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MIXED PROBLEM FOR THE NEGATIVE ORDER MODIFIED KORTEWEG-DE VRIES–COSINE GORDON EQUATION

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Abstract. In this paper, the negative order modified Korteweg-de Vries–cosine Gordon equation (nmKdV–coshG) in the class of periodic infinite-gap functions is integrated using the inverse spectral problem method. It is demonstrated that the Cauchy problem for the infinite Dubrovin differential equation system is solvable in the class of twice continuously differentiable periodic infinite-gap functions. It is demonstrated that the mKdV–coshG equation is satisfied by the sum of the uniformly convergent functional series constructed by solving the Dubrovin differential equation system. Furthermore, it is demonstrated that a global solution to the mixed problem for the negative order modified Korteweg-de Vries–cosine Gordon equation exists for sufficiently smooth initial data.

1. Introduction and Statement of the Problem

In this work, we consider a mixed problem for the negative order modified Korteweg-de Vries–cosine Gordon equation (nmKdV–coshG) of the form

$$\begin{cases} a(t)\,(u_{xt}-\cosh(2u))-b(t)\,(u_{xxxt}-(2u_x\mu_{xt})_x)=0, \\ \mu_{xx}=u_x^2, \end{cases} \tag{1.1}$$

with the conditions

$$\begin{cases} u(x,t)|_{t=0}=u_0(x),\ u_0(x+\pi)=u_0(x)\in C^5(\mathbb{R}), \\ u(x,t)|_{x=0}=\alpha(t),\ \mu_x(x,t)|_{x=0}=\beta(t), \\ \left[u_{xt}(x,t)-\mu_{xt}(x,t)\right]|_{x=0}=\zeta(t), \end{cases} \tag{1.2}$$

in the class of real infinite-gap π periodic with respect to x functions satisfying the smoothness conditions

$$u(x,t)\in C^{4,1}_{x,t}(t>0)\cap C(t\geq 0),\ \ \mu(x,t)\in C^{2,1}_{x,t}(t>0)\cap C(t\geq 0). \tag{1.3}$$

Here $a(t),b(t)\in C[0;\infty)$ and $\alpha(t),\beta(t),\zeta(t)\in C^1(t>0)\cap C(t\geq 0)$ are given continuously differentiable bounded functions.

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Note that if in equation (1.1) the coefficients $a(t) = 1, b(t) = 0$, then (1.1) takes the form of the hyperbolic cosine Gordon equation popular in the literature ([23]):

$$u_{xt} = \cosh(2u), u = u(x,t), x \in \mathbb{R}, t > 0.$$

From equation (1.1) for the case $a(t) = 0, b(t) = 1$, we obtain the negative order modified Korteweg-de Vries (nmKdV) equation of the form ([53]):

$$\begin{cases} u_{xxxt} - (2u_x\mu_{xt})_x = 0, \\ \mu_{xx} = u_x^2. \end{cases}$$

For simplicity, we study the equation (1.1), in the case $a(t) = 1, b(t) \neq 0$, $B_1 \leq b(t) \leq B_2, B_1 > 0, t \geq 0$.

In this paper, we propose an algorithm for constructing periodic infinite-gap solutions $u(x,t), \mu_x(x,t), x \in \mathbb{R}, t > 0$ of the mixed problem (1.1)–(1.3) by reducing it to an inverse spectral problem for a self-adjoint periodic Dirac operator of the form:

$$\mathfrak{L}(\tau,t)y \equiv B\frac{dy}{dx} + \Omega(x+\tau,t)y = \lambda y, \;\; x,\tau \in \mathbb{R}, \; t > 0, \tag{1.4}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \; \Omega(x,t) = \begin{pmatrix} P(x,t) & Q(x,t) \\ Q(x,t) & -P(x,t) \end{pmatrix}, y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix},$$

$$P(x,t) = 0, \;\; Q(x,t) = u_x'(x,t).$$

Inverse spectral problems play a significant role in integrating some important evolution equations of mathematical physics and are one of the most important achievements of the last century. An important breakthrough was made in the paper by Gardner, Green, Kruskal and Miura [13] in 1967, where they discovered a deep connection between the well-known nonlinear Korteweg-de Vries (KdV) equation

$$q_t = 6qq_x - q_{xxx}, \;\; q(x,t)|_{t=0} = q_0(x), \;\; x \in \mathbb{R}, t > 0,$$

and the spectral theory of the Sturm–Liouville operator

$$\mathfrak{L}(t)y \equiv -y'' + q(x,t)y = \lambda y, \;\; x \in \mathbb{R}, t > 0.$$

The authors of [13] succeeded in finding a global solution to the Cauchy problem for the KdV equation by reducing it to the inverse scattering problem. The inverse scattering problem for the Sturm–Liouville operator on the entire axis was studied in the works of Faddeev [11], Levitan [41], Marchenko [44] and others. In [38], Lax proved the universality of the inverse scattering method (ISM) and generalized the KdV equation by introducing the concept of a higher KdV equation. In their work [57], Zakharov and Shabat showed that the nonlinear Schrodinger equation (NLS)

$$iu_t \pm 2|u|^2u + u_{xx} = 0,$$

can also be included in the ISM formalism. Using a technique suggested by Lax, they found a solution to the NLS equation for given initial functions $u(x,0)$ that decay sufficiently rapidly as $|x| \to \infty$. Soon Wadati [54], using the ideas of Zakharov and Shabat, proposed a method for solving the Cauchy problem for the modified KdV equation (mKdV):

$$u_t \pm 6u^2u_x + u_{xxx} = 0, \;\; u_t \pm 6|u|^2u_x + u_{xxx} = 0.$$

Combining the focusing nonlinear Schrodinger equation (FNLS) and the complex modified Korteweg-de Vries equation (cmKdV), Hirota [16] proposed a method for finding exact solution to the equation titled "Hirota equation":

$$iq_t + \alpha\left(q_{xx} + 2|q|^2 q\right) + i\beta\left(q_{xxx} + 6|q|^2 q_x\right) = 0,\ \alpha, \beta \in \mathbb{R},\ x \in \mathbb{R},\ t > 0.$$

Ablowitz, Kaup, Newell, Sigur [1] and Zakharov, Takhtadzhyan, Faddeev [58] showed that the ISM can also be applied to the solution of the sine-Gordon (sG) equation:

$$u_{xt} = \sin u,\ \ u = u(x,t),\ \ x \in \mathbb{R},\ \ t > 0.$$

It should be noted that in the works [35], [51], [55], the modified Korteweg-de Vries–sine Gordon (mKdV–sG) equation of the form

$$u_{xt} + a\left\{\frac{3}{2}u_x^2 u_{xx} + u_{xxxx}\right\} = b\sin u,$$

where $a, b = const$, was integrated by a direct method in the class of rapidly decreasing functions.

The application of the ISM to the NLS, mKdV, sG, FNLS–cmKdV and mKdV–sG equations is based on the scattering problem for the Dirac operator on the entire axis:

$$\mathfrak{L} = i\begin{pmatrix} \frac{d}{dx} & -q(x) \\ r(x) & -\frac{d}{dx} \end{pmatrix},\ x \in \mathbb{R}.$$

The inverse scattering problem for the Dirac operator $\mathfrak{L}$ on the entire axis was studied in [12]. It is known that the operator $\mathfrak{L}$ is not self-adjoint, in the "rapidly decreasing"case it has a finite number of multiple complex eigenvalues and can have spectral singularities that lie in the continuous spectrum. The scattering data of a non-self-adjoint Dirac operator, in addition to the characteristics of the continuous spectrum, include a discrete spectrum and spectral singularities. In papers [1], [16], [28]– [30], such nonlinear evolution equations were integrated in the case when all eigenvalues of the corresponding Dirac operator $\mathfrak{L}$ are simple and without spectral singularities. In this regard, the search for a solution to nonlinear evolution equations without a source and with a self-consistent source corresponding to multiple eigenvalues of the Dirac operator $\mathfrak{L}$ is relevant. These problems are discussed in papers [28] and [33].

Using the method of the inverse spectral problem for the Sturm–Liouville operator with a periodic potential, when the spectrum contains only a finite number of nontrivial gaps, in the works of Dubrovin–Novikov [9], Its [19], Its–Kotlyarov [20], Its–Matveev [21], Matveev–Smirnov [46]– [47], Smirnov [52], the complete integrability of the KdV, NLS, mKdV, sG, FNLS–cmKdV equations in the class of finite-gap periodic and quasiperiodic functions was established. In addition, an explicit formula in terms of Riemann theta functions was derived for finite-gap solutions of such nonlinear evolution equations. In these papers, the solvability of the Cauchy problem for nonlinear evolution equations for any finite-gap initial data was proved. This theory is presented in more detail in monographs [14], [56], as well as in papers [4], [45].

It is well known that finding an explicit formula for solving a nonlinear evolution equation in the class of periodic functions depends significantly on the number of nontrivial gaps in the spectrum of the periodic Sturm–Liouville operator and the Dirac operator. In this regard, it is convenient to split the class of periodic functions into two sets:

1) the class of periodic finite-gap functions;
2) the class of periodic infinite-gap functions.

The following n–gap periodic Lame–Ince potentials are introduced in [2] and [18]:

$$q_n(x) = n(n+1)\wp(x),\ n \in \mathbb{N}$$

where $\wp(x)$ is the Weierstrass elliptic function.

