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GLOBAL STABILIZATION OF HYPERBOLIC FITZHUGH-NAGUMO EQUATIONS

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Abstract. The present paper is concerned with the internal stabilization of the zero steady-state solutions to hyperbolic FitzHugh-Nagumo equations via internal finite-dimensional feedback controllers involving finitely many Fourier modes and finitely many volume elements.

1. Introduction

We consider the problem of global stabilization of the hyperbolic FitzHugh-Nagumo equations

$$\begin{cases} \tau\partial_t^2 u + \partial_t u - \partial_x u + f(u) + v = w, \ x \in (0,L), t > 0, \\ \partial_t v + dv - bu = 0, \ x \in (0,L), t > 0, \\ u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x), v(x,0) = v_0(x), \ x \in (0,L), \\ u(x,t) = 0, \ x \in \partial G, t > 0, \end{cases} \tag{1.1}$$

where u and v are unknown functions, w is the control input, $f : \mathbb{R} \to \mathbb{R}$ is a given continuously differentiable function that satisfies the conditions

$$f(0) = 0, \quad f(s)s - \mathcal{F}(s) \geq -r_1 s^2, \ \forall s \in \mathbb{R}, \tag{1.2}$$

$$a_1|s|^{p+2} - r_2 s^2 \leq \mathcal{F}(s) \leq a_2|s|^{p+2} + r_3 s^2, \ \forall s \in \mathbb{R}, \tag{1.3}$$

where a_1, a_2, r_1, r_2, r_3 are given positive numbers, $p \geq 2$ and $\mathcal{F}(s) = \int_0^s f(y)dy$. Our aim is to stabilize the zero equilibrium $\{0,0\}$ with finite dimensional controllers w.

The original FitzHugh-Nagumo system

$$\begin{cases} \partial_t u - \partial_x^2 u + f(u) + v = 0, \\ \partial_t v + dv - bu = 0, \end{cases} \tag{1.4}$$

where u and v are unknown functions, a, b, d, α, β are positive parameters and $f(u) = u(\alpha - u)(\beta - u)$, introduced by FitzHugh [8] and Nagumo et al. [18] is a simplification of the Hodjgin-Huxley model [10] in neurobiology describing the process of transmission of an impulse along an axon.

The FitzHugh-Nagumo model has been widely adopted not only in neuroscience

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but also in the study of other systems, such as cardiac rhythms and chemical reactions.

There are many works devoted to mathematical analysis of FitzHugh-Nagumo equations where the authors discussed the questions of global solvability and long time behavior of solutions of the Cauchy problem and initial boundary value problems for the system (1.4) and its generalizations (see, e.g., [12], [16], [20], [2] and references therein).

Some papers are devoted to the problem of stabilization of FitzHugh-Nagumo equations. The authors of [6] used the backstepping method to establish local exponential stabilization of the FitzHugh-Nagumo equations by boundary feedback. In [22], the authors proved that the FitzHugh-Nagumo system can be exponentially stabilized by a feedback controller acting on the subdomain in the reaction-diffusion equation.

Several papers are devoted to mathematical analysis of the hyperbolic FitzHug-Nagumo equations (i.e. the equations taking into account the effects of relaxation) (see, e.g., [21], [14], [19],[4], [7], [11], [9], [13], [17], [23]). In these papers the authors studied the problems of existence of solitary waves, existence and uniqueness of solution to the Cauchy problem and initial boundary value problems, existence of bounded solutions of considered systems, the asymptotic spatial behavior of solutions and some other qualitative properties of the system.

Few works are devoted to the problem of stabilization hyperbolic FitzHugh-Nagumo equations. In [1] it is shown that the hyperbolic FitzHugh-Nagumo system can be exponentially stabilized by a feedback controller acting on subdomain.

Our main goal in this short note is to show that the hyperbolic FitzHugh-Nagumo system also can be exponentially stabilized by a feedback controller depending on finitely many Fourier modes and controllers depending on finitely many volume elements.

Throughout this paper, we will use the following notations:
$Q_T = G \times (0,T)$; $L^p(G), 1 \le p \le \infty$, and $H^s(G),\ s>0$, are the usual Lebesgue and Sobolev spaces, respectively. With $(\cdot,\cdot)$ and $\|\cdot\|$ we denote the inner product and norm of $L^2(G)$.

We will need below the following inequalities.

Young's inequality:

$$ab \le \frac{\varepsilon}{p}a^p + \frac{1}{q\varepsilon^{1/(p-1)}}b^q, \text{ for all } a,b,\varepsilon>0, \text{ with } q=p/(p-1), 1<p<\infty. \tag{1.5}$$

Interpolation inequality:

$$\|u_x\|^2 \le \|u\|\|u_{xx}\|, \quad \forall u \in H^2(0,L)\cap H_0^1(0,L). \tag{1.6}$$

Poincaré type inequality:

$$\left\| v - \sum_{k=1}^{N}(v,w_k)^2 \right\|^2 \le \lambda_{N+1}^{-1}\|\nabla v\|^2, \quad \forall v \in H^2(0,L)\cap H_0^1(0,L), \tag{1.7}$$

where w_k are eigenfunctions of the problem

$$\begin{cases} -w''(x) = \lambda w(x), \ x \in G, \\ w(0,t) = w(L,t) = 0, \ t > 0, \end{cases} \tag{1.8}$$

corresponding to eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

Sobolev inequality:

$$\|u\|_{L^p(G)} \le c_L \|u'\|, \quad \forall u \in H_0^1(0,L), \tag{1.9}$$

where $p > 0$ is arbitrary number, c_L that depends on L.

