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UNIFORM CONVERGENCE OF EXPANSIONS IN THE FOURIER SERIES IN THE SYSTEM OF ROOT FUNCTIONS OF SOME FOURTH-ORDER SPECTRAL PROBLEM

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Abstract. In this paper we consider an eigenvalue problem for ordinary differential equations of fourth order with a spectral parameter in three of the boundary conditions. This problem describes small bending vibrations of a homogeneous Euler-Bernoulli beam of constant rigidity, in the cross sections of which a longitudinal force acts, at the left end of which a load is concentrated, and at the right end an inertial load is concentrated. Sufficient conditions are established for the uniform convergence of expansions in the Fourier series in the system of root functions of this spectral problem.

1. Introduction

We consider the following eigenvalue problem

$$\ell(y)(x) \equiv y^{(4)}(x) - (q(x)y'(x))' = \lambda y(x),\ 0 < x < 1, \tag{1.1}$$

$$U_1(\lambda, y) \equiv y''(0) = 0, \tag{1.2}$$

$$U_2(\lambda, y) \equiv Ty(0) - a\lambda y(0) = 0, \tag{1.3}$$

$$U_3(\lambda, y) \equiv y''(1) - b\lambda y'(1) = 0, \tag{1.4}$$

$$U_4(\lambda, y) \equiv Ty(1) - c\lambda y(1) = 0, \tag{1.5}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, $Ty \equiv y''' - qy'$, q is a positive absolutely continuous function on $[0, 1]$, a, b, c are real constants such that $abc \neq 0$.

This problem arises when applying the method of separation of variables to a boundary value problem for partial differential equations describing small bending vibrations of a homogeneous Euler-Bernoulli beam of constant rigidity, in the sections of which a longitudinal force acts. In addition, at the left end of which either the load is concentrated, or a tracking force acts, and at the right end either the inertial load is concentrated, or a tracking force acts, and to this end a load is attached by means of a weightless rod, held in equilibrium by means of an elastic spring (see, e.g., [7, 22]). Note that in order to justify this

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method in the boundary value problem for partial differential equations, it is necessary to study the convergence expansions in the Fourier series in the system of root functions of problem (1.1)-(1.5) in various functional spaces, for example, in $L_p(0,1)$, $1 < p < \infty$, or $C[0,1]$.

The convergence expansions in the Fourier series in the system of root functions of eigenvalue problems for ordinary differential equations of second and fourth order in L_p, $1 < p < \infty$, was studied in papers [1-3, 5, 6, 8, 10, 17-19]. Uniform convergence of spectral expansions in the system of root functions of such problems was studied in papers [2, 4, 9, 11, 12, 14-16, 21] (see also bibliography therein).

Problem (1.1)-(1.5) for $a > 0$, $b > 0$ and $c < 0$ was studied in [5], and for $a > 0$, $b > 0$ and $c > 0$ in [19]. In the case of $a > 0$, $b > 0$ and $c < 0$ the eigenvalues of problem (1.1)-(1.5) are nonnegative, simple and forms an infinitely increasing sequence, in the case $a > 0$, $b > 0$ and $c > 0$ the eigenvalues of this problem are real, simple, with except, for the case $c > 1$ and $a = c - 1$, when the eigenvalue $\lambda = 0$ which has algebraic multiplicity 2, and form an unboundedly nondecreasing sequence. In [5] and [19] the authors establish sufficient conditions for the system of root functions of problem (1.1)-(1.5) after removing three functions from this system, to form a basis in the space L_p, $1 < p < \infty$. However, the uniform convergence of expansions in Fourier series for the system of root functions of problem (1.1)-(1.5) has not yet been studied.

Note that in [2, 4, 9, 11-15, 19] sufficient conditions were established for the uniform convergence of the expansion of continuous functions in the subsystem of root functions of the problems considered there.

The rest of the article is organized as follows. In Section 2, we first refine the asymptotic formulas for the eigenvalues and eigenfunctions of the fourth-order spectral problem without the presence of a potential at the first derivative and with asymptotic boundary conditions. Then we obtain refined asymptotic formulas for the eigenvalues and eigenfunctions of problem (1.1)-(1.5). In Section 3 using these asymptotic formulas we find sufficient conditions for the uniform convergence of expansions in the Fourier series of continuous functions in the system of root functions of problem (1.1)-(1.5) after removing three functions.

2. Refined asymptotic formulas for eigenvalues and eigenfunctions of problem (1.1)-(1.5) and an auxiliary problem

Recall that problem (1.1)-(1.5) in the case of $a > 0$, $b > 0$ and $c > 0$ was considered in [18], where it was shown that the eigenvalues of problem (1.1)-(1.5) are real, simple, with except, for the case $c > 1$ and $a = c-1$, when the eigenvalue 0 which has algebraic multiplicity 2, and form an unbounded sequence $\{\lambda_k\}_{k=1}^{\infty}$ such that

$$\lambda_1 \leq \lambda_2 < \lambda_3 < \ \dots \ < \lambda_k < \ \dots \ ,$$

and are located on the real axis in the following order:

