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NEUMANN BOUNDARY VALUE PROBLEMS FOR LOWER SEMI-CONTINUOUS NON-CONVEX DIFFERENTIAL INCLUSIONS WITH ϕ–LAPLACIAN

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Abstract. Using a combination of upper and lower solutions method with the topological degree approach, we establish the existence of solutions that satisfy the Neumann conditions for the given differential inclusion $(\phi(x'(t)))' \in F(t, x(t))$, where F denotes a lower semi-continuous multi-valued map and ϕ represents an homeomorphism.

1. Introduction

In this paper, we shall prove the existence of solutions to the following differential inclusions:

$$\begin{cases} \big(\phi(x'(t))\big)' \in F(t, x(t)), & \text{a.e. on } [0, T]; \\ x'(0) = r, \ x'(T) = r'; \end{cases} \tag{1.1}$$

where F is a multi-valued map, $\phi :]-a, a[\to \mathbb{R}$ is a function and $(r, r') \in \mathbb{R}^2$.

Neumann boundary value problems have received the attention of many authors. Mawhin and Ruiz, in [10], have proved the existence of a solution of the following Neumann boundary value problem

$$\begin{cases} \big(|x'|^{p-2}x'\big)' + f(t, x) + h(t, x) = 0 \ \text{ a.e. on }]0, 1[; \\ x'(0) = x'(1) = 0, \end{cases}$$

where $p \geq 2$ and $f, h :]0, 1[\times\mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions. Their work is based on topological degree techniques.

Mawhin and Bereanu [5] have studied the problem $\big(\phi(x'(t))\big)' = f\big(t, x(t), x'(t)\big)$. They have addressed this problem, under periodic, Dirichlet, and Neumann boundary conditions, in the case where ϕ is a singular homeomorphism and f is a continuous function. Frigon, El Khattabi, and Ayyadi [8] have considered the same problem, under the same boundary conditions, in the case where f is a Carathéodory function satisfying a Wintner-Nagumo type growth condition. They have applied the method of upper and lower solutions, combined with the fixed point index theory.

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Aitalioubrahim, see [4], has showed the existence of solutions to the boundary value problem

$$\begin{cases} x''(t) \in F(t, x(t), x'(t)) \text{ a.e. on } [0,1]; \\ x'(0) = r, \ x'(1) = s, \end{cases}$$

where F is measurable in the first argument and Lipschitz continuous in the second argument and r and s are in a Banach space E.

Recently, Aitalioubrahim and Tebbaa, see [2, 3], have proved the existence of solutions for the problem $\big(\phi(x'(t))\big)' \in F(t, x(t))$ satisfying periodic, Dirichlet, Cauchy or terminal boundary conditions, where F is lower semicontinuous with respect to the second argument and ϕ is an increasing homeomorphism. They have supposed the existence of upper and lower solutions and have used some properties of the topological degree.

In this work, we establish the existence result for Problem (1.1) with Neumann boundary value conditions, in the lower semi-continuous case. We use the method of upper and lower solutions combined with the fixed point index theory.

2. Preliminaries and statement of the main results

In this section, we present the definitions, notations and preliminary concepts that will be utilized throughout this paper. Let E be a Banach space equipped with the norm $\|.\|$. The notation $\mathcal{C}([0,T], E)$ refers to the Banach space of all continuous functions mapping from the interval $[0,T]$ into E, equipped with the norm defined by $\|x\|_\infty := \sup\big\{\|x(t)\|; t \in [0,T]\big\}$. The space $\mathcal{C}^1([0,T], E)$ indicates the Banach space of functions that are continuously differentiable on $[0,T]$. $L^1([0,T], \mathbb{R})$ denotes the Banach space of Lebesgue integrable functions from $[0,T]$ to $\mathbb{R}$. A subset U of $[0,T] \times \mathbb{R}$ is considered $\mathcal{L} \otimes \mathcal{B}$-measurable if it is part of the σ-algebra generated by sets of the form $I \times X$, where I is Lebesgue measurable in $[0,T]$ and X is Borel measurable in $\mathbb{R}$. A multifunction is defined as measurable if its graph is measurable.

Definition 2.1. A subset U of $L^1([0,T], \mathbb{R})$ is considered decomposable if for any elements u, v in U and any measurable set $I \subset [0,T]$, the function $u\chi_I + v\chi_{[0,T]\setminus I}$ also belongs to U, where χ represents the characteristic function.

Definition 2.2. Let E be a separable Banach space, X denote a nonempty closed subset of E and $G : X \to 2^E$ be a multi-valued function with nonempty closed values. We define G as lower semi-continuous if for every open set C in E, the set $\{x \in X : G(x) \cap C \neq \emptyset\}$ is an open set.

