ID 020027.

[6] M. Aripov, A. Matyakubov, J. Khasanov and M. Bobokandov, Mathematical modeling of double nonlinear problem of reaction diffusion in not divergent form with a source and variable density, *J. Phys.: Conf. Ser.* **2131** (2021), no. 3, ID 032043.

[7] M. Aripov, M. Bobokandov, M. Mamatkulova, To numerical solution of the non-divergent diffusion equation in non-homogeneous medium with source or absorption, *AIP Conf. Proc.* **3085** (2024), no. 1, ID 020024.

[8] M. Aripov, M. Sayfullayeva, F. Kabiljanova, and M. Bobokandov, About one exact solution to the nonlinear problem of a biological population with absorption in a heterogeneous medium, *AIP Conf. Proc.* **3085** (2024), no. 1, ID 020031.

[9] S. Bolys, T. Berikbol, Global existence and blow-up of solutions to the double nonlinear porous medium equation, *Discrete Contin. Dyn. Syst.* **44** (2023), no. 3, 743-767.

[10] H. Brezis and A. Friedman, Nonlinear parabolic equations involving measures as initial conditions, *J. Math. Pures Appl.* **62** (1983), 73-97.

[11] H. Brezis, L.A. Peletier, and D. Terman, A very singular solution of the heat equation with absorption, *Arch. Ration. Mech. Anal.* **95** (1986), 185-209.

[12] H. Brezis, L. Nirenberg, Removable singularities for nonlinear elliptic equations, *Topol. Methods Nonlinear Anal.* **9** (1997), 201-219.

[13] J. Chunhua, Y. Jingxue, Asymptotic behavior of solutions for a doubly degenerate parabolic non-divergence form equation, *Rocky Mt. J. Math.* **47** (2017), 479-510.

[14] E. DiBenedetto, A. Friedman, Holder estimates for nonlinear degenerate parabolic system, *J. Reine Angew. Math.* **357** (1985), 1-22.

[15] E. DiBenedetto, *Degenerate Parabolic Equations*, Springer, New York, 1993.

[16] V. Galaktionov, J. Vazquez, Asymptotic behavior of nonlinear parabolic equations with critical exponents, A dynamical systems approach, *J. Funct. Anal.* **100** (1991), no. 2, 435-462.

[17] V. Galaktionov, H. Levine, A general approach to critical Fujita exponents in nonlinear parabolic problems, *Nonlinear Anal.* **34** (1998), 1005-1027.

[18] R. Gianni, A. Tedeev, V. Vincenzo, Asymptotic expansion of solutions to the Cauchy problem for doubly degenerate parabolic equations with measurable coefficients, Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods **138** (2015), 111-126.

[19] K. Hui, Another proof for the removable singularities of the heat equation *Proc. Amer. Math. Soc.* **138** (2010), no. 7: 2397-2402.

[20] S. Kamin, P. Rosenau, Propagation of thermal waves in an inhomogeneous medium, *Commun. Pure Appl. Math.* **34** (1981), no. 6, 831-852.

[21] S. Kamin, L.A. Peletier, J.L. Vazquez, Classification of singular solutions of a nonlinear heat equation, *Duke Math. J.* **58** (1989), 601-615.

[22] A. Martynenko, A. Tedeev, Cauchy problem for a quasilinear parabolic equation with a source term and an inhomogeneous density, *Comput. Math. Math. Phys.* **47** (2007), 238-248.

[23] A. Martynenko, A. Tedeev, On the Behavior of Solutions to the Cauchy Problem for a Degenerate Parabolic Equation with Inhomogeneous Density and a Source, *Comput. Math. Math. Phys.* **48** (2008), 1145-1160.

[24] A. Martynenko A. Tedeev, and V. Shramenko, The Cauchy problem for a degenerate parabolic equation with inhomogeneous density and source in the class of slowly decaying initial data, *Izv. Math.* **76** (2012), no. 3, 563-580.