$$\lambda_1 < 0 = \lambda_2 \ \text{ for } \ c \leq 1 \text{ and } c > 1, \ a > c-1,$$

$$\lambda_1 = 0 = \lambda_2 \ \text{ for } \ c > 1 \text{ and } a = c-1,$$

$$\lambda_1 = 0 < \lambda_2 \text{ for } c > 1 \text{ and } a < c - 1.$$

Along with the boundary value problem (1.1)-(1.5), we consider the following spectral problem

$$y^{(4)}(x) = \lambda y(x),\ 0 < x < 1, \tag{2.1}$$

$$\begin{array}{l} V_1(y) \equiv y''(0) = 0,\ V_2(y) \equiv y(0) = 0, \\ V_3(y) \equiv y'(1) = 0,\ \ V_4(y) \equiv y(1) = 0, \end{array} \tag{2.2}$$

the eigenvalue of which are positive and simple and form an unboundedly increasing sequence $\{\mu_k\}_{k=1}^{\infty}$ (see [13]). Moreover, it follows from [13, Theorem 3.1] one has the following asymptotic formulas

$$\sqrt[4]{\mu_k} = \left(k + \frac{1}{4}\right)\pi + O\left(\frac{1}{k}\right), \tag{2.3}$$

$$\vartheta_k(x) = \sin\left(k + \tfrac{1}{4}\right)\pi x - (-1)^k \tfrac{\sqrt{2}}{2}\, e^{\left(k+\frac{1}{4}\right)\pi(1-x)} + O\left(\tfrac{1}{k}\right), \tag{2.4}$$

where relation (2.3) holds uniformly for $x \in [0, 1]$.

Theorem 2.1. *The following asymptotic formulas hold*

$$\sqrt[4]{\mu_k} = \left(k + \frac{1}{4}\right)\pi + O\left(\frac{1}{e^{k\pi}}\right), \tag{2.5}$$

$$\vartheta_k(x) = \sin\left(k + \tfrac{1}{4}\right)\pi x - (-1)^k \tfrac{\sqrt{2}}{2}\, e^{\left(k+\frac{1}{4}\right)\pi(1-x)} + O\left(\tfrac{1}{e^{k\pi}}\right), \tag{2.6}$$

where relation (2.6) *holds uniformly for* $x \in [0, 1]$.

Proof. In Eq. (2.1) let $\lambda = \rho^4$, $\rho > 0$. Note that Eq. (2.1) has the four linearly independent solutions

$$\phi_j(x, \rho) = e^{\rho\omega_j(x)},\ j = 1,\ 2,\ 3,\ 4, \tag{2.7}$$

where

$$\omega_1 = -1,\ \omega_2 = -\,i,\ \omega_3 = i,\ \omega_4 = 1.$$

By (2.7) we have

$$\phi_j^{(s)}(x, \rho) = (\rho\omega_j)^s e^{\rho\omega_j x},\ j = 1,\ 2,\ 3,\ 4,\ s = 0,\ 1,\ 2,\ 3,$$

which implies that

$$\phi_j^{(s)}(0, \rho) = (\rho\omega_j)^s,\ \phi_j^{(s)}(1, \varrho) = (\varrho\omega_j)^s e^{\varrho\omega_j},\ j = 1,\ 2,\ 3,\ 4,\ s = 0,\ 1,\ 2,\ 3. \tag{2.8}$$

By (2.8) it follows from (2.2) that

$$\begin{array}{l} V_1(\phi_j) \equiv \phi_j''(0, \rho) = \rho^2\omega_j^2,\ V_2(\phi_j) \equiv \phi_j(0, \varrho) = 1, \\ \\ V_3(\phi_j) \equiv \phi_j'(1, \rho) = \rho\omega_j e^{\rho\omega_j},\ V_4(\phi_j) \equiv \phi_j(1, \rho) = e^{\rho\omega_j},\ j = 1,\ 2,\ 3,\ 4. \end{array} \tag{2.9}$$

It is obvious that the eigenvalues of problem (2.1), (2.2) are the zeros of characteristic determinant

$$\Delta_0(\lambda) = \begin{vmatrix} V_1(\phi_1) & V_1(\phi_2) & V_1(\phi_3) & V_1(\phi_4) \\ V_2(\phi_1) & V_2(\phi_2) & V_2(\phi_3) & V_2(\phi_4) \\ V_3(\phi_1) & V_3(\phi_2) & V_3(\phi_3) & V_3(\phi_4) \\ V_4(\phi_1) & V_4(\phi_2) & V_4(\phi_3) & V_4(\phi_4) \end{vmatrix}. \tag{2.10}$$

Taking (2.8) and (2.9) into account from (2.10) we get

$$\Delta_0(\lambda) = \rho^3 \begin{vmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ e^{-\rho} & e^{-i\rho} & e^{i\rho} & e^{\rho} \\ -e^{-\rho} & -ie^{-i\rho} & ie^{i\rho} & e^{\rho} \end{vmatrix} =$$

$$2\rho^3 e^{\rho}\left\{(1-i)e^{i\rho} - (1+i)e^{-i\rho} + O\left(\frac{1}{e^{\rho}}\right)\right\}.$$