Now, let $F : [0,T] \times \mathbb{R} \to 2^{\mathbb{R}}$ be a multi-valued map characterized by nonempty compact values. We define the multi-valued operator $\mathcal{F} : \mathcal{C}([0,T], \mathbb{R}) \to 2^{L^1([0,T],\mathbb{R})}$ as follows

$$\mathcal{F}(x) = \left\{ y \in L^1([0,T], \mathbb{R}) : y(t) \in F(t, x(t)) \text{ for a.e. } t \in [0,T] \right\}.$$

This operator known as the Niemytzki operator associated with F is classified as of the lower semi-continuous type if it is lower semi-continuous and possesses nonempty closed and decomposable values. In this context, we require the following lemma.

Lemma 2.1. [6] *Let E be a separable metric space and $\Gamma : E \to 2^{L^1([0,T],\mathbb{R})}$ a multi-valued operator which is lower semi-continuous and has nonempty closed and decomposable values. Then Γ has a continuous selection, i.e. there exists a continuous function $g : E \to L^1([0,T],\mathbb{R})$ such that $g(y) \in \Gamma(y)$ for every $y \in E$.*

In the sequel, it is essential to introduce the notions of compact and completely continuous functions.

Definition 2.3. Let X and Y be topological spaces. A function $f : X \to Y$ is defined as compact if the image $f(X)$ is relatively compact. In the case where X is a metric space, the function $f : X \to Y$ is referred to as completely continuous if for every bounded subset $B \subset X$, the image $f(B)$ is relatively compact.

This work will be based on the subsequent assumptions.

(H1) $F : [0,T] \times \mathbb{R} \to 2^{\mathbb{R}}$ is a set-valued map with nonempty compact values satisfying
  (i) $(t,x) \mapsto F(t,x)$ is $\mathcal{L} \otimes \mathcal{B}-$measurable,
  (ii) $x \mapsto F(t,x)$ is lower semi-continuous for almost all $t \in [0,T]$.
(H2) There exists $m \in L^1([0,T],\mathbb{R}^+)$ such that, for almost all $t \in [0,T]$ and all $x \in \mathbb{R}$, $\|F(t,x)\| \leq m(t)$.
(H3) $\phi :]-a,a[\to \mathbb{R}$ is an increasing homeomorphism, where $0 < a < +\infty$.

We define the space $W_a^{2,1}([0,T],\mathbb{R})$ as the set of all functions $x \in C^1([0,T],\mathbb{R})$ for which the condition $\|x'\|_\infty < a$ holds, $\phi(x')$ is absolutely continuous.

In the following sections, we will use the following important lemmas.

Lemma 2.2. [7] *If assumptions (H1) and (H2) are satisfied, then F is of the lower semi-continuous type.*

Lemma 2.3. [8] *Assume that (H3) holds. Let $v, w \in W_a^{2,1}([0,T],\mathbb{R})$ be such that*

$$(\phi(v'(t)))' \leq (\phi(w'(t)))' \quad a.e. \text{ on } \{t \in [0,T] : v(t) < w(t)\}.$$

If one of the following conditions holds:
  (a) $v(0) \geq w(0)$, $v(T) \geq w(T)$,
  (b) $\phi(v'(0)) \leq \phi(w'(0))$, $\phi(v'(T)) \geq \phi(w'(T))$,
  (c) $v(0) = v(T)$, $w(0) = w(T)$, $\phi(v'(0)) - \phi(v'(T)) \leq \phi(w'(0)) - \phi(w'(T))$,

then $v(t) \geq w(t)$ for all $t \in [0,T]$, or there exists $c > 0$ such that $v(t) = w(t) - c$ for all $t \in [0,T]$.

We will now present some definitions and basic results related to fixed point index theory. Let X and Y be two spaces and $A \subset X$. A function $f : A \to Y$ is said to be extendable over X if there exists a function $f^* : X \to Y$ such that $f^* = f|_A$.

Definition 2.4.
- A space Y is said to be an absolute retract (AR) if Y is metrizable and for any metrizable X and every closed $A \subset X$, each continuous $f : A \to Y$ is extendable over X.
- A space Y is said to be an absolute neighborhood retract (ANR) if Y is metrizable and for any metrizable X and every closed $A \subset X$, each continuous $f : A \to Y$ is extendable over some neighborhood U of A.

The class of all absolute retracts is represented by AR, while the class of all absolute neighborhood retracts is represented by ANR. It is evident that AR is a subset of ANR. For convex set, we have the following result.