It is known from [17] that if $q(x) = 2a\cos 2x, a \neq 0$, then in the spectrum of the Sturm–Liouville operator $\mathfrak{L}y \equiv -y'' + q(x)y, x \in \mathbb{R}$, all gaps are open, in other words, $q(x)$ is a periodic infinite-gap potential. Similar examples exist for the periodic Dirac operator [8].

Using the ideas of [16], in papers [10], [26,27], a method for constructing exact solutions of the Cauchy problem for equations composed of a combination of the defocusing nonlinear Schrodinger equation and the complex modified Korteweg-de Vries equation (DNLS–cmKdV) of the following forms is proposed

$$iu_t + b(t)\left(u_{xx} - 2|u|^2u\right) - ia(t)\left(u_{xxx} - 6|u|^2u_x\right) = 0,$$
$$iu_t + b(t)\left(u_{xx} - 2\left(|u|^2 - \rho^2\right)u\right) - ia(t)\left(u_{xxx} - 6\left(|u|^2 - \rho^2\right)u_x\right) = 0,$$
$$iu_t + b(t)\left(u_{xx} - 2|u|^2u\right) - ia(t)\left(u_{xxx} - 6|u|^2u_x\right) - ic(t)u_x - d(t)u = 0,$$
$$0 \leq \rho < \infty,\ u(x,t) = q(x,t) - ip(x,t),\ x \in \mathbb{R}, t > 0.$$

in the class π – periodic infinite-gap functions. Based on the ideas of the above works, in [31,32] the solvability of the Cauchy problem was established for nonlinear mKdV–coshG, Liouville(L) and mKdV–L equations in the class of real infinite-gap π – periodic in x functions.

Note that the KdV and mKdV equations of negative order, as well as their hierarchies, were studied in [22]. The integrability of the negative order mKdV–Liouville equation in the class of periodic infinite-gap functions was proved in [24]. In addition, the Cauchy problem in the class of periodic, almost periodic infinite-gap functions for nonlinear evolution equations without and with a self-consistent source, as well as with an additional term in various settings was studied in [3], [25], [29,30], [34], [39,40].

2. Fundamental concepts of a Dirac Operator

Let us denote by $c(x,\lambda,\tau,t) = (c_1(x,\lambda,\tau,t), c_2(x,\lambda,\tau,t))^T$ and $s(x,\lambda,\tau,t) = (s_1(x,\lambda,\tau,t), s_2(x,\lambda,\tau,t))^T$ solutions of the equation (1.4) with initial conditions $c(0,\lambda,\tau,t) = (1,0)^T$ and $s(0,\lambda,\tau,t) = (0,1)^T$, respectively. The function $\Delta(\lambda,\tau,t) = c_1(\pi,\lambda,\tau,t)+\ s_2(\pi,\lambda,\tau,t)$ is called the *Lyapunov function* for equation (1.4). The spectrum of the Dirac operator $\mathfrak{L}(\tau,t)$ in (1.4) is purely continuous:

$$\sigma(\mathfrak{L}) = \{\lambda \in \mathbb{R} : |\Delta(\lambda)| \leq 2\} = \mathbb{R}\setminus\left(\bigcup_{n=-\infty}^{+\infty}(\lambda_{2n-1},\lambda_{2n})\right).$$

The intervals $(\lambda_{2n-1},\lambda_{2n})$, $n \in \mathbb{Z}\setminus\{0\}$, are called gaps, where λ_n are the roots of the equation $\Delta(\lambda) \mp 2 = 0$. They coincide with the eigenvalues of the periodic or antiperiodic ($y(0,\tau,t) = \pm y(\pi,\tau,t)$) problem for equation (1.4). It is easy to prove that $\lambda_{-1} = \lambda_0 = 0$, that is, $\lambda = 0$ is a double eigenvalue of the periodic problem for equation (1.4).

The roots of the equation $s_1(\pi,\lambda,\tau,t) = 0$ are denoted by $\xi_n(\tau,t), n \in \mathbb{Z}\backslash\{0\}$, while $\xi_n(\tau,t) \in [\lambda_{2n-1},\lambda_{2n}], n \in \mathbb{Z}\backslash\{0\}$. They coincide with the eigenvalues of the Dirichlet problem for system (1.4) with boundary conditions $y_1(0,\tau,t) = 0$, $y_1(\pi,\tau,t) = 0$.

Definition 2.1. Numbers $\xi_n(\tau,t), n \in \mathbb{Z}\backslash\{0\}$, and signs $\sigma_n(\tau,t) = \operatorname{sgn}\{s_2(\pi,\xi_n,\tau,t) - c_1(\pi,\xi_n,\tau,t)\} = \pm 1,\ n \in \mathbb{Z}\backslash\{0\}$ are called *the spectral parameters* of the operator $\mathfrak{L}(\tau,t)$. The spectral parameters $\xi_n(\tau,t)$, $\sigma_n(\tau,t) = \pm 1, n \in \mathbb{Z}\backslash\{0\}$, and the spectral boundaries $\lambda_n(\tau,t), n \in \mathbb{Z}\backslash\{0\}$, are called *the spectral data* of the Dirac operator $\mathfrak{L}(\tau,t)$.

Definition 2.2. The problem of reconstructing the coefficient $\Omega(x,t)$ of the operator $\mathfrak{L}(\tau,t)$ from the spectral data is called *the inverse problem.*

If we construct the Dirac operator $\mathfrak{L}(\tau,0)$ using the initial function $u_0(x+\tau)$, $x,\tau \in \mathbb{R}$, we will see that the boundaries of the spectrum $\lambda_n(\tau), n \in \mathbb{Z}$ of the resulting problem do not depend on the parameter $\tau \in \mathbb{R}$, that is, $\lambda_n(\tau) = \lambda_n, n \in \mathbb{Z}$, and the spectral parameters depend on the parameter τ: $\xi_n^0 = \xi_n^0(\tau), \sigma_n^0 = \sigma_n^0(\tau) = \pm 1, n \in \mathbb{Z}$, and are periodic functions:

$$\xi_n^0(\tau+\pi) = \xi_n^0(\tau), \sigma_n^0(\tau+\pi) = \sigma_n^0(\tau), \tau \in \mathbb{R}.$$

Solving the direct spectral problem, we find the spectral data $\left\{\lambda_n, \xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1, n \in \mathbb{Z}\backslash\{0\}\right\}$ of the operator $\mathfrak{L}(\tau,0)$.

Definition 2.3. The coefficients $P(x,t) \equiv 0, Q(x,t) = u_x'(x,t)$ of the periodic Dirac operator $\mathfrak{L}(\tau,t)$ are called *infinite-gap functions* if the boundaries of the gap $(\lambda_{2n-1},\lambda_{2n}), n \in \mathbb{Z}$, satisfy the conditions

$$\ldots < \lambda_{-3} \leq \xi_{-1} \leq \lambda_{-2} < \lambda_{-1} \leq \xi_0 \leq \lambda_0 < \lambda_1 \leq \xi_1 \leq \lambda_2 < \ldots$$

where $\lambda_{-1} = \lambda_0 = \xi_0 = 0$.

Definition 2.4. The coefficients $P(x,t) \equiv 0, Q(x,t) = u_x'(x,t)$ of the periodic Dirac operator $\mathfrak{L}(\tau,t)$ are called *finite-gap functions* if there exists a finite number N such that for all $|n| > N$ the equalities $\lambda_{2n-1} = \lambda_{2n} = \xi_n$, $n = N+1, N+2, \ldots$ hold.

Inverse problems for the Dirac operator of the form

$$\mathfrak{L}y \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \ x \in \mathbb{R}$$

with periodic coefficients $p(x+\pi) = p(x), q(x+\pi) = q(x), x \in \mathbb{R}$ in various formulations are studied in the works [5–8], [15], [36, 37], [42], [48–50].

3. Main result

The main result of this paper is contained in the following theorem.

Theorem 3.1. *Let $u(x,t), \mu_x(x,t), x \in \mathbb{R}, t > 0$ be solutions of the mixed problem (1.1)–(1.3). Then the boundaries of the spectrum $\lambda_n(\tau,t), n \in \mathbb{Z}\backslash\{0\}$, of the operator $\mathfrak{L}(\tau,t)$ do not depend on parameters τ and t, that is, $\lambda_n(\tau,t) = \lambda_n, n \in \mathbb{Z}\backslash\{0\}$, and the spectral parameters $\xi_n(\tau,t), \sigma_n(\tau,t) = \pm 1, n \in \mathbb{Z}\backslash\{0\}$, satisfy the first and second systems of Dubrovin differential equations, respectively:*

$$1.\frac{\partial \xi_n(\tau,t)}{\partial \tau} = 2(-1)^{n-1}\sigma_n(\tau,t)h_n(\xi(\tau,t))\xi_n(\tau,t),\ n \in \mathbb{Z}\backslash\{0\}, \tag{3.1}$$

$$2.\frac{\partial \xi_n(\tau,t)}{\partial t} = 2(-1)^{n}\sigma_n(\tau,t)h_n(\xi(\tau,t))g_n(\xi(\tau,t)),\ n \in \mathbb{Z}\backslash\{0\}. \tag{3.2}$$

In addition, the following initial conditions are satisfied:

$$\xi_n(\tau,t)|_{t=0} = \xi_n^0(\tau),\ \sigma_n(\tau,t)|_{t=0} = \sigma_n^0(\tau),\ n \in \mathbb{Z}\backslash\{0\}, \tag{3.3}$$

where $\xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1, n \in \mathbb{Z}\backslash\{0\}$ are the spectral parameters of the Dirac operator $\mathfrak{L}(\tau,0)$.