Therefore, the eigenvalues of (2.1), (2.2) are the roots of the equation

$$e^{2i\rho} = i + O\left(\frac{1}{e^{\rho}}\right). \tag{2.11}$$

In view of (2.3) we get

$$\rho_k = \sqrt[4]{\mu_k} = \left(k + \frac{1}{4}\right)\pi + \varepsilon_k, \tag{2.12}$$

where $\varepsilon_k = O\left(\frac{1}{k}\right)$. Then it follows from (2.11) that

$$e^{2i\rho_k} = ie^{2i\varepsilon_k} = i + O\left(\frac{1}{e^{\rho}}\right),$$

and consequently,

$$\varepsilon_k = O\left(\frac{1}{e^{k\pi}}\right). \tag{2.13}$$

Taking into account (2.13) in (2.12) we obtain (2.5).

By (2.5) we have the following relations

$$e^{i\varrho_k} = (-1)^k\frac{\sqrt{2}}{2}(1+i) + O\left(\frac{1}{e^{k\pi}}\right),\; e^{-i\varrho_k} = (-1)^k\frac{\sqrt{2}}{2}(1-i) + O\left(\frac{1}{e^{k\pi}}\right). \tag{2.14}$$

Note that the eigenfunction $\vartheta(x,\varrho)$ corresponding to the eigenvalue $\lambda = \rho^4$ of problem (2.1), (2.2) has the following representation

$$v(x,\rho_k) = B_{\rho_k}\begin{vmatrix} \phi_1(x,\rho_k) & \phi_2(x,\rho_k) & \phi_3(x,\rho_k) & \phi_4(x,\rho_k) \\ V_1(\phi_1) & V_1(\phi_2) & V_1(\phi_3) & V_1(\phi_4) \\ V_2(\phi_1) & V_2(\phi_2) & V_2(\phi_3) & V_2(\phi_4) \\ V_3(\phi_1) & V_3(\phi_2) & V_3(\phi_3) & V_3(\phi_4) \end{vmatrix}, \tag{2.15}$$

where $B_k = B_{\rho_k}$ is a nonzero constant depending on ρ_k.

By (2.7)-(2.9) it follows from (2.15) that

$$\vartheta_k(x) = \vartheta(x,\rho_k) = B_{\rho_k}\rho_k^2 e^{\varrho_k}\begin{vmatrix} e^{-\varrho_k x} & e^{-i\varrho_k x} & e^{i\varrho_k x} & e^{\varrho_k(x-1)} \\ 1 & -1 & -1 & e^{-\rho_k} \\ 1 & 1 & 1 & e^{-\rho_k} \\ e^{-\varrho_k} & e^{-i\varrho_k} & e^{i\varrho_k} & 1 \end{vmatrix} =$$

$$= B_{\rho_k}\rho_k^2 e^{\rho_k}$$

$$\left\{\begin{vmatrix} e^{-\left(k+\frac{1}{4}\right)\pi x} & e^{-i\left(k+\frac{1}{4}\right)\pi x} & e^{i\left(k+\frac{1}{4}\right)\pi x} & e^{\left(k+\frac{1}{4}\right)\pi(x-1)} \\ 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & (-1)^k\frac{\sqrt{2}}{2}(1-i) & (-1)^k\frac{\sqrt{2}}{2}(1+i) & 1 \end{vmatrix} + O\left(\frac{1}{e^{k\pi}}\right)\right\} =$$

$$= 4iB_{\rho_k}\rho_k^2\, e^{\rho_k}\left\{\sin\left(k+\frac{1}{4}\right)\pi x-(-1)^k\frac{\sqrt{2}}{2}\,e^{\left(k+\frac{1}{4}\right)\pi(x-1)}+O\left(\frac{1}{e^{k\pi}}\right)\right\}. \tag{2.16}$$

In view of (2.4) we choose the constant B_k as follows:

$$B_k=\frac{1}{4iB_{\rho_k}\rho_k^2e^{\rho_k}}.$$

Then we have

$$\vartheta_k(x)=\sin\left(k+\frac{1}{4}\right)\pi x-(-1)^k\frac{\sqrt{2}}{2}\,e^{\left(k+\frac{1}{4}\right)\pi(x-1)}+O\left(\frac{1}{e^{k\pi}}\right).$$

The proof of this theorem is complete.

It follows from (2.6) by a straightforward computation that

$$||\vartheta_k||_2^2=\int_0^1\vartheta_k^2(x)\,dx=1+O\left(\frac{1}{e^{k\pi}}\right). \tag{2.17}$$

Let

$$\psi_k(x)=\vartheta_k(x)||\vartheta_k||_2^{-1},\ k\in\mathbb{N}.$$

Then, by (2.6) and (2.17), we obtain

$$\psi_k(x)=\sin\left(k+\tfrac{1}{4}\right)\pi x-(-1)^k\tfrac{\sqrt{2}}{2}e^{\left(k+\frac{1}{4}\right)\pi(x-1)}+O\left(\tfrac{1}{e^{k\pi}}\right). \tag{2.18}$$

We introduce the following notation:

$$q_0=\int_0^1 q(t)\,dt,\ \ q_0(x)=\int_0^x q(t)\,dt \text{ and } q_1(x)=\int_x^1 q(t)\,dt.$$

Now we demonstrate the refined asymptotic formulas for eigenvalues and eigenfunctions of problem (1.1)-(1.5).