Lemma 2.4. [9] *Let C be a convex subset of a locally convex linear topological space. If C is metrizable, then C is an AR. In particular C is an ANR.*

Now, let X be an ANR and U open in X. By $\mathcal{K}(\overline{U}, X)$ we denote the set of all compact maps from $\overline{U}$ to X, and by $\mathcal{K}_{\partial U}(\overline{U}, X)$ the set of all maps $f \in \mathcal{K}(\overline{U}, X)$ that have no fixed points on ∂U. We say that f is an admissible map from $\overline{U}$ to X if $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$. The set of all fixed points of f is denoted by $\mathrm{Fix}(f)$. A homotopy is a parametrized family $\{h_t : X \to Y\}$, where X and Y are two spaces, of maps indexed by $t \in [0,1]$ such that the map $h : X \times [0,1] \to Y$ given by $h(x,t) = h_t(x)$ is continuous. We say that an homotopy $h_t : X \to Y$ is compact if the map $h : X \times [0,1] \to Y$ is compact. We are, now, ready to give the definition of the fixed point index and their properties.

Theorem 2.1. [9] *Let X be an ANR and $U \subset X$ an arbitrary open subset. Then there exists an integer-valued fixed point index function*

$$f \mapsto i(f, U) := index(f, U)$$

for $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$ with the following properties:

(i) *(Normalization): If $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$ is a constant map $u \mapsto u_0$, then $i(f, U) = 1$ or 0 depending on whether or not $u_0 \in U$.*

(ii) *(Additivity): If $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$ and $\mathrm{Fix}(f) \subset U_1 \cup U_2 \subset U$ with U_1, U_2 are open and disjoint, then*

$$i(f, U) = i(f, U_1) + i(f, U_2).$$

(iii) *(Homotopy): If $h_t : \overline{U} \to X$, $t \in [0,1]$, is an admissible compact homotopy in $\mathcal{K}_{\partial U}(\overline{U}, X)$, then $i(h_0, U) = i(h_1, U)$.*

(iv) *(Existence): If $i(f, U) \neq 0$, then $\mathrm{Fix}(f) \neq \emptyset$.*

(v) *(Excision): If V is an open subset of U and if $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$ has no fixed points in $U \setminus V$, where $U \setminus V$ is the relative complement of V in U, then $i(f, U) = i(f, V)$.*

(vi) *(Multiplicativity): If $f_1 \in \mathcal{K}_{\partial U_1}\left(\overline{U}_1, X_1\right)$ and $f_2 \in \mathcal{K}_{\partial U_2}\left(\overline{U}_2, X_2\right)$, then $f_1 \times f_2 \in \mathcal{K}_{\partial(U_1 \times U_2)}\left(\overline{U}_1 \times \overline{U}_2, X_1 \times X_2\right)$ and*

$$i(f_1 \times f_2, U_1 \times U_2) = i(f_1, U_1) \cdot i(f_2, U_2).$$

(vii) *(Commutativity): Let X, X' be ANRs, $U \subset X$, $U' \subset X'$ be open and $f : \overline{U} \to X'$ and $g : \overline{U'} \to X$ be continuous maps, at least one of them being compact. Define $V = U \cap f^{-1}(U')$ and $V' = U' \cap g^{-1}(U)$. Then*
   (a) *the maps $g \circ f : \overline{V} \to X$ and $f \circ g : \overline{V'} \to X'$ are compact,*
   (b) *if $\mathrm{Fix}(g \circ f) \subset V$ and $\mathrm{Fix}(f \circ g) \subset V'$, then $i(g \circ f, V) = i(f \circ g, V')$.*

(viii) *(Contraction): Let (X, A) be a pair of ANRs with A closed in X, $U \subset X$ open, and $f \in \mathcal{K}_{\partial U}(\overline{U}, X)$ with $f(\overline{U}) \subset A$. Let $\widehat{f} = f_{\overline{U \cap A}} : \overline{U \cap A} \to A$ be the contraction of f. Then*

$$\widehat{f} \in \mathcal{K}_{\partial(U \cap A)}(\overline{U \cap A}, A) \text{ and } i(f, U) = i(\widehat{f}, U \cap A).$$

Next, let X be an ANR and $J = [a, b] \subset \mathbb{R}$. For a subset M of $X \times J$, the t-slice of M, denoted as M_t, is defined for each $t \in J$ by the set

$$M_t = \{x \in X \mid (x, t) \in M\}.$$

Consider an open set $U \subset X \times J$ with both U_a and U_b being nonempty. The vertical boundary of U is defined as

$$\partial U \setminus [(U \cap (X \times \{a\})) \cup (U \cap (X \times \{b\}))]$$

and is denoted by $\widehat{\partial} U$. If $f : \overline{U} \to X$ is a map, we define the set

$$S_U = \{(x, t) \in \overline{U} \mid f(x, t) = x\}.$$

For each $t \in J$, the function $f_t : \overline{U}_t \to X$ is defined by $f_t(x) = f(x, t)$.