The sequences $h_n(\xi)$ and $g_n(\xi), n \in \mathbb{Z}\backslash\{0\}$, in equation (3.2) are defined by the formulas:

$$h_n(\xi) = \sqrt{(\xi_n(\tau,t) - \lambda_{2n-1})(\lambda_{2n} - \xi_n(\tau,t))} \cdot f_n(\xi),$$

$$f_n(\xi) = \sqrt{\prod_{\substack{k=-\infty\\k\neq n}}^{\infty} \frac{(\lambda_{2k-1} - \xi_n(\tau,t))(\lambda_{2k} - \xi_n(\tau,t))}{(\xi_k(\tau,t) - \xi_n(\tau,t))^2}},$$

$$g_n(\xi) = \frac{1}{1 + 4b(t)\xi_n^2(\tau,t)}\left[\frac{e^{2u}}{2\xi_n(\tau,t)} + 2b(t)\xi_n(\tau,t)(u_{\tau t} + \mu_{\tau t})\right], \tag{3.4}$$

where $\xi \equiv \xi(\tau,t) = (\ldots, \xi_{-1}(\tau,t), \xi_1(\tau,t), \ldots)$, $\sigma \equiv \sigma(\tau,t) = (\ldots, \sigma_{-1}(\tau,t), \sigma_1(\tau,t), \ldots)$.

Proof. Let π periodic in x functions $u(x,t), \mu_x(x,t), x \in \mathbb{R}, t > 0$ satisfy the equation (1.1). Denote by $y_n = (y_{n,1}(x,\tau,t), y_{n,2}(x,\tau,t))^T, n \in \mathbb{Z}$, the orthonormal eigenvector functions of the operator $\mathfrak{L}(\tau,t)$, considered on the interval $[0;\pi]$ with Dirichlet boundary conditions $y_1(0,\tau,t) = 0, y_1(\pi,\tau,t) = 0$ corresponding to the eigenvalues $\xi_n = \xi_n(\tau,t), n \in \mathbb{Z}\backslash\{0\}$. Differentiating with respect to the variable t, the identity

$$\xi_n(\tau,t) = (\mathfrak{L}(\tau,t)y_n, y_n), n \in \mathbb{Z}\backslash\{0\},$$

and using the symmetry of the operator $\mathfrak{L}(\tau,t)$, we have

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = \left(\frac{\partial \Omega(x+\tau,t)}{\partial t} y_n, y_n\right), n \in \mathbb{Z}\backslash\{0\}. \tag{3.5}$$

Using the explicit form of the scalar product, we write the equality (3.5) in the form

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = 2\int_0^{\pi} [y_{n,1}y_{n,2}u_{xt}]\,dx. \tag{3.6}$$

Substituting (1.1) into (3.6), we obtain

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = 2\int_0^{\pi} y_{n,1}y_{n,2}\left(\cosh(2u) + b(t)(u_{xxt} - 2u_x\mu_{xt})_x\right)dx. \tag{3.7}$$

Using the following identities

$$\begin{cases} u(\pi,t)=u(0,t),\ \ \mu(\pi,t)=\mu(0,t), \\ y_{n,1}(\pi,\tau,t)=y_{n,1}(0,\tau,t)=0, \\ y_{n,1}'=u_x y_{n,1}-\xi_n y_{n,2},\ y_{n,2}'=-u_x y_{n,2}+\xi_n y_{n,1}, \\ (y_{n,1}y_{n,2})'=\xi_n\left(y_{n,1}^2-y_{n,2}^2\right), \\ \left(y_{n,1}^2-y_{n,2}^2\right)'=2u_x\left(y_{n,1}^2+y_{n,2}^2\right)-4\xi_n y_{n,1}y_{n,2}, \\ \left(y_{n,1}^2+y_{n,2}^2\right)'=2u_x\left(y_{n,1}^2-y_{n,2}^2\right) \end{cases}$$

and taking into account

$$\mu_{xxt}=2u_x u_{xt},$$

it is easy to derive the equality

$$\begin{aligned} &2\int_0^\pi y_{n,1}y_{n,2}\left(\cosh(2u)+b(t)\left(u_{xxt}-2u_x\mu_{xt}\right)_x\right)dx=\int_0^\pi y_{n,1}y_{n,2}e^{2u}dx+ \\ &+\int_0^\pi y_{n,1}y_{n,2}e^{-2u}dx+2b(t)\int_0^\pi y_{n,1}y_{n,2}d\left(u_{xxt}-2u_x\mu_{xt}\right)= \\ &=\frac{1}{2\xi_n}\int_0^\pi 2y_{n,2}\left(y_{n,2}'+u_x y_{n,2}\right)e^{2u}dx+\frac{1}{2\xi_n}\int_0^\pi 2y_{n,1}\left(-y_{n,1}'+u_x y_{n,1}\right)e^{-2u}dx- \\ &-2b(t)\xi_n\int_0^\pi\left(u_{xxt}-2u_x\mu_{xt}\right)\left(y_{n,1}^2-y_{n,2}^2\right)dx=\frac{1}{2\xi_n}\int_0^\pi\left(y_{n,2}^2e^{2u}\right)'dx- \\ &-\frac{1}{2\xi_n}\int_0^\pi\left(y_{n,1}^2e^{-2u}\right)'dx-2b(t)\xi_n\int_0^\pi u_{xxt}\left(y_{n,1}^2-y_{n,2}^2\right)dx+ \\ &+2b(t)\xi_n\int_0^\pi 2u_x\left(y_{n,1}^2-y_{u,2}^2\right)\mu_{xt}dx=\frac{e^{2u}}{2\xi_n}y_{n,2}^2\Big|_{x=0}^{x=\pi}-\frac{e^{-2u}}{2\xi_n}y_{n,1}^2\Big|_{x=0}^{x=\pi}- \\ &-2b(t)\xi_n\int_0^\pi\left(y_{n,1}^2-y_{n,2}^2\right)du_{xt}+2b(t)\xi_n\int_0^\pi\mu_{xt}d\left(y_{n,1}^2+y_{n,2}^2\right)= \\ &=\frac{1}{2\xi_n}e^{2u}y_{n,2}^2\Big|_{x=0}^{x=\pi}+2b(t)\xi_n u_{xt}y_{n,2}^2\big|_{x=0}^{x=\pi}+2b(t)\xi_n\int_0^\pi 2u_x u_{xt}\left(y_{n,1}^2+y_{n,2}^2\right)dx- \\ &-4b(t)\xi_n^2\int_0^\pi 2y_{n,1}y_{n,2}u_{xt}dx+2b(t)\xi_n\mu_{xt}y_{n,2}^2\big|_{x=0}^{x=\pi}- \\ &-2b(t)\xi_n\int_0^\pi\mu_{xxt}\left(y_{n,1}^2+y_{n,2}^2\right)dx=\left(\frac{e^{2u}}{2\xi_n}+2b(t)\xi_n\left(u_{xt}+\mu_{xt}\right)\right)y_{n,2}^2\Big|_{x=0}^{x=\pi}- \\ &-4b(t)\xi_n^2\int_a^\pi 2y_{n,1}y_{n,2}u_{xt}dx. \end{aligned}$$

Thus, we have

$$2\int_0^\pi y_{n,1}y_{n,2}u_{xt}dx=\frac{1}{1+4b(t)\xi_n^2}\left[\frac{e^{2u}}{2\xi_n}+2b(t)\xi_n\left(u_{xt}+\mu_{xt}\right)\right]y_{n,2}^2\Big|_{x=0}^{x=\pi}. \tag{3.8}$$

Substituting (3.8) into the identity (3.7), we have

$$\frac{\partial\xi_n(\tau,t)}{\partial t}=\frac{(2\xi_n)^{-1}e^{2u}+2b(t)\xi_n\left(u_{xt}+\mu_{xt}\right)}{1+4b(t)\xi_n^2}\left[y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)\right]. \tag{3.9}$$

In [28], the following equality is proved:

$$y_{n,2}^2(\pi,\tau,t)-y_{n,2}^2(0,\tau,t)=2(-1)^n\sigma_n(\tau,t)h_n(\xi(\tau,t)).$$

Substituting this expression into the identity (3.9), we obtain (3.2). Similarly, we can prove (3.1).

If we replace the Dirichlet boundary conditions with periodic ($y(0,\tau,t) = y(\pi,\tau,t)$) or with antiperiodic ($y(0,\tau,t) = -y(\pi,\tau,t)$) boundary conditions, then instead of the equation (3.9), we obtain

$$\frac{\partial \lambda_n(\tau,t)}{\partial t} = 0, \ \ \lambda_n(\tau,t) = \lambda_n(\tau,0).$$

Now in the equation $\mathfrak{L}(\tau,t)\nu_n = \lambda_n(\tau,t)\nu_n, n \in \mathbb{Z}$, we get $t = 0$. Since the eigenvalues $\lambda_n(\tau) = \lambda_n(\tau,0), n \in \mathbb{Z}$, do not depend on the parameter $\tau \in \mathbb{R}$, we have $\lambda_n(\tau,t) = \lambda_n(\tau) = \lambda_n, n \in \mathbb{Z}$.