Theorem 2.2. *One has the following asymptotic formulas*

$$\sqrt[4]{\lambda_k}=\left(k-\frac{11}{4}\right)\pi+\frac{q_0+2/a-4/c}{4k\pi}+O\left(\frac{1}{k^2}\right), \tag{2.19}$$

$$\begin{gathered}y_k(x)=\sin\left(k-\tfrac{11}{4}\right)\pi x-(-1)^{k+1}\tfrac{\sqrt{2}}{2}\,e^{\left(k-\frac{11}{4}\right)\pi(x-1)}+\\ \tfrac{2/a}{4k\pi}\sin\left(k-\tfrac{11}{4}\right)\pi x+\tfrac{(q_0+2/a-4/c)\,x-(q_0(x)+2/a)}{4k\pi}\cos\left(k-\tfrac{11}{4}\right)\pi x-\\ \tfrac{2/a}{4k\pi}\,e^{-\left(k-\frac{11}{4}\right)\pi x}-(-1)^{k+1}\tfrac{\sqrt{2}}{2}\,e^{\left(k-\frac{11}{4}\right)\pi(x-1)}\times\\ \tfrac{(q_0+2/a-4/c)\,(x-1)+(2/a+4/c-q_1(x))}{4k\pi}+O\left(\tfrac{1}{k^2}\right).\end{gathered} \tag{2.20}$$

where relation (2.20) *holds uniformly for* $x\in[0,1]$.

Proof. In Eq. (1.1) let $\lambda=\varrho^4$, where $\varrho>0$. By [20, Ch. 2, §4.5, Theorem 1] this equation has four linearly independent solutions $\varphi_j(x,\varrho)$, $j=1,\,2,\,3,\,4$, which are regular in ϱ (for sufficiently large $|\varrho|$) and satisfy the following relations

$$\begin{gathered}\varphi_j^{(s)}(x,\varrho)=(\varrho\omega_j)^s e^{\varrho\omega_j x}\left\{1+\tfrac{q_0(x)}{4\rho\omega_j}+O\left(\tfrac{1}{\rho^2}\right)\right\},\ j=1,\,2,\,3,\,4,\\ s=0,\,1,\,2,\,3.\end{gathered} \tag{2.21}$$

where $\omega_1=-1$, $\omega_2=-\,i$, $\omega_3=i$ and $\omega_4=1$.

In view of (2.21), by boundary conditions (1.2)-(1.5) we obtain

$$\begin{aligned}
&U_1(\lambda,\varphi_j)=1+O\left(\tfrac{1}{\varrho^2}\right),\\
&U_2(\lambda,\varphi_j)=-\,a\varrho^4\left(1-\tfrac{1}{a\varrho\omega_j}+O\left(\tfrac{1}{\varrho^2}\right)\right),\\
&U_3(\lambda,\varphi_j)=-\,b\varrho^5\omega_j\,e^{\varrho\omega_j}\left(1+\tfrac{q_0}{4\varrho\omega_j}+O\left(\tfrac{1}{\varrho^2}\right)\right),\\
&U_4(\lambda,\varphi_j)=-\,c\varrho^4\,e^{\varrho\omega_j}\left(1+\tfrac{q_0-4/c}{4\rho\omega_j}+O\left(\tfrac{1}{\varrho^2}\right)\right),\ j=1,\,2,\,3,\,4.
\end{aligned}\tag{2.22}$$

Let $\lambda=\rho^4$ is an eigenvalue of the spectral problem (1.1)-(1.5). Then ϱ is a zero of the characteristic determinant

$$\Delta(\lambda)=\begin{vmatrix} U_1(\lambda,\varphi_1) & U_1(\lambda,\varphi_2) & U_1(\lambda,\varphi_3) & U_1(\lambda,\varphi_4)\\ U_2(\lambda,\varphi_1) & U_2(\lambda,\varphi_2) & U_2(\lambda,\varphi_3) & U_2(\lambda,\varphi_4)\\ U_3(\lambda,\varphi_1) & U_3(\lambda,\varphi_2) & U_3(\lambda,\varphi_3) & U_3(\lambda,\varphi_4)\\ U_4(\lambda,\varphi_1) & U_4(\lambda,\varphi_2) & U_4(\lambda,\varphi_3) & U_4(\lambda,\varphi_4)\end{vmatrix}.\tag{2.23}$$