Theorem 2.2. [9] *Let X be an ANR and $U \subset X \times [a, b]$ be open. If $f : \overline{U} \to X$ is a compact map such that $S_U \cap \widehat{\partial} U = \emptyset$, then $s \mapsto i(f_s, U_s)$ is a constant function on $[a, b]$.*

Consider the function $\operatorname{sgn} : \mathbb{R}^* \to \{-1, 1\}$ defined by $\operatorname{sgn}(x) = 1$ if $x > 0$ and $\operatorname{sgn}(x) = -1$ if $x < 0$.

Proposition 2.1. [9] *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(a) \neq a$ and $f(b) \neq b$, where $a < b$. Then*

$$i(f,]a, b[) = \frac{1}{2} \operatorname{sgn}[b - f(b)] - \frac{1}{2} \operatorname{sgn}[a - f(a)].$$

To give our result, we need the following notions.

Definition 2.5.
- A function $\alpha \in W_a^{2,1}([0, T], \mathbb{R})$ is called a lower solution of (1.1), if there exists $v_1 \in L^1([0, T], \mathbb{R})$ such that $v_1(t) \in F(t, \alpha(t))$, $(\phi(\alpha'(t)))' \geq v_1(t)$ a.e. $t \in [0, T]$, $\alpha'(0) \geq r$ and $\alpha'(T) \leq r'$.
- A function $\beta \in W_a^{2,1}([0, T], \mathbb{R})$ is said an upper solution of (1.1), if there exists $v_2 \in L^1([0, T], \mathbb{R})$ such that $v_2(t) \in F(t, \beta(t))$, $(\phi(\beta'(t)))' \leq v_2(t)$ a.e. $t \in [0, T]$, $\beta'(0) \leq r$ and $\beta'(T) \geq r'$.
- A function $x : [0, T] \to \mathbb{R}$ is a solution of (1.1) if $x \in W_a^{2,1}([0, T], \mathbb{R})$ and x satisfies the conditions of (1.1).

In the sequel, we will prove the following theorem which is the main result of this paper.

Theorem 2.3. *If assumptions (H1), (H2) and (H3) are satisfied, and if $(r, r') \in]-a, a[^2$, and the problem (1.1) has a lower solution α and an upper solution β with $\alpha \leq \beta$, then the problem (1.1) has at least one solution x such that $\alpha \leq x \leq \beta$ on $[0, T]$.*

3. Proof of Theorem 2.3

Consider first the set-valued map $\overline{F}$ defined by

$$\overline{F}(t, x) = F(t, h(t, x)) \cap \Gamma(t, x), \text{ for all } (t, x) \in [0, T] \times \mathbb{R},$$

where

$$h(t, x) = \begin{cases} \alpha(t) & \text{if } x < \alpha(t), \\ x & \text{if } \alpha(t) \leq x \leq \beta(t), \\ \beta(t) & \text{if } x > \beta(t), \end{cases}$$

and

$$\Gamma(t,x)=\begin{cases}[(\phi(\beta'(t)))',+\infty[& \text{if } x\geq\beta(t),\\ \mathbb{R} & \text{if } \alpha(t)<x<\beta(t),\\ ]-\infty,(\phi(\alpha'(t)))'] & \text{if } x\leq\alpha(t).\end{cases}$$

The set-valued map $x\mapsto\overline{F}(t,x)$ is lower semicontinuous (see [1]) and $\overline{F}$ satisfies all the other assumptions imposed on F. Therefore, by Lemma 2.2, $\overline{F}$ is of the lower semi-continuous type. Furthermore, by Lemma 2.1, there exists a continuous function $g:\mathcal{C}([0,T],\mathbb{R})\to L^1([0,T],\mathbb{R})$ such that $g(y)\in\mathcal{F}(y)$ for all $y\in\mathcal{C}([0,T],\mathbb{R})$, where $\mathcal{F}$ is the Niemytzki operator associated with $\overline{F}$. Next, consider the problem

$$\begin{cases}(\phi(y'(t)))'=g(y)(t)\ \text{ a.e. on } [0,T];\\ y'(0)=r,\ y'(T)=r'.\end{cases}\tag{3.1}$$

