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ON τ-INEQUALITIES AND EQUATIONS EMERGING FROM SQUARE ROOT MAPPINGS

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Abstract. This work aims to present equations and τ-inequalities resulting from square root mappings. The noteworthy stability results related to various inequalities and equations with additive, quadratic, cubic and quartic mappings as approximate solutions motivated us to introduce novel form of functional inequalities and equations involved with radical arguments. We prove the stabilities of the inequalities and equations dealt in this study through Hyers' method (direct method) in the domain of non-negative real numbers.

1. Introduction

The approximation of various mappings emanated through an inquisitive query raised in [13] pertinent to homomorphisms occurring in group theory. The first brilliant reply to this query was presented in [6] in the setting of Banach spaces. Further, this stability problem was dealt in various directions in [1, 11] by considering the unbounded Cauchy difference. Moreover, a generalized form of the stability result was discussed in [3]. The solutions and non-Archimedean stabilities of ρ-inequalities and equations arising from linear mappings have been studied in [8, 9] and their non-Archimedean 2-normed stabilities in [14]. Under the setting of complex Banach spaces, the analytic solutions and pertinent stabilities of the ρ-functional inequalities and equations arising from quadratic mapping have been established in [7].

It was proved in [4] that if a mapping φ satisfies the inequality

$$\left\|2\varphi(a) + 2\varphi(b) - \varphi(ab^{-1})\right\| \leq \|\varphi(ab)\| \tag{1.1}$$

then φ satisfies the equation $2\varphi(a) + 2\varphi(b) = \varphi(ab) + \varphi(ab^{-1})$. Further, the stability results of inequality (1.1) have been obtained in [2, 5]. The stabilities of Cauchy-Jensen kind additive ρ-functional inequalities have been studied in [12]. The ρ-inequalities arising from cubic and quartic functions were dealt in [10] to prove their Ulam stabilities.

The interesting and significant results associated with several inequalities

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and equations arising from additive, quadratic, cubic and quartic mappings motivated us to introduce the following τ-inequalities

$$\left|\varphi(a+b+2\sqrt{ab})+\varphi(a+b-2\sqrt{ab})-2\varphi(a)\right| \leq \left|\tau\left(4\varphi\left(\frac{a+b+2\sqrt{ab}}{4}\right)+4\varphi\left(\frac{a+b-2\sqrt{ab}}{4}\right)-4\varphi(a)\right)\right| \quad (1.2)$$

and

$$\left|4\varphi\left(\frac{a+b+2\sqrt{ab}}{4}\right)+4\varphi\left(\frac{a+b-2\sqrt{ab}}{4}\right)-4\varphi(a)\right| \leq \left|\tau\left(\varphi(a+b+2\sqrt{ab})+\varphi(a+b-2\sqrt{ab})-2\varphi(a)\right)\right| \quad (1.3)$$

where τ is a fixed real number with the condition $|\tau| < \frac{1}{2}$. We find the solution of the above inequalities (1.2) and (1.3). Further, we also discuss the stability problems of the functional inequalities (1.2) and (1.3) by taking the domain as $\mathbb{R}^+ \cup \{0\}$.

In our entire investigation, we assume that $\mathbb{R}$ is the space of real numbers and $A = \mathbb{R}^+ \cup \{0\}$. To perform the computations in a simpler manner, let us denote

$$D_1\varphi(a,b) = \varphi(a+b+2\sqrt{ab})+\varphi(a+b-2\sqrt{ab})-2\varphi(a)$$

and

$$D_2\varphi(a,b) = 4\varphi\left(\frac{a+b+2\sqrt{ab}}{4}\right)+4\varphi\left(\frac{a+b-2\sqrt{ab}}{4}\right)-4\varphi(a).$$

Then the inequalities (1.2) and (1.3), respectively can be written as

$$|D_1\varphi(a,b)| \leq |\tau\left(D_2\varphi(a,b)\right)|$$

and

$$|D_2\varphi(a,b)| \leq |\tau\left(D_1\varphi(a,b)\right)|.$$

In the next sections, we shall assume that $\varphi : A \longrightarrow \mathbb{R}$ is a function.