By (2.22) it follows from (2.23) that

$$\Delta(\lambda)=-\,abc\varrho^{15}e^{\varrho}\left(1+\frac{q_0-4/c}{4\varrho}\right)$$

$$\left\{\begin{vmatrix} 1 & -\,1 & -\,1 & 0\\ 1+\frac{1}{a\varrho} & 1+\frac{1}{ia\varrho} & 1-\frac{1}{ia\varrho} & 0\\ 0 & -\,ie^{-i\rho}\left(1-\frac{q_0}{4i\rho}\right) & ie^{i\rho}\left(1+\frac{q_0}{4i\rho}\right) & 1\\ 0 & e^{-i\rho}\left(1-\frac{q_0-4/c(1+i)}{4i\rho}\right) & e^{i\rho}\left(1+\frac{q_0-4/c(1-i)}{4i\rho}\right) & 1\end{vmatrix}+O\left(\frac{1}{\varrho^2}\right)\right\}=$$

$$-\,2abc\varrho^{15}e^{\varrho}\left(1+\frac{q_0-4/c}{4\varrho}\right)$$

$$\left\{\begin{vmatrix} 1+\frac{1}{2ia\varrho}(1+i) & 1-\frac{1}{2ia\varrho}(1-i) & 0\\ -\,ie^{-i\rho}\left(1-\frac{q_0}{4i\rho}\right) & ie^{i\rho}\left(1+\frac{q_0}{4i\rho}\right) & 1\\ e^{-i\rho}\left(1-\frac{q_0-4/c(1+i)}{4i\rho}\right) & e^{i\rho}\left(1+\frac{q_0-4/c(1-i)}{4i\rho}\right) & 1\end{vmatrix}+O\left(\frac{1}{\varrho^2}\right)\right\}=$$

$$2abc\varrho^{15}e^{\varrho}\left(1+\frac{q_0-4/c}{4\varrho}\right)$$

$$\left\{\begin{vmatrix} 1+\frac{1}{2ia\varrho}(1+i) & 1-\frac{1}{2ia\varrho}(1-i)\\ e^{-i\rho}(1+i)\left(1-\frac{q_0-4/c}{4i\rho}\right) & e^{i\rho}(1-i)\left(1+\frac{q_0-4/c}{4i\rho}\right)\end{vmatrix}+O\left(\frac{1}{\varrho^2}\right)\right\}=$$

$$2abc\varrho^{15}e^{\varrho}\left(1+\frac{q_0-4/c}{4\varrho}\right)$$

$$\left\{e^{i\rho}(1-i)(1+\frac{1}{2ia\varrho}(1+i))\left(1+\frac{q_0-4/c}{4i\rho}\right)-e^{-i\rho}(1+i)(1-\frac{1}{2ia\varrho}(1-i))\right.$$

$$\left.\left(1-\frac{q_0-4/c}{4i\rho}\right)+O\left(\frac{1}{\varrho^2}\right)\right\}.$$

Then it follows from the last relation that the zeros of the characteristic determinant are the roots of the equation

$$e^{i\rho}(1-i)(1+\tfrac{1}{2ia\varrho}(1+i))\left(1+\tfrac{q_0-4/c}{4i\rho}\right)-e^{-i\rho}(1+i)(1-\tfrac{1}{2ia\varrho}(1-i))\times$$
$$\left(1-\tfrac{q_0-4/c}{4i\rho}\right)+O\left(\tfrac{1}{\varrho^2}\right)=0. \tag{2.24}$$

From (2.24) we obtain

$$e^{2\varrho}=i\left(1-\frac{q_0+2/a\,-\,4/c}{2i\rho}\right)+O\left(\frac{1}{\varrho^2}\right). \tag{2.25}$$

By [5, formula (3.13)] we have

$$\varrho_k=\sqrt[4]{\lambda_k}=\left(k-\frac{11}{4}\right)\pi+\epsilon_k, \tag{2.26}$$

where $\epsilon_k=O\left(\frac{1}{k}\right)$.

In view of (2.26), by (2.25) we get

$$e^{2i\rho_k}=i\,e^{2i\varepsilon_k}=i\,(1+2i\varepsilon_k+o(\varepsilon_k^2))=i\left(1-\frac{q_0+2/a\,-\,4/c}{2ik\pi}\right)+O\left(\frac{1}{k^2}\right),$$

whence implies that

$$\varepsilon_k=\frac{q_0+2/a\,-\,4/c}{4k\pi}+O\left(\frac{1}{k^2}\right). \tag{2.27}$$

Now (2.26) and (2.27) yield (2.19).

By (2.19) we have the following relations

$$\begin{aligned} e^{i\varrho_k}&=(-1)^{k+1}\tfrac{\sqrt{2}}{2}\,(1+i)\left(1-\tfrac{q_0+2/a-4/c}{4i\varrho_k}+O\left(\tfrac{1}{\varrho_k^2}\right)\right),\\ e^{-\,i\varrho_k}&=(-1)^{k+1}\tfrac{\sqrt{2}}{2}\,(1-i)\left(1+\tfrac{q_0+2/a-4/c}{4i\varrho_k}+O\left(\tfrac{1}{\varrho_k^2}\right)\right). \end{aligned} \tag{2.28}$$

The eigenfunction $y(x,\varrho_k)$ corresponding to the eigenvalue $\lambda=\varrho_k^4$ of the spectral problem (1.1)-(1.5) has the following form

$$y_k(x)=y(x,\varrho_k)=$$
$$C_k\begin{vmatrix} \phi_1(x,\varrho_k) & \phi_2(x,\varrho_k) & \phi_3(x,\varrho_k) & \phi_4(x,\varrho_k)\\ U_1(\lambda_k,\phi_1) & U_1(\lambda_k,\phi_2) & U_1(\lambda_k,\phi_3) & U_1(\lambda_k,\phi_4)\\ U_2(\lambda_k,\phi_1) & U_2(\lambda_k,\phi_2) & U_2(\lambda_k,\phi_3) & U_2(\lambda_k,\phi_4)\\ U_3(\lambda_k,\phi_1) & U_3(\lambda_k,\phi_2) & U_3(\lambda_k,\phi_3) & U_3(\lambda_k,\phi_4) \end{vmatrix}. \tag{2.29}$$

where $C_k=C_{\varrho_k}$ is a nonzero constant depending on ϱ_k.