Since $g(y)(t)\in\overline{F}(t,y(t))$ a.e. on $[0,T]$, any solution to the problem (3.1) also serves as a solution to the following problem

$$\begin{cases}(\phi(y'(t)))'\in\overline{F}(t,y(t))\ \text{ a.e. on } [0,T];\\ y'(0)=r,\ y'(T)=r'.\end{cases}\tag{3.2}$$

Furthermore, any solution y of (3.2) that satisfies $\alpha\leq y\leq\beta$ on $[0,T]$ is also a solution of (1.1). Now, for every $\lambda\in[0,1]$, let us consider the following modified problem

$$\begin{cases}(\phi(y'(t)))'=\lambda\overline{g}(y)(t)+\dfrac{1-\lambda}{T}\displaystyle\int_0^T\overline{g}(y)(s)ds\ \text{ a.e. on } [0,T];\\ y'(0)=r,\ y'(T)=r',\end{cases}\tag{3.3}$$

where $\overline{g}:\mathcal{C}([0,T],\mathbb{R})\to L^1([0,T],\mathbb{R})$ is defined by

$$\overline{g}(y)(t)\ =\ \begin{cases}g(\beta)(t)-M_\beta(t)(y(t)-\beta(t)) & \text{if } y(t)>\beta(t),\\ g(y)(t) & \text{if } \alpha(t)\leq y(t)\leq\beta(t),\\ g(\alpha)(t)+m_\alpha(t)(y(t)-\alpha(t)) & \text{if } y(t)<\alpha(t),\end{cases}$$

with $m_\alpha,\ M_\beta\in L^1([0,T],\mathbb{R})$ such that

$$m_\alpha(t)>\max\Big\{0,g(\alpha)(t)-\frac{1}{T}\big(\phi(r')-\phi(r)\big)\Big\}$$

and

$$M_\beta(t)<\min\Big\{0,g(\beta)(t)-\frac{1}{T}\big(\phi(r')-\phi(r)\big)\Big\}.$$

Let $w\in\mathcal{C}^1([0,T],\mathbb{R})$ and $v\in W_a^{2,1}([0,T],\mathbb{R})$ be defined for all $t\in[0,T]$ by

$$w(t)=\phi(r)+\frac{t}{T}(\phi(r')-\phi(r))\quad\text{and}\quad v(t)=\int_0^t\phi^{-1}\big(w(s)\big)ds.$$

Let us consider the operators $N_{\overline{g}}:\mathcal{C}([0,T],\mathbb{R})\to\mathcal{C}([0,T],\mathbb{R})$ and

$$\mathcal{H}:[0,1]\times\mathcal{C}([0,T],\mathbb{R})\to\mathcal{C}([0,T],\mathbb{R})$$

defined, for all $(t,\lambda,u)\in[0,T]\times[0,1]\times\mathcal{C}([0,T],\mathbb{R})$, by

$$N_{\overline{g}}(u)(t)=\int_0^t\overline{g}(u)(s)ds$$

and

$$\begin{aligned}\mathcal{H}(\lambda,u)(t) &= u(0)+\frac{1}{T}\big(\phi(r)-\phi(r')+N_{\overline{g}}(u)(T)\big)\\ &\quad+\int_0^t \phi^{-1}\big(w(s)+\lambda(N_{\overline{g}}(u)(s)-\frac{s}{T}N_{\overline{g}}(u)(T))\big)ds.\end{aligned}$$

Next, under the assumptions (H1), (H2) and (H3), the function $\mathcal{H}$ is both continuous and completely continuous. Moreover, note that the fixed points of $\mathcal{H}(\lambda,\cdot)$ correspond to the solutions of (3.3). Indeed, if $u=\mathcal{H}(\lambda,u)$, then, for all $t\in[0,T]$, we have

$$\begin{aligned}u(t) &= u(0)+\frac{1}{T}\big(\phi(r)-\phi(r')+N_{\overline{g}}(u)(T)\big)\\ &\quad+\int_0^t \phi^{-1}\big(w(s)+\lambda(N_{\overline{g}}(u)(s)-\frac{s}{T}N_{\overline{g}}(u)(T))\big)ds.\end{aligned}$$

Thus, in particular for $t=0$ one has $u(0)=u(0)+\frac{1}{T}\left(\phi(r)-\phi(r')+N_{\overline{g}}(u)(T)\right)$. So $\phi(r')-\phi(r)=N_{\overline{g}}(u)(T)$. Also by derivation, we get

$$u'(t)=\phi^{-1}\big(w(t)+\lambda(N_{\overline{g}}(u)(t)-\frac{t}{T}N_{\overline{g}}(u)(T))\big) \text{ for almost all } t\in[0,T],$$

which gives

$$(\phi(u'(t)))'=\frac{1}{T}(\phi(r')-\phi(r))+\lambda(\overline{g}(u)(t)-\frac{1}{T}N_{\overline{g}}(u)(T)) \text{ for almost all } t\in[0,T].$$