2. Function satisfying inequality (1.2)

Here, we find the function which satisfies the inequality (1.2). We begin with the following definition.

Definition 2.1. Suppose a function φ satisfies the following equation

$$\varphi(a+b+2\sqrt{ab})+\varphi(a+b-2\sqrt{ab}) = 2\varphi(a) \quad (2.1)$$

for all $a, b \in A$. Then φ is called a square root function. Also, we call equation (2.1) as a square root functional equation.

Remark 2.1. The function φ satisfying equation (2.1) is given by $\varphi(a) = \sqrt{a}$, for all $a \in A$

Theorem 2.1. *A function φ with $\varphi(0) = 0$ is a square root function if and only if it satisfies the inequality* (1.2).

Proof. Firstly, let us show the necessary part. Suppose φ satisfies (1.2). Letting $b = a$ in (1.2), we get

$$|\varphi(4a) - 2\varphi(a)| \leq 0$$

and therefore it leads to

$$\varphi\left(\frac{a}{4}\right) = \frac{1}{2}\varphi(a) \tag{2.2}$$

for all $a \in A$. The inequality (1.2) together with (2.2) produces

$$\begin{aligned}|D_1\varphi(a,b)| &\leq |\tau\left(D_2\varphi(a,b)\right)| \\ &\leq 2|\tau|\,|D_1\varphi(a,b)|\end{aligned}$$

which induces

$$D_1\varphi(a,b) = 0.$$

Hence φ is a square root function.

The sufficient part of proof is obviously true. □

Corollary 2.1. *A function φ is a square root function if and only if it is a solution of the ensuing equation*

$$D_1\varphi(a,b) = \tau\left(D_2\varphi(a,b)\right) \tag{2.3}$$

for all $a, b \in A$.

Remark 2.2. The equation (2.3) is termed to be a square root τ-functional equation.

3. Function satisfying inequality (1.3)

In the following results, we find the function which satisfies the inequality (1.3).

Theorem 3.1. *A function φ is a square root function if and only if it satisfies the inequality* (1.3)*.*

Proof. Firstly, let us show the necessary part of the proof. For this, suppose φ satisfies the inequality (1.3). Letting $a = b = 0$ in (1.3), we find that

$$4\,|\varphi(0)| \leq 0$$

which produces that $\varphi(0) = 0$. Now, replacing b by 0 in (1.3), we obtain that

$$\left|8\varphi\left(\frac{a}{4}\right) - 4\varphi(a)\right| \leq 0$$

which turns out to be

$$\varphi\left(\frac{a}{4}\right) = \frac{1}{2}\varphi(a) \tag{3.1}$$

for all $a \in A$. On applying the outcome from equation (3.1) in (1.3), we get

$$\begin{aligned}2\,|D_1\varphi(a,b)| &= |D_2\varphi(a,b)| \\ &\leq |\tau|\,|D_1\varphi(a,b)|\end{aligned}$$

which ends up with

$$D_1\varphi(a,b) = 0$$

and hence equation (2.1) is arrived.

The sufficient part of proof is obviously true. □

Corollary 3.1. *A function φ is a square root function if and only if it is a solution of the equation*

$$D_2\varphi(a,b) = \tau\left(D_1\varphi(a,b)\right)$$

for all $a, b \in A$.

4. Solution to the stability problems of inequality (1.2)

In the following theorem, we find the solution to the stability problem of (1.2).

Theorem 4.1. *Suppose a function $\zeta : A \times A \longrightarrow [0,\infty)$ satisfies*

$$\sum_{\beta=0}^{\infty} \frac{1}{2^\beta}\zeta\left(4^\beta a, 4^\beta b\right) < \infty \tag{4.1}$$

for all $a, b \in A$. If a function φ with $\varphi(0) = 0$ satisfies the following inequality

$$|D_1\varphi(a,b)| \leq |\tau\left(D_2\varphi(a,b)\right)| + \zeta(a,b) \tag{4.2}$$

for all $a, b \in A$, then a unique approximate square root function $\Phi : A \longrightarrow \mathbb{R}$ exists and satisfies (1.2) with the condition that

$$|\varphi(a) - \Phi(a)| \leq \frac{1}{2}\sum_{\beta=0}^{\infty} \frac{1}{2^\beta}\zeta\left(4^\beta a, 4^\beta a\right) \tag{4.3}$$

for all $a \in A$.