The theorem is proved. □

Now taking into account the conditions (1.2) and integrating the equalities

$$u_\tau(\tau,t) = \sum_{k=-\infty}^{+\infty} (-1)^{k-1}\sigma_k(\tau,t)h_k(\xi(\tau,t)), \tag{3.10}$$

$$\mu_{\tau\tau}(\tau,t) = u_\tau^2(\tau,t),$$

we have

$$u(\tau,t) = \alpha(t) + \int_0^\tau \left\{ \sum_{k=-\infty}^{+\infty} (-1)^{k-1}\sigma_k(s,t)h_k(\xi(s,t)) \right\} ds, \tag{3.11}$$

$$\mu_\tau(\tau,t) = \beta(t) + \int_0^\tau u_\tau^2(s,t)ds. \tag{3.12}$$

Lemma 3.1. *The following formulas hold:*

$$\begin{gathered} w(\tau,t) \equiv u_{\tau t}(\tau,t) + \mu_{\tau t}(\tau,t) = \exp\left\{ \int_0^\tau A(s,t)ds \right\} \times \\ \times \left(\zeta(t) + \int_0^\tau B(s,t)\exp\left\{ 2u(s,t) - \int_0^s A(y,t)dy \right\} ds \right\}, \end{gathered} \tag{3.13}$$

where

$$\begin{aligned} A(\tau,t) &= \sum_{k=-\infty}^{+\infty} \frac{8(-1)^{k-1}b(t)\sigma_k(\tau,t)h_k(\xi(\tau,t))\xi_k^2(\tau,t)}{1+4b(t)\xi_k^2(\tau,t)}, \\ B(\tau,t) &= \sum_{k=-\infty}^{+\infty} \frac{2(-1)^{k-1}\sigma_k(\tau,t)h_k(\xi(\tau,t))}{1+4b(t)\xi_k^2(\tau,t)}. \end{aligned} \tag{3.14}$$

Proof. If differentiating with respect to t the second trace formula

$$\mu_{\tau\tau}(\tau,t) + u_{\tau\tau}(\tau,t) = \sum_{k=-\infty}^{+\infty} \left(\frac{\lambda_{2k-1}^2 + \lambda_{2k}^2}{2} - \xi_k^2(\tau,t) \right),$$

and using the system of Dubrovin differential equations (3.2), then with respect to the function $w(\tau, t)$ we obtain the linear differential equations:

$$(u_{\tau t} + \mu_{\tau t})_\tau = \sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} -2\xi_k \frac{\partial \xi_k}{\partial t} = \sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} \frac{4(-1)^{k-1}\sigma_k(\tau,t)h_k(\xi)\xi_k}{1+4b(t)\xi_k^2} \times$$
$$\times \left(\frac{e^{2u}}{2\xi_k} + 2b(t)\xi_k\left(u_{\tau t} + \mu_{\tau t}\right)\right) = e^{2u} \sum_{k=-\infty}^{+\infty} \frac{2(-1)^{k-1}\sigma_k(\tau,t)h_k(\xi)}{1+4b(t)\xi_k^2} +$$
$$+ \left(u_{\tau t} + \mu_{\tau t}\right) \sum_{k=-\infty}^{+\infty} \frac{8(-1)^{k-1}b(t)\sigma_k(\tau,t)h_k(\xi)\xi_k^2}{1+4b(t)\xi_k^2},$$

that is,

$$w_\tau(\tau,t) - A(\tau,t)w(\tau,t) = B(\tau,t)e^{2u}. \tag{3.15}$$

Solving the linear equation (3.15), we obtain (3.13).

The lemma is proved □

Next, taking into account formulas (3.11) and (3.13), the system of differential equations (3.2) can be written in closed form:

$$\begin{aligned} &\frac{\partial \xi_n(\tau,t)}{\partial t} = 2(-1)^n \sigma_n(\tau,t)\sqrt{\left(\xi_n(\tau,t) - \lambda_{2n-1}\right)\left(\lambda_{2n} - \xi_n(\tau,t)\right)} f_n(\xi) g_n(\xi), \\ &\xi_n(\tau,t)|_{t=0} = \xi_n^0(\tau),\ \sigma_n(\tau,t)|_{t=0} = \sigma_n^0(\tau), n \in \mathbb{Z}\backslash\{0\}, \end{aligned} \tag{3.16}$$

where

$$g_n(\xi) = \frac{1}{2\xi_n\left(1+4b(t)\xi_n^2\right)} \cdot \exp\left\{2\alpha(t) + 2\int_0^\tau \left(\sum_{k=-\infty}^{+\infty} (-1)^{k-1}\sigma_k(s,t)h_k(\xi)\right) ds\right\}$$
$$+ \frac{2b(t)\xi_n \exp\left\{\int_0^\tau A(s,t)ds\right\}}{1+4b(t)\xi_n^2} \cdot \left(\zeta(t) + \int_0^\tau B(s,t)\exp\{2\alpha(t) + \right.$$
$$\left. +2\int_0^s \left(\sum_{k=-\infty}^{+\infty} (-1)^{k-1}\sigma_k(y,t)h_k(\xi) - \frac{1}{2}A(y,t)\right) dy\right\} ds\Bigg),$$
$$\xi \equiv \xi(\tau,t) = \left(\ldots, \xi_{-1}(\tau,t), \xi_1(\tau,t), \ldots\right),$$
$$\sigma \equiv \sigma(\tau,t) = \left(\ldots, \sigma_{-1}(\tau,t), \sigma_1(\tau,t), \ldots\right).$$

As a result of the change of variables

$$\xi_n(\tau,t) = \lambda_{2n-1} + (\lambda_{2n} - \lambda_{2n-1})\sin^2 x_n(\tau,t),\ n \in \mathbb{Z}\backslash\{0\}, \tag{3.17}$$

the Cauchy problem (3.16) can be written as a single equation in the Banach space $\mathbb{K}$:

$$\frac{dx(\tau,t)}{dt} = H(x(\tau,t)),\ x(\tau,t)|_{t=0} = x^0(\tau) \in \mathbb{K}, \tag{3.18}$$

where

$$\mathbb{K} = \{x = (\dots, x_{-1}(\tau,t), x_1(\tau,t), \dots) : \\ \left. \|x(\tau,t)\| = \sum_{\substack{n=-\infty,\\ n\neq 0}}^{+\infty} (\lambda_{2n} - \lambda_{2n-1}) \left(\frac{1}{|n|^2} + 1\right) |x_n(\tau,t)| < \infty \right\}.$$

$$H(x) = (\dots, H_{-1}(x), H_1(x), \dots), \; H_n(x) = (-1)^n \sigma_n^0(\tau) f_n(x(\tau,t)) g_n(x(\tau,t)).$$

It is known that if $u_0(x+\pi) = u_0(x) \in C^5(\mathbb{R})$, then $u_0'(x) \in C^4(\mathbb{R})$. Therefore, for the length of the gaps of the operator $\mathfrak{L}(\tau,0)$ the following estimate holds (see [48]):

$$\gamma_k \equiv \lambda_{2k} - \lambda_{2k-1} = \frac{|q_{2k}^4|}{2^3|k|^4} + \frac{\delta_k}{|k|^5}. \tag{3.19}$$

Here

$$\lambda_{2k} = k + \sum_{j=1}^{4} c_j k^{-j} + 2^{-4}|k|^{-4} \left|q_{2k}^4\right| + |k|^{-5}\varepsilon_k^+,$$
$$\lambda_{2k-1} = k + \sum_{j=1}^{4} c_j k^{-j} - 2^{-4}|k|^{-4} \left|q_{2k}^4\right| + |k|^{-5}\varepsilon_k^-, \tag{3.20}$$
$$\sum_{k=-\infty}^{+\infty} \left|q_{2k}^4\right|^2 < \infty, \; \sum_{k=-\infty}^{+\infty} \left(\varepsilon_k^\pm\right)^2 < \infty, \; \delta_k = \varepsilon_k^+ - \varepsilon_k^-.$$

From here, taking into account $\xi_n(\tau,t) \in [\lambda_{2n-1}, \lambda_{2n}]$, we obtain

$$\inf_{k\neq n} |\xi_n(\tau,t) - \xi_k(\tau,t)| \geq a_1, \; a_2|n| \leq \xi_n(\tau,t) \leq a_3|n|, \; a_i > 0, i = 1,2,3.$$

Lemma 3.2. *The following estimates hold:*

$$C_1 \leq |f_n(x(\tau,t))| \leq C_2, \qquad \left|\frac{\partial f_n(x(\tau,t))}{\partial x_m}\right| \leq C_3\gamma_m, \tag{3.21}$$

$$|g_n(x(\tau,t))| \leq \frac{C_4}{|n|}, \qquad \left|\frac{\partial g_n(x(\tau,t))}{\partial x_m}\right| \leq \frac{C_5\gamma_m}{|n|}\left(\frac{1}{|m|^2} + 1\right), \tag{3.22}$$

where $C_j > 0, j = \overline{1,5}$, *do not depend on the parameters* m *and* n.