Hence

$$(\phi(u'(t)))'=\lambda\overline{g}(u)(t)+\frac{(1-\lambda)}{T}\int_0^T \overline{g}(u)(s)ds \text{ for almost all } t\in[0,T].$$

Moreover, $u'(0)=\phi^{-1}(\phi(r))=r$ and $u'(T)=\phi^{-1}(\phi(r'))=r'$.

Now, fix $\widetilde{R}>0$ such that

$$\widetilde{R}>\max\{v(t)-\min_{s\in[0,T]}\alpha(s)+1,-v(t)+\max_{s\in[0,T]}\beta(s)+1\},\quad \forall t\in[0,T].$$

For every $u\in W_a^{2,1}([0,T],\mathbb{R})$, one has a.e on $\left\{t\in[0,T]: u(t)<v(t)-\widetilde{R}\right\}$

$$\overline{g}(u)(t)=g(\alpha)(t)+m_\alpha(t)(u(t)-\alpha(t))<g(\alpha)(t)-m_\alpha(t)<\frac{\phi(r')-\phi(r)}{T}. \tag{3.4}$$

Similarly,

$$\overline{g}(u)(t)>\frac{\phi(r')-\phi(r)}{T} \quad \text{a.e. on } \left\{t\in[0,T]: u(t)>v(t)+\widetilde{R}\right\}. \tag{3.5}$$

$\|u-v\|_\infty\le\widetilde{R}$ for any solution u of (3.3).

Proof. Let u be a solution of (3.3). One has

$$\begin{aligned}
\phi(r') - \phi(r) &= \phi(u'(T)) - \phi(u'(0)) \\
&= \int_0^T (\phi(u'(s)))' ds \\
&= \int_0^T \left(\lambda \overline{g}(u)(s) + \frac{(1-\lambda)}{T} N_{\overline{g}}(u)(T) \right) ds \\
&= \lambda \int_0^T \overline{g}(u)(s) ds + \frac{T\,(1-\lambda)}{T} N_{\overline{g}}(u)(T) \\
&= \lambda N_{\overline{g}}(u)(T) + (1-\lambda)\, N_{\overline{g}}(u)(T).
\end{aligned}$$

Then

$$\phi(r') - \phi(r) = N_{\overline{g}}(u)(T) = \int_0^T \overline{g}(u)(s) ds. \tag{3.6}$$

Combining (3.4) and (3.6), we deduce that a.e on $\left\{ t \in [0,T] : u(t) < v(t) - \widetilde{R} \right\}$

$$\begin{aligned}
(\phi(u'(t)))' &= \lambda \overline{g}(u)(t) + \frac{(1-\lambda)}{T} \big(\phi(r') - \phi(r) \big) \\
&\leq \frac{\lambda}{T} \big(\phi(r') - \phi(r) \big) + \frac{(1-\lambda)}{T} \big(\phi(r') - \phi(r) \big) \\
&= (\phi(v'(t)))'.
\end{aligned}$$

Similarly,

$$(\phi(u'(t)))' \geq (\phi(v'(t)))' \text{ a.e. on } \left\{ t \in [0,T] : u(t) > v(t) + \widetilde{R} \right\}.$$

Moreover $\phi(u'(0)) = \phi(r) = \phi(v'(0))$ and $\phi(u'(T)) = \phi(r') = \phi(v'(T))$. It follows from Lemma 2.3 that $\|u - v\|_\infty \leq \widetilde{R}$, or there exists $c > 0$ such that

$$|u(t) - v(t)| = \widetilde{R} + c$$

for all $t \in [0,T]$. If $u(t) = v(t) + \widetilde{R} + c$, for all $t \in [0,T]$, then

$$\begin{aligned}
\phi(r') - \phi(r) &= \int_0^T \overline{g}(u)(s) ds \\
&= \int_0^T \left(g(\beta)(s) - M_\beta(s)(v(s) + \widetilde{R} + c - \beta(s)) \right) ds \\
&> \int_0^T g(\beta)(s) ds \\
&\geq \int_0^T (\phi(\beta'(s)))' ds \\
&= \phi(\beta'(T)) - \phi(\beta'(0)) \\
&\geq \phi(r') - \phi(r).
\end{aligned}$$