Proof. Firstly, we determine that there exists a square root function Φ which satisfies (4.3). To achieve this, let us replace a with b in (4.2), we obtain

$$\left|\frac{1}{2}\varphi(4a) - \varphi(a)\right| \leq \frac{1}{2}\zeta(a,a)$$

for all $a \in A$. Suppose λ, μ are positive integers with $\mu > \lambda$. Then, we have

$$\begin{aligned}\left|\frac{1}{2^\lambda}\varphi(4^\beta a) - \frac{1}{2^\mu}\varphi(4^\mu a)\right| &\leq \sum_{\beta=\lambda}^{\mu-1}\left|\frac{1}{2^\beta}\varphi(4^\beta a) - \frac{1}{2^{\beta+1}}\varphi\left(4^{\beta+1}a\right)\right| \\ &\leq \frac{1}{2}\sum_{\beta=\lambda}^{\mu-1}\frac{1}{2^\beta}\zeta(4^\beta a, 4^\beta a) \end{aligned}\tag{4.4}$$

for all $a \in A$. By the reasoning of the description of ζ in (4.1) and utilizing (4.4), the sequence $\left\{\frac{1}{2^\beta}\varphi(4^\beta a)\right\}$ ends up as Cauchy. Since $\mathbb{R}$ is complete, this sequence converges to a function $\Phi : A \longrightarrow \mathbb{R}$ given by $\Phi(a) = \lim_{\beta\to\infty} \frac{1}{2^\beta}\varphi(4^\beta a)$, for each $a \in A$. Furthermore, letting $\lambda = 0$ and allowing $\mu \to \infty$ in (4.4), we arrive at

(4.3). Using (4.2), we have

$$\begin{aligned}
&|D_1\Phi(a,b)| \\
&= \lim_{\beta\to\infty}\frac{1}{2^\beta}\left|\varphi\left(4^\beta(a+b+2\sqrt{ab})\right)+\varphi\left(4^\beta(a+b-2\sqrt{ab})\right)-2\varphi(4^\beta a)\right| \\
&\leq \lim_{\beta\to\infty}\frac{1}{2^\beta}|\tau|\left|4\varphi\left(4^\beta\left(\frac{a+b+2\sqrt{ab}}{4}\right)\right)+4\varphi\left(4^\beta\left(\frac{a+b-2\sqrt{ab}}{4}\right)\right)-4\varphi(4^\beta a)\right| \\
&\qquad + \lim_{\beta\to\infty}\frac{1}{2^\beta}\zeta(4^\beta a, 4^\beta b) \\
&= |\tau|\,|D_2\Phi(a,b)|
\end{aligned}$$

for all $a, b \in A$. Hence, we conclude that

$$|D_1\Phi(a,b)| \leq |\tau|\,|D_2\Phi(a,b)|$$

for all $a, b \in A$. Due to Theorem 2.1, the mapping Φ is a square root function. In order to show that Φ is unique, let $\Gamma : A \longrightarrow \mathbb{R}$ be another square root function satisfying (4.3). Now, we obtain for all $a \in A$ that

$$\begin{aligned}
|\Phi(a)-\Gamma(a)| &= \frac{1}{2^\eta}\,|\Phi(4^\eta a)-\Gamma(4^\eta a)| \\
&\leq \frac{1}{2^\eta}\left(|\Phi(4^\eta a)-\varphi(4^\eta a)|+|\varphi(4^\eta a)-\Gamma(4^\eta a)|\right) \\
&\leq \frac{1}{2^\eta}\sum_{\beta=0}^{\infty}\frac{1}{2^{\eta+\beta}}\zeta(4^{\eta+\beta}a, 4^{\eta+\beta}a) \longrightarrow 0.
\end{aligned}$$

Therefore, we conclude with $\Phi(a) = \Gamma(a)$, for all $a \in A$. Hence, Φ is unique and satisfies (4.3). ☐

The outcome of the following corollary shows that the solution exists to the stability problem of inequality (1.2) involved with sum of powers.