Proof. Estimates (3.21) are proved in the paper [43], so we prove (3.22). Since the functions $\alpha(t), \beta(t), \zeta(t)$ are bounded, there exist numbers $M_j > 0, j = \overline{1,3}$, such that the following inequalities hold:

$$|\alpha(t)| \leq M_1, |\beta(t)| \leq M_2, |\zeta(t)| \leq M_3.$$

Now, using the closed form system of differential equations (3.2), the change of variables (3.17) and the estimates (3.19)–(3.21), we obtain

1. $|A(\tau,t)| =$

$$= \left| \sum_{k=-\infty}^{+\infty} \frac{4(-1)^{k-1} b(t)\sigma_k^0(\tau)\gamma_k \sin 2x_k(\tau,t) f_k(x) \left(\lambda_{2k-1} + \gamma_k \sin^2 x_k(\tau,t)\right)^2}{1 + 4b(t)\left(\lambda_{2k-1} + \gamma_k \sin^2 x_k(\tau,t)\right)^2} \right| \le$$

$$\le \sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} \frac{4|b(t)|\gamma_k \left|\sin 2x_k(\tau,t)\right| \left|f_k(x)\right| \left|\lambda_{2k-1} + \gamma_k \sin^2 x_k(\tau,t)\right|^2}{4|b(t)| \left|\lambda_{2k-1} + \gamma_k \sin^2 x_k(\tau,t)\right|^2} \le$$

$$\le \sum_{k=-\infty}^{+\infty} \gamma_k \left|f_k(x)\right| = A_1;$$

2. $$|B(\tau,t)| = \left| \sum_{k=-\infty}^{+\infty} \frac{(-1)^{k-1}\sigma_k^0(\tau)\gamma_k \sin 2x_k(\tau,t) f_k(x)}{1 + 4b(t)\left(\lambda_{2k-1} + \gamma_k \sin^2 x_k(\tau,t)\right)^2} \right| \le$$

$$\le \sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} \frac{\gamma_k \left|\sin 2x_k(\tau,t)\right| \left|f_k(x)\right|}{|b(t)|} \le \sum_{k=-\infty}^{+\infty} \frac{C_2\gamma_k}{4B_1 a_2 |k|^2} = A_2;$$

3. $$\left| \sum_{k=-\infty}^{+\infty} (-1)^{k-1}\sigma_k^0(\tau)\gamma_k \sin 2x_k(\tau,t) f_k(x) \right| \le \sum_{k=-\infty}^{+\infty} \gamma_k \left|f_k(x)\right| \le$$

$$\le C_2 \sum_{k=-\infty}^{+\infty} \gamma_k = A_3;$$

4. $$\left| \sum_{k=-\infty}^{+\infty} (-1)^{k-1}\sigma_k^0(\tau)\gamma_k \sin 2x_k(\tau,t) \frac{\partial f_k(x)}{\partial x_m} \right| \le \sum_{k=-\infty}^{+\infty} \gamma_k \left| \frac{\partial f_k(x)}{\partial x_m} \right| \le$$

$$\le C_3\gamma_m \sum_{k=-\infty}^{+\infty} \gamma_k \le A_4\gamma_m;$$

5. $$\frac{\partial A(\tau,t)}{\partial x_m} = \frac{8(-1)^{m-1} b(t)\sigma_m^0(\tau)\gamma_m f_m(x)\left(\lambda_{2m-1} + \gamma_m \sin^2 x_m(\tau,t)\right)}{\left[1 + 4b(t)\left(\lambda_{2m-1} + \gamma_m \sin^2 x_m(\tau,t)\right)^2\right]^2} \times$$

$$\times \Big[4b(t)\cos 2x_m(\tau,t)\left(\lambda_{2m-1} + \gamma_m \sin^2 x_m(\tau,t)\right)^3 +$$

$$+ \cos 2x_m(\tau,t)\left(\lambda_{2m-1} + \gamma_m \sin^2 x_m(\tau,t)\right) + \gamma_m \sin^2 2x_m(\tau,t)\Big] +$$

$$+ \sum_{k=-\infty}^{+\infty} \frac{4(-1)^{k-1} b(t)\sigma_k^0(\tau)\gamma_k \sin 2x_k(\tau,t) \frac{\partial f_k(x)}{\partial x_m} \left(\lambda_{2k-1} + \gamma_k \sin^2 x_k(\tau,t)\right)^2}{1 + 4b(t)\left(\lambda_{2k-1} + \gamma_k \sin^2 x_k(\tau,t)\right)^2},$$

that is,

$$\left| \frac{\partial A(\tau,t)}{\partial x_m} \right| \le A_5\gamma_m \left(\frac{1}{|m|^2} + 1 \right);$$

6. $$\frac{\partial B(\tau,t)}{\partial x_m} = (-1)^{m-1}\sigma_m^0(\tau)\gamma_m f_m(x) \left[\frac{2\cos 2x_m(\tau,t)}{1 + 4b(t)\left(\lambda_{2m-1} + \gamma_m \sin^2 x_m(\tau,t)\right)^2} - \right.$$

$$\left.-\frac{8b(t)\sin^2 2x_m(\tau,t)\gamma_m\left(\lambda_{2m-1}+\gamma_m\sin^2 x_m(\tau,t)\right)}{\left[1+4b(t)\left(\lambda_{2m-1}+\gamma_m\sin^2 x_m(\tau,t)\right)^2\right]^2}\right]+$$

$$+\sum_{k=-\infty}^{+\infty}\frac{(-1)^{k-1}\sigma_k^0(\tau)\gamma_k\sin 2x_k(\tau,t)\frac{\partial f_k(x)}{\partial x_m}}{1+4b(t)\left(\lambda_{2k-1}+\gamma_k\sin^2 x_k(\tau,t)\right)^2},$$

that is,

$$\left|\frac{\partial B(\tau,t)}{\partial x_m}\right|\le A_6\gamma_m\left(\frac{1}{|m|^2}+1\right);$$

7. $|g_n(x(\tau,t))|\le$

$$\le\left|\frac{\exp\left\{2\alpha(t)+\int_0^\tau\left(\sum_{k=-\infty}^{+\infty}(-1)^{k-1}\sigma_k^0(s)\gamma_k\sin 2x_k(s,t)f_k(x)\right)ds\right\}}{2\left(\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)\right)\left(1+4b(t)\left(\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)\right)^2\right)}\right|+$$

$$+\left|\frac{2b(t)\left(\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)\right)}{1+4b(t)\left(\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)\right)^2}\right|\cdot\exp\left\{\int_0^\tau A(s,t)ds\right\}\cdot|\zeta(t)+$$

$$+\int_0^\tau B(s,t)\cdot\exp\{2\alpha(t)+$$

$$+\int_0^s\left(\sum_{k=-\infty}^{+\infty}(-1)^{k-1}\sigma_k^0(y)\gamma_k\sin 2x_k(y,t)f_k(x)-A(y,t)\right)dy\Bigg\}ds\Bigg|\le$$

$$\le\frac{C_4}{|n|};$$

8. $\dfrac{\partial g_n(x(\tau,t))}{\partial x_m}=$

$$=\frac{\exp\left\{2\alpha(t)+\int_0^\tau\left[\sum_{k=-\infty}^{+\infty}(-1)^{k-1}\sigma_k^0(s)\gamma_k\sin 2x_k(s,t)f_k(x)\right]ds\right\}}{2\left(\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)\right)\left(1+4b(t)\left(\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)\right)^2\right)}\times$$

$$\times\int_0^\tau 2(-1)^{m-1}\sigma_m^0(s)\gamma_m\cos 2x_m(s,t)f_m(x)ds+$$

$$+\int_0^\tau\left(\sum_{k=-\infty}^{+\infty}(-1)^{k-1}\sigma_k^0(s)\gamma_k\sin 2x_k(s,t)\frac{\partial f_k(x)}{\partial x_m}\right)ds+$$

$$+\frac{2b(t)\left(\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)\right)}{1+4b(t)\left(\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)\right)^2}\cdot\exp\left\{\int_0^\tau A(s,t)ds\right\}\times$$

$$\times\left(\int_0^\tau\frac{\partial A(s,t)}{\partial x_m}ds\right)\cdot[\zeta(t)+\int_0^\tau B(s,t)\exp\{2\alpha(t)+$$

$$\left.+\int_0^s\left(\sum_{k=-\infty}^{+\infty}(-1)^{k-1}\sigma_k^0(y)\gamma_k\sin 2x_k(y,t)f_k(x)-A(y,t)\right)dy\Bigg\}ds\right]+$$

$$+\frac{2b(t)\left(\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)\right)}{1+4b(t)\left(\lambda_{2n-1}+\gamma_n\sin^2 x_n(\tau,t)\right)^2}\cdot\exp\left\{\int_0^\tau A(s,t)ds\right\}\times$$

$$\times \left[\int_0^\tau \frac{\partial B(s,t)}{\partial x_m} \cdot \exp\{2\alpha(t) + \right.$$
$$+ \int_0^s \left(\sum_{k=-\infty}^{+\infty} (-1)^{k-1} \sigma_k^0(y) \gamma_k \sin 2x_k(y,t) f_k(x) - A(y,t) \right) dy \Big\} ds +$$
$$+ \int_0^\tau B(s,t) \cdot \exp\{2\alpha(t) +$$
$$+ \int_0^s \left(\sum_{k=-\infty}^{+\infty} (-1)^{k-1} \sigma_k^0(y) \gamma_k \sin 2x_k(y,t) f_k(x) - A(y,t) \right) dy \Big\} \times$$
$$\times \left(\int_0^s \left(2(-1)^{m-1} \sigma_m^0(y) \gamma_m \cos 2x_m(y,t) f_m(x) + \right.\right.$$
$$\left.\left.\left. + \sum_{k=-\infty}^{+\infty} (-1)^{k-1} \sigma_k^0(y) \gamma_k \sin 2x_k(y,t) \frac{\partial f_k(x)}{\partial x_m} - \frac{\partial A(y,t)}{\partial x_m} \right) dy \right) ds \right].$$

Thus,

$$\left| \frac{\partial g_n(x(\tau,t))}{\partial x_m} \right| \le A_7 \frac{\gamma_m}{|n|^3} + A_8 \frac{\gamma_m}{|n|} \left(\frac{1}{|m|^2} + 1 \right) \le \frac{C_5 \gamma_m}{|n|} \left(\frac{1}{|m|^2} + 1 \right).$$

Here $A_i = \text{const}, i = \overline{1,8}$.