This is a contradiction. Similarly, we cannot have $u(t) = v(t) - \widetilde{R} - c$, for all $t \in [0,T]$. Hence $\|u - v\|_\infty \leq \widetilde{R}$. □

Now, let $R > \widetilde{R}$ and $\mathcal{U} = \Big\{u \in \mathcal{C}([0,T],\mathbb{R}) : \|u - v\|_\infty < R\Big\}$. By Claim 3, we get $u \neq \mathcal{H}(\lambda, u)$ for all $(\lambda, u) \in [0,1] \times \partial\mathcal{U}$. By the general homotopy invariance property of the fixed point index,

$$i(\mathcal{H}(\lambda,\cdot),\mathcal{U}) = i(\mathcal{H}(0,\cdot),\mathcal{U}), \quad \forall\lambda \in [0,1]. \tag{3.7}$$

Observe that

$$\mathcal{H}(0,u) = u(0) + \frac{1}{T}\left(\phi(r) - \phi(r') + N_{\overline{g}}(u)(T)\right) + v \in \mathbb{R} + v.$$

Let $\mathcal{C} = \{u = l + v : l \in \mathbb{R}\} \subset \mathcal{C}([0,T],\mathbb{R})$. Notice that $\mathcal{C}$ is a convex subset of a locally convex linear topological space. In view of Lemma 2.4, $\mathcal{C}$ is an ANR. By the contraction property of the fixed point index,

$$i(\mathcal{H}(0,\cdot),\mathcal{U}) = i(\mathcal{H}(0,\cdot),\mathcal{U} \cap \mathcal{C}). \tag{3.8}$$

One has $\partial(\mathcal{U} \cap \mathcal{C}) = \{-R + v, R + v\}$. Consider $\mathcal{N} : [-R,R] \to \mathbb{R}$ defined by $\mathcal{N}(c) = \mathcal{H}(0, c + v) - v$. By (3.5), one has

$$\mathcal{N}(R) = \mathcal{H}(0,R+v) - v = R + \frac{1}{T}\left(\phi(r) - \phi(r') + \int_0^T \overline{g}(R+v)(s)ds\right) > R.$$

Similarly, $\mathcal{N}(-R) < -R$. Now, let us consider the maps $f_1 : \overline{\mathcal{U} \cap \mathcal{C}} \to [-R,R]$ and $g_1 : [-R,R] \to \mathcal{C}$ defined by $f_1(u) = u - v$, $\forall u \in \overline{\mathcal{U} \cap \mathcal{C}}$ and $g_1(c) = H(0, c+v)$, $\forall c \in [-R,R]$. Observe that,

$$f_1 \circ g_1(c) = \mathcal{N}(c), \quad \forall c \in [-R,R] \text{ and } g_1 \circ f_1(u) = \mathcal{H}(0,u), \quad \forall u \in \overline{\mathcal{U} \cap \mathcal{C}}.$$

Let $V = \mathcal{U} \cap \mathcal{C} \cap f_1^{-1}(]-R,R[)$ and $V' =]-R,R\left[\cap g_1^{-1}(\mathcal{U} \cap \mathcal{C})\right.$. It is clear that f_1 and g_1 are continuous maps and g_1 is a compact map. Then by the commutativity property of fixed point index,

$$i(g_1 \circ f_1, V) = i(f_1 \circ g_1, V'). \tag{3.9}$$

One has

$$i(\mathcal{H}(0,\cdot),\mathcal{U} \cap \mathcal{C}) = i(g_1 \circ f_1, \mathcal{U} \cap \mathcal{C}). \tag{3.10}$$

Since $\mathrm{Fix}(g_1 \circ f_1) \subset V$, $g_1 \circ f_1$ has no fixed points on $(\mathcal{U} \cap \mathcal{C}) \setminus V$. By the excision property of fixed point index,

$$i(g_1 \circ f_1, \mathcal{U} \cap \mathcal{C}) = i(g_1 \circ f_1, V). \tag{3.11}$$

Similarly,

$$i(f_1 \circ g_1, V') = i(f_1 \circ g_1,]-R,R[). \tag{3.12}$$

On the other hand

$$i(f_1 \circ g_1,]-R,R[) = i(\mathcal{N},]-R,R[). \tag{3.13}$$

Combining (3.8), (3.9), (3.10), (3.11), (3.12) and (3.13), we get

$$i(\mathcal{H}(0,\cdot),\mathcal{U}) = i(\mathcal{N},]-R,R[) = \frac{1}{2}\operatorname{sgn}[R - \mathcal{N}(R)] - \frac{1}{2}\operatorname{sgn}[-R - \mathcal{N}(-R)] = -1.$$

By (3.7), we deduce that, for every $\lambda \in [0,1]$, $i(\mathcal{H}(\lambda,\cdot),\mathcal{U}) = i(\mathcal{H}(0,\cdot),\mathcal{U}) = -1$. Thus, $\mathcal{H}(\lambda,\cdot)$ has a fixed point, and hence (3.3) has a solution.