We define the numbers τ_k, $k\in\mathbb{N}$, and the functions $q_{i,k}(x)$, $x\in[0,1]$, $i=1,2,3,4$, $k\in\mathbb{N}$, as follows:

$$\tau_k=(-1)^k\frac{\sqrt{2}}{2}\,(1+i),\;\; q_{1,k}(x)=1-\frac{q_0(x)}{4\varrho_k},\;\; q_{2,k}(x)=1-\frac{q_0(x)}{4i\varrho_k},$$

$$q_{3,k}(x)=1+\frac{q_0(x)}{4i\varrho_k},\;\; q_{4,k,}(x)=1+\frac{q_0(x)}{4\varrho_k},\;\; x\in[0,1]$$

Then, by (2.21), (2.22) and (2.28), from (2.29) we get

$$y(x,\varrho_k)=ab\varrho_k^{11}e^{\varrho_k}\left(1+\frac{q_0}{4\varrho_k}\right)C_{\varrho_k}$$

$$\left\{\begin{vmatrix} e^{-\rho_k x}q_{k,1}(x) & e^{-i\rho_k x}q_{2,k}(x) & e^{i\rho_k x}q_{3,k}(x) & e^{\rho_k(x-1)}q_{4,k}(x) \\ 1 & -1 & -1 & 0 \\ 1+\frac{1}{a\varrho_k} & 1+\frac{1}{ai\varrho_k} & 1-\frac{1}{ai\varrho_k} & 0 \\ 0 & \left(1+\frac{2/a-4/c}{4i\varrho_k}\right)\tau_k & \left(1-\frac{2/a-4/c}{4i\varrho_k}\right)\tau_k & 1 \end{vmatrix} + \right.$$

$$\left. O\left(\frac{1}{\varrho_k^2}\right)\right\} = 2ab\varrho_k^{11}e^{\varrho_k}C_{\varrho_k}\left(1+\frac{q_0}{4\varrho_k}\right)\times$$

$$\left\{\frac{1}{ai\varrho_k}\,e^{-\rho_k x} - \left(1-\frac{q_0(x)}{4i\varrho_k}\right)\left(1-\frac{1}{2ai\varrho_k}\,(1-i)\right)e^{-i\rho_k x} + \right.$$

$$\left(1+\frac{q_0(x)}{4i\varrho_k}\right)\left(1+\frac{1}{2ai\varrho_k}\,(1+i)\right)e^{i\varrho_k x} -$$

$$\left.(-1)^{k+1}\sqrt{2}\,i\left(1+\frac{q_0(x)-q_0}{4\rho_k}\right)\left(1+\frac{2/a\,+4/c}{4\varrho_k}\right)e^{\varrho_k(x-1)} + O\left(\frac{1}{\varrho_k^2}\right)\right\} =$$

$$4iab\varrho_k^{11}e^{\varrho_k}C_{\varrho_k}\left(1+\frac{q_0}{4\varrho_k}\right)\times$$

$$\left\{\sin\varrho_k x - (-1)^{k+1}\frac{\sqrt{2}}{2}\,e^{\varrho_k(x-1)} + \frac{2/a}{4\rho_k}\sin\varrho_k x - \frac{q_0(x)+2/a}{4\rho_k}\cos\varrho_k x - \right.$$

$$\left.\frac{2/a}{4\rho_k}\,e^{-\,\varrho_k x} - (-1)^{k+1}\frac{\sqrt{2}}{2}\,\frac{2/a\,+4/c-q_1(x)}{4\varrho_k}\,e^{\varrho_k(x-1)} + O\left(\frac{1}{\varrho_k^2}\right)\right\}. \tag{2.30}$$

In view of (2.19) we can choose C_{ϱ_k} as follows:

$$C_{\varrho_k} = \frac{(1-q_0/4\rho_k)\varrho_k^{-\,11}e^{-\,\varrho_k}}{4iab}.$$

Then it follows from (2.30) that

$$y(x,\varrho_k) = \sin\varrho_k x - (-1)^{k+1}\tfrac{\sqrt{2}}{2}\,e^{\varrho_k(x-1)} + \tfrac{2/a}{4\rho_k}\sin\varrho_k x - \tfrac{q_0(x)+2/a}{4\rho_k}\cos\varrho_k x -$$

$$\tfrac{2/a}{4\rho_k}\,e^{-\,\varrho_k x} - (-1)^{k+1}\tfrac{\sqrt{2}}{2}\,\tfrac{2/a\,+4/c-q_1(x)}{4\varrho_k}\,e^{\varrho_k(x-1)} + O\left(\tfrac{1}{\varrho_k^2}\right). \tag{2.31}$$