Corollary 4.1. *Assume that $k(>0), p\left(>\frac{1}{2}\right)$ are real numbers. If a function φ satisfies*

$$|D_1\varphi(a,b)| \leq |\tau\left(D_2\varphi(a,b)\right)| + k\left(a^p+b^p\right)$$

for all $a, b \in A$, then a unique approximate square root function $\Phi : A \longrightarrow \mathbb{R}$ exists and satisfies (1.2) *such that*

$$|\varphi(a)-\Phi(a)| \leq \frac{k}{1-2^{2p+1}}a^p \tag{4.5}$$

for all $a \in A$.

Proof. The proof is obtained by considering $\zeta(a,b) = k\left(a^p+b^p\right)$ in Theorem 4.1 and advancing with the parallel arguments as in Theorem 4.1. ☐

5. Solution to the stability problems of inequality (1.3)

In the following theorem, we find the solution to the stability problem of inequality (1.3).

Theorem 5.1. *Suppose a function $\zeta : A \times A \longrightarrow [0,\infty)$ satisfies*

$$\sum_{\beta=0}^{\infty} 2^{\beta}\zeta\left(\frac{a}{4^{\beta}}, \frac{b}{4^{\beta}}\right) < \infty \tag{5.1}$$

for all $a,b \in A$. If a function φ with $\varphi(0)=0$ satisfies the following inequality

$$|D_2\varphi(a,b)| \leq |\tau\left(D_1\varphi(a,b)\right)| + \zeta(a,b) \tag{5.2}$$

for all $a,b \in A$, then a unique approximate square root function $\Phi : A \longrightarrow \mathbb{R}$ exists and satisfies (1.3) *with the condition that*

$$|\varphi(a) - \Phi(a)| \leq \frac{1}{4}\sum_{\beta=0}^{\infty} 2^{\beta}\zeta\left(\frac{a}{4^{\beta}}, 0\right) \tag{5.3}$$

for all $a \in A$.

Proof. Firstly, we determine that there exists a square root function Φ which satisfies (5.3). To achieve this, letting $b = 0$ in (5.2), we obtain

$$\left|2\varphi\left(\frac{a}{4}\right) - \varphi(a)\right| \leq \frac{1}{4}\zeta(a,0)$$

for all $a \in A$. Suppose λ, μ are positive integers with $\mu > \lambda$. Then, we have

$$\begin{aligned}\left|2^{\lambda}\varphi\left(\frac{a}{4^{\beta}}\right) - 2^{\mu}\varphi\left(\frac{a}{4^{\mu}}\right)\right| &\leq \sum_{\beta=\lambda}^{\mu-1}\left|2^{\beta}\varphi\left(\frac{a}{4^{\beta}}\right) - 2^{\beta+1}\varphi\left(\frac{a}{4^{\beta+1}}\right)\right| \\ &\leq \frac{1}{4}\sum_{\beta=\lambda}^{\mu-1} 2^{\beta}\zeta\left(\frac{a}{4^{\beta}}, 0\right) \end{aligned}\tag{5.4}$$

for all $a \in A$. By the reasoning of the description of ζ in (5.1) and utilizing (5.4), the sequence $\left\{2^{\beta}\varphi\left(\frac{a}{4^{\beta}}\right)\right\}$ ends up as Cauchy. Since $\mathbb{R}$ is complete, this sequence converges to a function $\Phi : A \longrightarrow \mathbb{R}$ given by $\Phi(a) = \lim\limits_{\beta\to\infty} 2^{\beta}\varphi\left(\frac{a}{4^{\beta}}\right)$, for each $a \in A$. Furthermore, letting $\lambda = 0$ and allowing $\mu \to \infty$ in (5.4), we arrive at (5.3). Using (5.2), we have