The lemma is proved. □

Lemma 3.3. *If a periodic infinite-gap function $u_0(x)$ satisfies the condition $u_0(x+\pi) = u_0(x) \in C^2(\mathbb{R})$ and $\alpha(t), \beta(t), \zeta(t) \in C([0;\infty))$ are continuously differentiable bounded functions, then the vector function $H(x(\tau,t))$ satisfies the Lipschitz condition in the Banach space $\mathbb{K}$, that is, there exists a constant $L_1 > 0$ such that for arbitrary elements $x(\tau,t), y(\tau,t) \in \mathbb{K}_1$, the following inequality holds*

$$||H(x(\tau,t)) - H(y(\tau,t))|| \le L_1 ||x(\tau,t) - y(\tau,t)||,$$

where

$$L_1 = C \sum_{\substack{n=-\infty,\\ n\neq 0}}^{+\infty} \frac{\gamma_n}{|n|} \left(\frac{1}{|n|^2} + 1 \right) < \infty. \tag{3.23}$$

Proof. First, using Lemma (3.2), we estimate the derivative of the function $F_n(x) = f_n(x) g_n(x), n \in \mathbb{Z} \backslash \{0\}$:

$$\left| \frac{\partial F_n(x)}{\partial x_m} \right| \le \left| \frac{\partial f_n(x)}{\partial x_m} \right| |g_n(x)| + |f_n(x)| \left| \frac{\partial g_n(x)}{\partial x_m} \right| \le C_3 \gamma_m \frac{C_4}{|n|} +$$
$$+ C_2 \frac{C_5 \gamma_m}{|n|} \left(\frac{1}{|m|^2} + 1 \right) \le C_3 \gamma_m \frac{C_4}{|n|} \left(\frac{1}{|m|^2} + 1 \right) + C_2 \frac{C_5 \gamma_m}{|n|} \left(\frac{1}{|m|^2} + 1 \right) \le$$
$$\le C \frac{\gamma_m}{|n|} \left(\frac{1}{|m|^2} + 1 \right),$$

where the positive constant $C = C_3 C_4 + C_2 C_5$ do not depend on m and n. Further, using the expression

$$H_n(x(\tau,t)) = (-1)^n \sigma_n^0(\tau) F_n(x(\tau,t)), n \in \mathbb{Z} \backslash \{0\},$$

we obtain

$$|H_n(x(\tau,t)) - H_n(y(\tau,t))| = |F_n(x(\tau,t)) - F_n(y(\tau,t))|.$$

Now let us apply Mean Value Theorem on the segment $t \in [0,1]$ to the function $\varphi(t) = F_n(y + t(x-y))$ and obtain $\varphi(1) - \varphi(0) = \varphi'(t^*)$, that is,

$$F_n(x) - F_n(y) = \sum_{\substack{m=-\infty,\\ m\neq 0}}^{+\infty} \frac{\partial F_n(\theta)}{\partial x_m} (x_m(\tau,t) - y_m(\tau,t)),$$

where $\theta = y + t^*(x-y)$. It follows that

$$|H_n(x(\tau,t)) - H_n(y(\tau,t))| = |F_n(x(\tau,t)) - F_n(y(\tau,t))| \le$$

$$\le \sum_{\substack{m=-\infty,\\ m\neq 0}}^{+\infty} \left|\frac{\partial F_n(\theta)}{\partial x_m}\right| |x_m(\tau,t) - y_m(\tau,t)| \le$$

$$\le \frac{C}{|n|} \sum_{\substack{m=-\infty,\\ m\neq 0}}^{+\infty} \gamma_m \left(\frac{1}{|m|^2} + 1\right) |x_m(\tau,t) - y_m(\tau,t)| = \frac{C}{|n|}\|x-y\|.$$

Next we estimate the norm:

$$\|H(x) - H(y)\| = \sum_{\substack{n=-\infty,\\ n\neq 0}}^{+\infty} \gamma_n \left(\frac{1}{|n|^2} + 1\right) |H_n(x) - H_n(y)| \le$$

$$\le \sum_{\substack{n=-\infty,\\ n\neq 0}}^{+\infty} \gamma_n \left(\frac{1}{|n|^2} + 1\right) \frac{C}{|n|}\|x-y\| = L_1\|x-y\|.$$

Here

$$L = C \sum_{\substack{n=-\infty,\\ n\neq 0}}^{+\infty} \frac{\gamma_n}{|n|} \left(\frac{1}{|n|^2} + 1\right).$$

Since $u_0(x) \in C^2(\mathbb{R})$ and $u_0'(x) \in C^1(\mathbb{R})$ the estimates (3.19) have the following form:

$$\gamma_n \equiv \lambda_{2n} - \lambda_{2n-1} = \frac{|q_{2n}^1|}{|n|} + \frac{\delta_n}{|n|^2}.$$

Thus, $L < \infty$, that is, the Lipschitz condition is satisfied. Therefore, the solution of the Cauchy problem (3.2), (3.3) for all $t > 0$ and $\tau \in \mathbb{R}$ exists and is unique.

The lemma is proved. □

In the same way as above, the following theorem is proved on the existence of a solution of the Cauchy problem for the first system of Dubrovin differential equations, that is, the Cauchy problem (3.1), (3.3).

Lemma 3.4. *If a periodic infinite-gap function $u_0(x)$ satisfies the condition $u_0(x+\pi) = u_0(x) \in C^3(\mathbb{R})$ and $\alpha(t), \beta(t), \zeta(t) \in C([0;\infty))$ are continuously differentiable bounded functions, then the solution of the Cauchy problem* (3.1), (3.3) *for all $t > 0$ and $\tau \in \mathbb{R}$ exists and is unique.*

Remark 3.1. Theorem (3.1), Lemma (3.3) and Lemma (3.4) give an algorithm for finding a solution of the mixed problem (1.1)–(1.3):

1. First, we find the spectral data $\lambda_n, \xi_n^0(\tau), \sigma_n^0(\tau) = \pm 1$, $n \in \mathbb{Z}\backslash\{0\}$, of the Dirac operator $\mathfrak{L}(\tau, 0)$. Let us denote the spectral data of the operator $\mathfrak{L}(\tau, t)$ by $\lambda_n, \xi_n(\tau, t), \sigma_n(\tau, t) = \pm 1, n \in \mathbb{Z}\backslash\{0\}$;
2. Now, having solved the Cauchy problem (3.16) for an arbitrary value of τ, we find $\xi_n(\tau, t), \sigma_n(\tau, t), n \in \mathbb{Z}\backslash\{0\}$;
3. From the formula (3.10)–(3.12) we define the functions $u(\tau, t), \mu_\tau(\tau, t), \tau \in \mathbb{R}$, $t > 0$, that is, solutions of the mixed problem (1.1)–(1.3).

So far we have assumed that the Cauchy problem (1.1)–(1.3) has a solution. It is easy to directly verifying that the functions $u(\tau, t), \mu_\tau(\tau, t), \tau \in \mathbb{R}$, $t > 0$ obtained in this way, satisfy equation (1.1).

Lemma 3.5. *The functions* $u(\tau, t), \mu_\tau(\tau, t), \tau \in \mathbb{R}, t > 0$, *constructed using the system of Dubrovin differential equations* (3.2), (3.3) *and formula* (3.10)–(3.12) *satisfy equation* (1.1).