[25] A. Matyakubov, D. Raupov, Explicit estimate for blow-up solutions of nonlinear parabolic systems of non-divergence form with variable density, *AIP Conf. Proc.* **2781** (2023), no.1, ID 020055.

[26] F. Nicolosi, I.I. Skrypnik, and I.V. Skrypnik, Removable isolated singularities for solutions of quasilinear parabolic equations, *Ukr. Math. Bull.* **6** (2009), 208-234.

[27] L.A. Peletier, J.Y. Wang, A very singular solution of a quasilinear degenerate diffusion equation with absorption, *Trans. Am. Math. Soc.* **307** (1988).

[28] F. Punzo, *Well-posedness of Degenerate Elliptic and Parabolic Problems*, Ph.D. thesis, 2008.

[29] Z. Rakhmonov, A. Alimov, Properties of solutions for a nonlinear diffusion problem with a gradient nonlinearity, *Int. J. Appl. Math.* **36** (2023), no. 3, 405-424.

[30] Z. Rakhmonov, J. Urunbaev, and A. Alimov, Properties of solutions of a system of nonlinear parabolic equations with nonlinear boundary conditions, *AIP Conf. Proc.* **2637** (2022), no. 1, ID 040008.

[31] N. Ravshanov, E. Nazirova, V. Pitolin, Numerical modelling of the liquid filtering process in a porous environment including the mobile boundary of the "oil-water" section Numerical modelling of the liquid filtering process in a porous environment including the mobile boundary of the "oil-water" section *J. Phys.: Conf. Ser.* **1399** (2019), no. 2, ID 022021.

[32] W. Reid, *Ordinary differential equations*, Wiley, New York, 1971.

[33] G. Reyes, J. Vazquez, The Cauchy problem for the inhomogeneous porous medium equation, *Netw. Heterog. Media* **1** (2006), 337–351.

[34] J. Saa, J. Díaz, Uniqueness of very singular self-similar solution of a quasilinear degenerate parabolic equation with absorption, *Publ. Mat.* **36** (1992), 19–38.

[35] A. Samarskii, V. Galaktionov, S. Kurdyumov, and A. Mikhailov, *Blow-Up in Quasilinear Parabolic Equations*, Walter de Grueter, Berlin, 1995.

[36] J. Shao, Z. Guo, X. Shan, Ch. Zhang, B. Wu, A new non-divergence diffusion equation with variable exponent for multiplicative noise removal, *Nonlinear Anal., Real World Appl.* **56** (2020), ID 103166.

[37] J.L. Vazquez, *Smoothing and decay estimates for nonlinear diffusion equations: equations of porous medium type*, Oxford University Press, Oxford, 2006.

[38] Z. Xiang, Ch. Mu, X. Hu, Support properties of solutions to a degenerate equation with absorption and variable density, *Nonlinear Anal.* **68** (2008), 1940-1953.

[39] P. Zheng, Ch. Mu, Global existence, large time behavior, and life span for a degenerate parabolic equation with inhomogeneous density and source, *Z. Angew. Math. Phys.* **65** (2014), 471-486.

[40] P. Zheng, Ch. Mu, X. Hu, and F. Zhang, Secondary critical exponent and life span for a doubly singular parabolic equation with a weighted source, *Acta Math. Sci.* **36** (2016), no. 1, 244-256.

[41] S. Zhou, X. Tang, C. Yang, A doubly degenerate diffusion equation not in divergence form with gradient term, *Bound. Value Probl.* **2016** (2016), ID 126.

Mersaid Aripov
National University of Uzbekistan, Tashkent, UZ 100174, Uzbekistan
E-mail address: mirsaidaripov@mail.ru

Makhmud Bobokandov
National University of Uzbekistan, Tashkent, UZ 100174, Uzbekistan,
Tashkent State Transport University, Tashkent, UZ 100167, Uzbekistan
E-mail address: m.boboqandov@nuu.uz

Received: July 23, 2024; Revised: November 11, 2024; Accepted: November 18, 2024