$$\begin{aligned}&|D_2\Phi(a,b)| \\ &= \lim_{\beta\to\infty} 2^{\beta}\left|4\varphi\left(\frac{1}{4^{\beta}}\left(\frac{a+b+2\sqrt{ab}}{4}\right)\right) + 4\varphi\left(\frac{1}{4^{\beta}}\left(\frac{a+b-2\sqrt{ab}}{4}\right)\right) - 4\varphi\left(\frac{a}{4^{\beta}}\right)\right| \\ &\leq \lim_{\beta\to\infty} 2^{\beta}|\tau|\left|\varphi\left(\frac{1}{4^{\beta}}(a+b+2\sqrt{ab})\right) + \varphi\left(\frac{1}{4^{\beta}}(a+b-2\sqrt{ab})\right) - 2\varphi\left(\frac{a}{4^{\beta}}\right)\right| \\ &\qquad + \lim_{\beta\to\infty} 2^{\beta}\zeta\left(\frac{a}{4^{\beta}}, 0\right) \\ &= |\tau|\,|D_2\Phi(a,b)|\end{aligned}$$

for all $a,b \in A$. Hence, we conclude that

$$|D_2\Phi(a,b)| \leq |\tau|\,|D_1\Phi(a,b)|$$

for all $a, b \in A$. Due to Theorem 3.1, the mapping Φ is a square root function. For the sake of showing that Φ is unique, let $\Gamma : A \longrightarrow \mathbb{R}$ be another square root function satisfying (5.3). Now, we obtain for all $a \in A$ that

$$\begin{aligned} |\Phi(a) - \Gamma(a)| &= 2^\eta \left|\Phi\left(\frac{a}{4^\eta}\right) - \Gamma\left(\frac{a}{4^\eta}\right)\right| \\ &\leq 2^\eta \left(\left|\Phi\left(\frac{a}{4^\eta}\right) - \varphi\left(\frac{a}{4^\eta}\right)\right| + \left|\varphi\left(\frac{a}{4^\eta}\right) - \Gamma\left(\frac{a}{4^\eta}\right)\right|\right) \\ &\leq \frac{1}{2} 2^\eta \sum_{\beta=0}^{\infty} 2^{\eta+\beta} \zeta\left(\frac{a}{4^{\eta+\beta}}, 0\right) \longrightarrow 0. \end{aligned}$$

Therefore, we conclude with $\Phi(a) = \Gamma(a)$, for all $a \in A$. Hence, Φ is unique and satisfies (5.3). □

The outcome of the following corollary shows that the solution exists to the stability problem of inequality (1.3) involved with sum of powers.

Corollary 5.1. *Assume that $k(>0), p\left(<\frac{1}{2}\right)$ are real numbers. If a function φ satisfies*

$$|D_2\varphi(a, b)| \leq |\tau(D_1\varphi(a, b))| + k(a^p + b^p)$$

for all $a, b \in A$, then a unique approximate square root mapping $\Phi : A \longrightarrow \mathbb{R}$ exists and satisfies (1.3) *such that*

$$|\varphi(a) - \Phi(a)| \leq \frac{2k}{8(1 - 2^{2p+1})} a^p \tag{5.5}$$

for all $a \in A$.

Proof. The required result follows by considering $\zeta(a, b) = k(a^p + b^p)$ in Theorem 5.1 and proceeding with similar arguments as in Theorem 5.1. □

6. Conclusion

This is the first time that inequalities and equations resulting from square root function are introduced and their approximate solutions are found using Ulam's stability theory. These radical τ-functional inequalities produce different stability results than other functional inequalities. Furthermore, the functional inequalities involve arguments in radical form, which is novel in the field of stability theory research. We conclude that the stability problems of the inequalities (1.2) and (1.3) have solutions in the setting of non-negative real numbers.

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