By (2.19) we have the following relations

$$\sin\varrho_k x = \sin\left(k-\frac{11}{4}\right)\pi x + \frac{(q_0+2/a-4/c)\,x}{4k\pi}\cos\left(k-\frac{11}{4}\right)\pi x + O\left(\frac{1}{k^2}\right),$$

$$\cos\varrho_k x = \cos\left(k-\frac{11}{4}\right)\pi x - \frac{(q_0+2/a-4/c)\,x}{4k\pi}\sin\left(k-\frac{11}{4}\right)\pi x + O\left(\frac{1}{k^2}\right),$$

$$e^{-\,\varrho_k x} = e^{-\,\left(k-\frac{11}{4}\right)\pi x}\left\{1-\frac{(q_0+2/a-4/c)\,x}{4k\pi}\right\} + O\left(\frac{1}{k^2}\right),$$

$$e^{\varrho_k(x-1)} = e^{\left(k-\frac{11}{4}\right)\pi(x-1)}\left\{1+\frac{(q_0+2/a-4/c)\,(x-1)}{4k\pi}\right\} + O\left(\frac{1}{k^2}\right).$$

Using these relations from (2.31) we obtain

$$y_k(x)=$$
$$\sin\left(k-\tfrac{11}{4}\right)\pi x-(-1)^{k+1}\tfrac{\sqrt{2}}{2}\,e^{\left(k-\frac{11}{4}\right)\pi(x-1)}+\tfrac{2/a}{4k\pi}\sin\left(k-\tfrac{11}{4}\right)\pi x+$$
$$\tfrac{(q_0+2/a-4/c)\,x-(q_0(x)+2/a)}{4k\pi}\cos\left(k-\tfrac{11}{4}\right)\pi x-\tfrac{2/a}{4k\pi}\,e^{-\left(k-\frac{11}{4}\right)\pi x}- \tag{2.32}$$
$$(-1)^{k+1}\tfrac{\sqrt{2}}{2}\,e^{\left(k-\frac{11}{4}\right)\pi(x-1)}\tfrac{(q_0+2/a-4/c)\,(x-1)+(2/a+4/c-q_1(x))}{4k\pi}+O\left(\tfrac{1}{k^2}\right).$$

The proof of this theorem is complete.

3. The uniform convergence of expansions in the root functions system of the spectral problem (1.1)-(1.5)

Asymptotic formulas (2.6) and (2.20) show that for $k\geq 4$ the following relation holds:

$$y_k(x)=\psi_{k-3}(x)+\tfrac{2/a}{4k\pi}\sin\left(k-\tfrac{11}{4}\right)\pi x+$$
$$\tfrac{(q_0+2/a-4/c)\,x-(q_0(x)+2/a)}{4k\pi}\cos\left(k-\tfrac{11}{4}\right)\pi x-\tfrac{2/a}{4k\pi}\,e^{-\left(k-\frac{11}{4}\right)\pi x}- \tag{3.1}$$
$$(-1)^{k+1}\tfrac{\sqrt{2}}{2}\,e^{\left(k-\frac{11}{4}\right)\pi(x-1)}\tfrac{(q_0+2/a-4/c)\,(x-1)+(2/a+4/c-q_1(x))}{4k\pi}+O\left(\tfrac{1}{k^2}\right).$$

It follows from (2.18), (2.20) and [4, estimate (8.3)] that

$$\psi_k(1)=O\left(\tfrac{1}{e^{k\pi}}\right),\ y_k(0)=-\tfrac{1}{ak\pi}+O\left(\tfrac{1}{k^2}\right),$$
$$y_k(1)=O\left(\tfrac{1}{k^2}\right),\ y_k'(1)=O\left(\tfrac{1}{k^2}\right). \tag{3.2}$$

Moreover, following the corresponding reasoning on pp. 282-284 of [4] we can show that

$$||y_k||_2^2=1+O\left(\frac{1}{k^2}\right), \tag{3.3}$$

where $||\cdot||_2$ is the norm in $L_2(0,1)$.

Let

$$\delta_k=||y_k||_2^2+ay_k^2(0)+by_k'^{\,2}(1)-cy_k^2(1),\ \ k\in\mathbb{N},\,k\geq 2. \tag{3.4}$$

Then it follows from [19, Lemma 8] that

$$\delta_k\neq 0,\ \ k\in\mathbb{N},\,k\geq 2.$$

We introduce the notations:

$$v_k(x)=\delta_k^{-1}y_k(x),\ x\in[0,1],\ s_k=\delta_k^{-1}ay_k(0),\ t_k=\delta_k^{-1}by_k'(1),$$
$$r_k=-\,\delta_k^{-1}cy_k(1),\ k\in\mathbb{N},\,k\geq 2. \tag{3.5}$$

Let i, j, l, be arbitrary different fixed positive integers such that $i, j, r \geq 3$ and

$$\Delta_{i,\,j,\,l} = \begin{vmatrix} s_i & s_j & s_l \\ t_i & t_j & t_l \\ r_i & r_j & r_l \end{vmatrix}. \tag{3.6}$$