In particular, there exists a solution $u \in W_a^{2,1}\big([0,T],\mathbb{R}\big)$ of (3.3) for $\lambda = 1$. To conclude, we have to show that $\alpha(t) \leq u(t) \leq \beta(t)$ for all $t \in [0,T]$. For almost every $t \in \{t \in [0,T] : u(t) < \alpha(t)\}$

$$(\phi(u'(t)))' = g(\alpha)(t) + m_\alpha(t)(u(t) - \alpha(t)) < g(\alpha)(t) \leq (\phi(\alpha'(t)))'.$$

In addition, we have $\phi(u'(T)) = \phi(r') \geq \phi(\alpha'(T))$ and $\phi(u'(0)) = \phi(r) \leq \phi(\alpha'(0))$. It follows, from Lemma 2.3, that $u(t) \geq \alpha(t)$ for all $t \in [0,T]$, or there exists $c > 0$ such that $u(t) = \alpha(t) - c$ for all $t \in [0,T]$. In this case, $u(t) < \alpha(t)$ for all $t \in [0,T]$ and for $t \in [0,T]$, one has

$$g(\alpha)(t) \leq (\phi(\alpha'(t)))' = (\phi(u'(t)))' = g(\alpha)(t) + m_\alpha(t)(u(t) - \alpha(t)) < g(\alpha)(t).$$

This is a contradiction. As consequence, we have $\alpha(t) \leq u(t)$ for all $t \in [0,T]$. A similar argument yields $u(t) \leq \beta(t)$ for all $t \in [0,T]$.

4. Illustrative example

Consider the following problem

$$\begin{cases} \big(x'(t)^3\big)' \in F(t,x(t)), & \text{a.e. on } [0,\pi]; \\ x'(0) = 0, \ x'(\pi) = 0; \end{cases} \tag{4.1}$$

where $F : [0,\pi] \times \mathbb{R} \to 2^{\mathbb{R}}$ is defined by $F(t,x) = [f_1(t,x), f_2(t,x)]$ with

$$f_1, f_2 : [0,\pi] \times \mathbb{R} \to \mathbb{R}$$

are defined by

$$f_1(t,x) = \frac{2h(t)}{1+x^2} - 3 \text{ and } f_2(t,x) = \frac{2h(t)}{1+x^2} + 3, \quad \forall (t,x) \in [0,\pi] \times \mathbb{R},$$

and $h : [0,\pi] \to \mathbb{R}$ is defined by $h(t) = \frac{-3}{2}\cos(t)\sin^2(t)(1+\cos^2(t))$, for each $t \in [0,\pi]$. In this example $\phi(x) = x^3$.

It is clear that F has nonempty compact values and is measurable and $F(t,.)$ is lower semi-continuous on $\mathbb{R}$. Moreover, for almost all $t \in [0,\pi]$ and all $x \in \mathbb{R}$

$$\begin{aligned} \big\|F(t,x)\big\| &= \sup\Big\{|y| : y \in [f_1(t,x), f_2(t,x)]\Big\} \\ &\leq max\,\{|f_1(t,x)|, |f_2(t,x)|\} \\ &\leq 6\sin^2(t) + 3 \end{aligned}$$

For all $t \in [0,\pi]$, set $\alpha(t) = -1$ and $\beta(t) = 1$. Let v_1 be such that, $v_1(t) = 0$, for all $t \in [0,\pi]$. Clearly, one has $|h(t)| \leq 3$, for all $t \in [0,\pi]$, then

$$v_1(t) = 0 \in [h(t) - 3, h(t) + 3] = F(t,\alpha(t)), \quad \forall t \in [0,\pi].$$

In addition, $\phi(\alpha'(0)) \geq \phi(0)$ and $\phi(\alpha'(\pi)) \leq \phi(0)$. Then α is a lower solution of (4.1). Similarly β is an upper solution of (4.1). Also, observe that $\alpha \leq \beta$ on $[0,\pi]$. We conclude that all assumptions of Theorem 2.3 are verified, then the problem (4.1) has at least one solution x such that $\alpha \leq x \leq \beta$ on $[0,\pi]$. For example the function u defined by $u(t) = \cos(t), \forall t \in [0,\pi]$, is a solution of the problem (4.1).

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