Proof. In this case, we will also use the Dubrovin equation system (3.1), as well as formula (3.4). Then from (3.2), we obtain

$$\begin{aligned}
&2\xi_n(\tau, t)\left(1 + 4b(t)\xi_n^2(\tau, t)\right)\frac{\partial \xi_n(\tau, t)}{\partial t} = 2(-1)^n \sigma_n(\tau, t) h_n(\xi) e^{2u} + \\
&+ 8(-1)^n b(t)\sigma_n(\tau, t) h_n(\xi)\xi_n^2(\tau, t)\left(u_{\tau t} + \mu_{\tau t}\right), \\
&2\xi_n(\tau, t)\frac{\partial \xi_n(\tau, t)}{\partial t} + 8b(t)\xi_n^3(\tau, t)\frac{\partial \xi_n(\tau, t)}{\partial t} = \\
&= 2(-1)^n \sigma_n(\tau, t) h_n(\xi(\tau, t)) e^{2u} - 4b(t)\xi_n(\tau, t)\frac{\partial \xi_n(\tau, t)}{\partial \tau}\left(u_{\tau t} + \mu_{\tau t}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\partial \xi_n^2(\tau, t)}{\partial t} + 2b(t)\frac{\partial \xi_n^4(\tau, t)}{\partial t} = \\
&\qquad = 2(-1)^n \sigma_n(\tau, t) h_n(\xi(\tau, t)) e^{2u} - 2b(t)\frac{\partial \xi_n^2(\tau, t)}{\partial \tau}\left(u_{\tau t}(\tau, t) + \mu_{\tau t}(\tau, t)\right).
\end{aligned}$$

Summing both parts of this equality, we have

$$\begin{aligned}
&\sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} \frac{\partial \xi_k^2(\tau, t)}{\partial t} + 2b(t)\sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} \frac{\partial \xi_k^4(\tau, t)}{\partial t} = \\
&= -2e^{2u}\sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} (-1)^{k-1}\sigma_k(\tau, t) h_k(\xi(\tau, t)) - \\
&- 2b(t)\left(u_{\tau t}(\tau, t) + \mu_{\tau t}(\tau, t)\right)\sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} \frac{\partial \xi_k^2(\tau, t)}{\partial \tau}.
\end{aligned}$$

Now we use the trace formula (3.10). Then we obtain the following equality:

$$\sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} \frac{\partial \xi_k^2(\tau,t)}{\partial t} + 2b(t) \sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} \frac{\partial \xi_k^4(\tau,t)}{\partial t} =$$

$$= -2e^{2u} u_\tau(\tau,t) - 2b(t)\left(u_{\tau t}(\tau,t) + \mu_{\tau t}(\tau,t)\right) \sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} \frac{\partial \xi_k^2(\tau,t)}{\partial \tau}. \quad (3.24)$$

Then, differentiating trace formulas:

$$u_\tau^2 + u_{\tau\tau} = \sum_{k=-\infty}^{+\infty} \left(\frac{\lambda_{2k-1}^2 + \lambda_{2k}^2}{2} - \xi_k^2 \right),$$

$$-u_{\tau\tau\tau\tau} - 2u_\tau u_{\tau\tau\tau} + 4u_{\tau\tau} u_\tau^2 + 2u_\tau^4 = 2 \sum_{k=-\infty}^{+\infty} \left(\frac{\lambda_{2k-1}^4 + \lambda_{2k}^4}{2} - \xi_k^4 \right),$$

with respect to τ and t, respectively, we have

$$-\left(2u_\tau u_{\tau\tau} + u_{\tau\tau\tau}\right) = \sum_{k=-\infty}^{+\infty} \frac{\partial \xi_k^2(\tau,t)}{\partial \tau}, \quad (3.25)$$

$$-\left(2u_\tau u_{\tau t} + u_{\tau\tau t}\right) = \sum_{\substack{k=-\infty,\\ k\neq 0}}^{+\infty} \frac{\partial \xi_k^2(\tau,t)}{\partial t}, \quad (3.26)$$

$$-u_{\tau\tau\tau\tau t} - 2u_{\tau t} u_{\tau\tau\tau} - 2u_\tau u_{\tau\tau\tau t} + 4u_\tau^2 u_{\tau\tau t} + 8u_\tau u_{\tau\tau} u_{\tau t} + 8u_\tau^3 u_{\tau t} =$$

$$= -2 \sum_{\substack{k=-\infty\\ k\neq 0}}^{\infty} \frac{\partial \xi_k^4(\tau,t)}{\partial t}. \quad (3.27)$$

Using (3.25)–(3.27), we can write formula (3.24) as follows:

$$u_{\tau\tau t} + 2u_\tau u_{\tau t} =$$
$$= b(t)\left(u_{\tau\tau\tau\tau t} + 2u_{\tau t} u_{\tau\tau\tau} + 2u_\tau u_{\tau\tau\tau t} - 4u_\tau^2 u_{\tau\tau t} - 8u_\tau u_{\tau\tau} u_{\tau t} - 8u_\tau^3 u_{\tau t}\right) -$$
$$- b(t)\left(4u_\tau u_{\tau\tau} u_{\tau t} + 2u_{\tau t} u_{\tau\tau\tau} + 4\mu_{\tau t} u_\tau u_{\tau\tau} + 2\mu_{\tau t} u_{\tau\tau\tau}\right) + 2e^{2u} u_\tau.$$

If we take the substitution

$$z(\tau,t) = u_{\tau t}(\tau,t), \quad (3.28)$$

then, with respect to $z(\tau,t)$ we obtain a linear equation

$$z_\tau + 2u_\tau z = b(t)\left(u_{\tau\tau\tau\tau t} + 2u_\tau u_{\tau\tau\tau t} - 4u_\tau^2 u_{\tau\tau t} - 12u_\tau u_{\tau\tau} u_{\tau t} - 8u_\tau^3 u_{\tau t} -\right.$$
$$\left. -4\mu_{\tau t} u_\tau u_{\tau\tau} - 2\mu_{\tau t} u_{\tau\tau\tau}\right) + 2e^{2u} u_\tau. \quad (3.29)$$

It is easy to verify that the function

$$z(\tau,t) = b(t)\left(u_{\tau\tau\tau t} - \left(2u_\tau \mu_{\tau t}\right)_\tau\right) + c(t)\left(e^{2u} + e^{-2u}\right)$$

is a solution of the linear equation (3.29). Choosing $c(t) = \frac{1}{2}$, we have

$$z(\tau,t) = b(t)\left(u_{\tau\tau\tau t} - \left(2u_\tau \mu_{\tau t}\right)_\tau\right) + \cosh(2u).$$

From here and from the notation (3.28), when $a(t)=1$, we obtain the equation (1.1):

$$u_{\tau t}=b(t)\left(u_{\tau\tau\tau t}-(2u_\tau\mu_{\tau t})_\tau\right)+\cosh(2u).$$

The lemma is proved.

Remark 3.2. It is directly verified that if the estimates (3.19), (3.20) are satisfied, that is, $u_0(x)\in C^5(\mathbb{R})$, then the series on the right-hand side of equality (3.27) will be uniformly convergent. In addition, the uniform convergence of the series in (3.10), (3.14) and (3.23) follows from equalities (3.19), (3.20) and estimate (3.21).

Thus, we have proved the following theorem.

Theorem 3.2. *If a periodic infinite-gap function $u_0(x)$ satisfies the condition*

$$u_0(x+\pi)=u_0(x)\in C^5(\mathbb{R}),$$

and $\alpha(t),\beta(t),\zeta(t)\in C^1(t>0)\bigcap C(t\geq 0)$ are bounded functions, then there exists a uniquely determined global solution $u(x,t),\mu_x(x,t)$, $x\in\mathbb{R},t>0$, of the mixed problem (1.1)–(1.3)*, which is determined by the formulas* (3.11) *and* (3.12)*, respectively, and belongs to the smoothness class* (1.3)*.*

□

4. Conclusion.

The inverse spectral problem method is applied to integrate the nonlinear negative modified Korteweg-de Vries–cosine Gordon equation (nmKdV–coshG) in the class of periodic infinite-gap functions. The solvability of the Cauchy problem for the first and second infinite system of Dubrovin differential equations is proved in the class of three and two times continuously differentiable periodic infinite-gap functions, respectively. The solvability of the Cauchy problem for the nmKdV–coshG equation in the class of five times continuously differentiable periodic infinite-gap functions is established.

References

[1] M.J. Ablowitz, D.J. Kaup, A.C. Newell and H. Segur, Method for Solving the Sine-Gordon Equation, *Phys. Rev. Lett.* **30** (1973), 1262.

[2] N.I. Akhiezer, Continuous analogues of orthogonal polynomials on a system of intervals, *Dokl. Akad. Nauk* **141** (1961), 263–266.

[3] B.A. Babajanov and A.B. Khasanov, Integration of equation of Toda periodic chain kind, *Ufa Math. J.* **9** (2017), no. 2, 17–24.

[4] B.A. Babazhanov and A.B. Khasanov, Inverse problem for a quadratic pencil of Sturm-Liouville operators with finite-gap periodic potential on the half-line, *Differ. Equ.* **43** (2007), 737–744.

[5] D. Battig, B. Grebert, J. Guillot and T. Kappeler, Foliation of phase space for the cubic nonlinear Schrodinger equation, *Compos. Math.* **85** (1993), no. 2, 163–199.

[6] S. Currie, T.T. Roth and B.A. Watson, Borg's Periodicity Theorems for First-Order Self-Adjoint Systems with Complex Potentials, *Proc. Edinb. Math. Soc. (2)* **60** (2017), no. 3, 615–633.