It follows from [19, Theorem 3] that if $\Delta_{i,\,j,\,l} \neq 0$, then the system $\{y_k\}_{k=1,\,k\neq i,j,\,l}^{\infty}$ of root functions of problem (1.1)-(1.5) forms a basis in $L_p(0,1)$, $1 < p < \infty$, which is an unconditional basis for $p = 2$. In this case by [1, Theorem 3.1] (see formula (3.8) in [1]) each element of the system $\{u_k\}_{k=1,\,k\neq i,j,\,l}^{\infty}$ conjugate to the system $\{y_k\}_{k=1,\,k\neq i,j,\,l}^{\infty}$ is determined as follows:

$$u_k = v_k - \frac{1}{\Delta_{i,j,\,l}} \left\{ v_i \Delta_{k,j,\,l} - v_j \Delta_{k,i,l} + v_l \Delta_{k,i,j} \right\}. \tag{3.7}$$

By (3.2)-(3.4) we get

$$\delta_k = 1 + O\left(\frac{1}{k^2}\right). \tag{3.8}$$

In view of (3.5) we have

$$\Delta_{i,\,j,\,l} = \begin{vmatrix} s_i & s_j & s_l \\ t_i & t_j & t_l \\ r_i & r_j & r_l \end{vmatrix} = -\,\delta_i^{-1}\delta_j^{-1}\delta_l^{-1} abc \begin{vmatrix} y_s(0) & y_j(0) & y_l(0) \\ y_i'(1) & y_j'(1) & y_l'(1) \\ y_i(1) & y_j(1) & y_l(1) \end{vmatrix}. \tag{3.9}$$

Let

$$\tilde{\Delta}_{i,\,j,\,l} = \begin{vmatrix} y_s(0) & y_j(0) & y_l(0) \\ y_i'(1) & y_j'(1) & y_l'(1) \\ y_i(1) & y_j(1) & y_l(1) \end{vmatrix}. \tag{3.10}$$

Then by (3.2), (3.2), (3.8) and (3.9), from (3.7) we obtain

$$\begin{gathered} u_k(x) = y_k(x) - \tilde{\Delta}_{i,j,\,l}^{-1} \left\{ y_i(x)\tilde{\Delta}_{k,j,\,l} - y_j(x)\tilde{\Delta}_{k,i,l} + y_l(x)\tilde{\Delta}_{k,i,j} \right\} + \\ O\left(\tfrac{1}{k^2}\right) = y_k(x) - y_k(0)\,\tilde{\Delta}_{i,j,\,l}^{-1}\,\tilde{\Delta}_{i,j,\,l}(x) + O\left(\tfrac{1}{k^2}\right), \end{gathered} \tag{3.11}$$

where

$$\tilde{\Delta}_{i,\,j,\,l}(x) = \begin{vmatrix} y_i(x) & y_j(x) & y_l(x) \\ y_i'(1) & y_j'(1) & y_l'(1) \\ y_i(1) & y_j(1) & y_l(1) \end{vmatrix}. \tag{3.12}$$

If $\tilde{\Delta}_{i,\,j,\,l} \neq 0$, then by (3.9) and (3.12) it follows from above arguments that for any function $f \in C[0,1]$ the Fourier series expansion

$$f(x) = \sum_{k=1,\,k\neq r,\,l}^{\infty} (f, u_k) y_k(x), \tag{3.13}$$

of this function in the system $\{y_k\}_{k=1,\,k\neq i,j,\,l}^{\infty}$ of root functions of problem (1.1)-(1.5) converges in space $L_p\,(0,1)$, $1 < p < \infty$, which converges unconditionally for $p = 2$.

We introduce notation:

$$\tilde{\Delta}_{i,j,\,l}^{*} = \begin{vmatrix} (f, y_i) & (f, y_j) & (f, y_l) \\ y_i'(1) & y_j'(1) & y_l'(1) \\ y_i(1) & y_j(1) & y_l(1) \end{vmatrix},$$

where $(\cdot\,,\,\cdot)$ is a scalar product in $L_2(0,1)$.

The following theorem is the main result of this paper.

Theorem 3.1. *Let i, j, l, be arbitrary different fixed positive integers such that $i, j, r \geq 3$ and $\tilde{\Delta}_{i,\,j,\,l} \neq 0$. Moreover, let the Fourier series expansion of a function $f(x) \in C[0,1]$ in the system $\{\vartheta_k\}_{k=1}^{\infty}$ of eigenfunctions of problem* (2.1), (2.2) *uniformly converges on the interval* $[0,1]$. *If $\tilde{\Delta}^*_{i,j,\,l} \neq 0$, then the series* (3.13) *uniformly converges on the interval $[0,\tau]$ for each $\tau \in (0,1)$, and if $\tilde{\Delta}^*_{i,j,\,l} = 0$, then the series* (3.13) *uniformly converges on the interval* $[0,1]$.

The proof of this theorem is similar to that of [4, Theorem 8.1] with the use of Theorems 2.1 and 2.2.

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