[7] P.B. Djakov and B.S. Mityagin, Instability zones of periodic 1-dimensional Schrodinger and Dirac operators, *Russian Math. Surveys* **61** (2006), 663–766.
[8] P. Djakov and B. Mityagin, Instability Zones of a Periodic 1D Dirac Operator and Smoothness of its Potential, *Comm. Math. Phys.* **259** (2005), 139–183.
[9] B.A. Dubrovin, S.P. Novikov, Periodic and conditionally periodic analogs of the many-soliton solutions of the Korteweg-de Vries equation, *J. Exp. Theor. Phys.* **67** (1975), no. 6, 2131–2143.
[10] R.Kh. Eshbekov and G.A. Mannonov, Integration of a Nonlinear Hirota Type Equation With Additional Terms, *Uzbek Math. J.* **68** (2024), no. 1, 46–56.
[11] L.D. Faddeev, Properties of the S–matrix of the one-dimensional Schrodinger equation, *Proc. Steklov Inst. Math.* **65** (1967), 139–166.
[12] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian Methods in the Theory of Solitons*, Springer Berlin, Heidelberg, 2007.
[13] C.S. Gardner, J.M. Greene, M.D. Kruskal and R.M. Miura, A method for solving the Korteweg-de Vries equation, *Phys. Rev. Lett.* **19** (1967), 1095–1098.
[14] F. Gestezy and H. Holden, *Soliton Equations and Their Algebro-Geometric Solutions*, Cambridge University Press, Cambridge, 2008.
[15] B. Grebert and J.C. Guillot, Gaps of one dimensional periodic AKNS systems, *Forum Math.* **5:5** (1993), 459–504.
[16] R. Hirota, Exact envelop-soliton solutions of a nonlinear wave equation, *J. Math. Phys.* **14** (1973), 805–809.
[17] E.L. Ince, A proof of the impossibility of the coexistence of two Mathien functions, *Math. Proc. Cambridge Philos. Soc.* **21** (1922), 117–120.
[18] E.L. Ince, VII–Further Investigations into the Periodic Lame Functions, *Proc. Roy. Soc. Edinburgh Sect. A* **60** (1940), no. 1, 83–99.
[19] A.R. Its, Inversion of hyperelliptic integrals and integration of nonlinear differential equations, *Bulletin of Leningrad State University* **7** (1976), no. 2, 39–46.
[20] A.R. Its and V.P. Kotlyarov, Explicit formulas for solutions of the nonlinear Schrodinger equation, *Dokl. Akad. Nauk* **10:11** (1976), 965–968.
[21] A.R. Its and V.B. Matveev, Schrodinger operators with finite-gap spectrum and N–soliton solutions of the Korteweg-de Vries equation, *Theoret. and Math. Phys.* **23** (1975), 343–355.
[22] J. Jie, J.B. Zhang, and D.J. Zhang, Soliton Solutions for a Negative Order AKNS Equation Hierarchy, *Commun. Theor. Phys. (Beijing)* **52** (2009), no. 3, 395–397.
[23] A.V. Jiber, R.D. Murtozina, I.T. Khabibullin and A.B. Shabat, *Characteristic Lie rings and nonlinear integrable equations*, Nauka, Moscow, 2012 [in Russian].
[24] A.B. Khasanov, A.A. Abdivokhidov and R.Kh. Eshbekov, Negative order modified Korteweg–de Vries–Liouville (nmKdV-L) Equation in the Class of Periodic Infinite-gap Functions, *Lobachevskii J. Math.* **45** (2024), no. 12, 6380-–6397.
[25] A.B. Khasanov and T.Z. Allanazarova, On the Modified Korteweg–De-Vries Equation with Loaded Term, *Ukrainian Math. J.* **73** (2022), 1783–1809.
[26] A.B. Khasanov, R.Kh. Eshbekov and T.G. Khasanov, Integration of a Non-Linear Hirota Type Equation with Additional Terms, *Izv. Ross. Akad. Nauk Ser. Mat.* **89** (2025), no. 1, 196–219.
[27] A.B. Khasanov, R.Kh. Eshbekov and Kh.N. Normurodov, Integration of a nonlinear Hirota type equation with finite density in the class of periodic functions, *Lobachevskii J. Math.* **44** (2023), no. 10, 4329–4347.
[28] Khasanov A. B. and Hoitmetov U. A., Integration of the loaded Korteweg-de Vries equation with a self-consistent source in the class of rapidly decreasing complex-valued functions, *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* **42** (2022), no. 4, 87–101.

[29] A.B. Khasanov and T.G. Khasanov, Integration of a Nonlinear Korteweg–de Vries Equation with a Loaded Term and a Source, *J. Appl. Ind. Math.* **16** (2022), no. 2, 227–239.

[30] A.B. Khasanov and T.G. Khasanov, Cauchy Problem for the Loaded Korteweg–de Vries Equation in the Class of Periodic Functions, *Differ. Equ.* **59** (2023), 1678–1690.

[31] A.B. Khasanov and U.O. Khudayorov, Integration of the Modified Korteweg–de Vries–Liouville Equation in the Class of Periodic Infinite-Gap Functions, *Math. Notes* **114** (2023), 1247–1259.

[32] A. Khasanov, U. Khudayarov and T. Khasanov, The Cauchy Problem for the Modified Korteeg-de Vries–Liouville(mKdV–L) Equation in the class of periodic infinite-gap functions, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **50** (2024), no. 1, 78–95.

[33] A.B. Khasanov and G.U. Urazboev, On the Sine-Gordon Equation with a Self-Consistent Source, *Siberian Adv. Math.* **19** (2009), 13–23.

[34] A.B. Khasanov, A.B. Yakhshimuratov, The Almost-Periodicity of infinite-Gap Potensials of the Dirac operator *Dokl. Math.* **54** (1996), 767–769.

[35] K. Konno, W. Kameyama and H. Sanuki, Effect of Weak Dislocation Potential on Nonlinear Wave Propagation in Anharmonic Crystal, *J. Phys. Soc. Jpn.* **37** (1974), no. 1, 171–176.

[36] E. Korotyaev, Inverse Problem and Estimates for Periodic-Shabat Systems, *J. fur Reine Angew. Math.* **2005** (2005), no. 583, 87–115.

[37] E. Korotyaev and D. Mokeev, Dubrovin Equation for Periodic Dirac Operator on the Half-Line, *Appl. Anal.* **101** (2022), no. 1, 337–365.

[38] P. D. Lax, Integrals of Nonlinear Equations of Evolution and Solitary Waves, *Comm. Pure Appl. Math* **21** (1968), no. 5, 467–490.

[39] P.D. Lax, Almost Periodic Solutions of the KdV Equation, *SIAM Rev.* **18** (1976), no. 3, 351–375.

[40] B.M. Levitan, Almost Periodicity of infinite-zone potentials, *Mathematics of the USSR-Izvestiya* **45** (1981) 291–320.

[41] B.M. Levitan, *Inverse Sturm-Liouville Problems*, De Gruyter, Berlin, 1987.

[42] A. Makin, On Periodic and Antiperiodic Spectra of Non-Self-Adjoint Dirac Operators, *J. Spectr. Theory* **12** (2022), no. 4, 1567–1588.

[43] G.A. Mannonov and A.B. Khasanov, Cauchy Problem for the Nonlinear Hirota Equation in the Class of Periodic Infinite-Zone Functions, *St. Petersburg Math. J.* **34** (2023), 821–845.

[44] V.A. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhauser Basel, Basel, 1986.

[45] V.B. Matveev, 30 Years of Finite-Gap Integration Theory, *Philos. Trans. Roy. Soc. A* **366** (2008), 837–875.

[46] V.B. Matveev and A.O. Smirnov, Solutions of the Ablowitz–Kaup–Newell–Segur Hierarchy Equations of the "Rogue Wave" Type: A Unified Approach, *Theoret. and Math. Phys.* **186** (2016), 156–182.

[47] V.B. Matveev and A.O. Smirnov, Two-Phase Periodic Solutions to the AKNS Hierarchy Equations, *J. Math. Sci.* **242** (2019), 722–741.

[48] T.V. Misura, Characteristics of the spectra of periodic and antiperiodic boundary value problems generated by the Dirac operation I, *Theory of function, functional analysis and their applications* **30** (1978), 90–101.

[49] I.M. Nabiev, The inverse periodic problem for the Dirac operator, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **19** (2003), 177-–180.

[50] I.M. Nabiev, Solution of the inverse quasiperiodic problem for the Dirac system, *Math. Notes* **89** (2011), no. 6, 845-–852.

[51] S.P. Popov, Nonautonomous Soliton Solutions of the Modified Korteweg-de Vries–Sine Gordon Equation, *Comput. Math. Math. Phys.* **56** (2016), 1929–1937.

[52] A.O. Smirnov, Elliptic solutions of the nonlinear Schrodinger equation and the modified Korteweg-de Vries equation, *Sbornik: Mathematics* **82** (1995), 461–470.

[53] G.U. Urazboev, A.B. Yakhshimuratov and M.M. Khasanov, Integration of Negative-Order Modified Korteweg–de Vries Equation in a Class of Periodic Functions, *Theoret. and Math. Phys.* **217** (2023), 1689–1699.

[54] M. Wadati, The Exact Solution of the Modified Korteweg-de Vries Equation, *J. Phys. Soc. Jpn.* **32** (1972), no. 6, 1681–1681.

[55] A.M. Wazwaz, N–Soliton Solutions for the Integrable Modified KdV–Sine–Gordon Equation, *Phys. Scr.* **89** (2014), no. 6, 1–5.

[56] V.E. Zakharov, S.V. Manakov, S.P. Novikov and L.P. Pitaevsky, *Inverse problem method*, Nauka, Moscow, 1980 [in Russian].

[57] V.E. Zakharov and A.B. Shabat, Exact Theory of Two-Dimensional Self-Focusing and One-Dimensional Self-Modulation of Waves in Nonlinear Media, *JETP* **34** (1970), 62–69.

[58] V.E. Zakharov, L.A. Takhtadzhyan and L.D. Faddeev, Complete description of solutions of the "sine-Gordon" equation, *Dokl. Akad. Nauk* **219** (1974), no. 6, 1334–1337.

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