Finally let us give the definition of a weak solution of the problem (1.1).

Definition 1.1. A pair of functions $[u, v]$ is called a weak solution of the problem(1.1) if $u \in C(0,T; H_0^1(0,L), \partial_t u \in C(0,T; L^2(0,L)), \ v \in C(0,T; L^2(0,L)),$ $\partial_t v \in L^2(0,T; L^2(\Omega), \ \forall T > 0$, and the equations (1.1) are satisfied in the sense of distributions.

2. Stabilization employing Fourier modes

To study the stabilization of the system, following [3], we apply the feedback controller involving the first N Fourier modes of the function $u(x,t)$

$$w = -\mu \sum_{k=1}^{N} (u, w_k) w_k \tag{2.1}$$

where $\mu > 0$ is the control parameter,

$$\tau \partial_t^2 u + \partial_t u - \partial_x^2 u + f(u) + v = -\mu \sum_{k=1}^{N} (u, w_k) w_k, \ x \in (0,L), t > 0, \tag{2.2}$$

$$\partial_t v + dv - bu = 0, \ x \in (0,L), t > 0, \tag{2.3}$$

$$u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x), v(x,0) = v_0(x), \ x \in (0,L), \tag{2.4}$$

$$u(0,t) = u(L,t) = 0, \ t > 0. \tag{2.5}$$

First, we multiply the equation (2.2) by $\partial_t u + \varepsilon u$, where $\varepsilon > 0$ is a parameter to be chosen below. Integrate the obtained relation over the interval $(0,L)$ and after some operations obtain the equality

$$\frac{d}{dt}\left[E_\varepsilon(t) + \frac{\mu}{2}\sum_{k=1}^{N}(u(t), w_k)^2\right] + (1 - \varepsilon\tau)\|\partial_t u(t)\|^2 + \varepsilon\|\partial_x u(t)\|^2 + \varepsilon(f(u), u)$$
$$+ \varepsilon(v, u) - (\partial_t v, u) = -\mu\varepsilon \sum_{k=1}^{N} (u(t), w_k)^2, \tag{2.6}$$

where

$$E_\varepsilon(t) := \frac{\tau}{2}\|\partial_t u\|^2 + \frac{1}{2}\|\partial_x u(t)\|^2 + (\mathcal{F}(u), 1) + \frac{\varepsilon}{2}\|u(t)\|^2 + \varepsilon\tau(\partial_t u, u) + (u, v). \tag{2.7}$$

Next we multiply the equation (2.3) by $\partial_t v + \frac{\varepsilon}{b} v$ and obtain the equality

$$\|\partial_t v\|^2 + (\frac{d}{2} + \frac{\varepsilon}{2b})\frac{d}{dt}\|v(t)\|^2 - b(\partial_t v, u) + \frac{\varepsilon d}{b}\|v\|^2 - \varepsilon(u,v) = 0 \tag{2.8}$$

Adding (2.6) and (2.8) then using the inequality

$$(1+b)|(\partial_t v, u)| \le \frac{1}{2}\|\partial_t v\|^2 + \frac{1}{2}(1+b)^2\|u\|^2$$

we get

$$\frac{d}{dt}\Phi_\varepsilon(t) + (1-\varepsilon\tau)\|\partial_t u(t)\|^2 + \varepsilon\|\partial_x u(t)\|^2 + \varepsilon(f(u),u) + \frac{1}{2}\|\partial_t v\|^2 + \frac{d\varepsilon}{b}\|v\|^2 - \frac{1}{2}(1+b)^2\|u\|^2$$
$$\le -\mu\varepsilon\sum_{k=1}^{N}(u(t),w_k)^2, \tag{2.9}$$

where

$$\Phi_\varepsilon(t) := E_\varepsilon(t) + \frac{\mu}{2}\sum_{k=1}^{N}(u(t),w_k)^2 + (\frac{d}{2} + \frac{\varepsilon}{2b})\|v(t)\|^2$$

Employing the inequalities

$$\varepsilon\tau(\partial_t u, u) \ge -\frac{\tau}{4}\|\partial_t u\|^2 - \tau\varepsilon^2\|u\|^2,$$

$$(u,v) \ge -\frac{d}{4}\|v\|^2 - \frac{1}{d}\|u\|^2$$

and the the condition (1.3) we get

$$\Phi_\varepsilon(t) \ge \frac{\tau}{4}\|\partial_t u\|^2 + \frac{1}{2}\|\partial_x u\|^2 + (\frac{\varepsilon}{2} - r_2 - \tau\varepsilon^2 - \frac{1}{d})\|u\|^2 + (\frac{d}{4} + \frac{\varepsilon}{2b})\|v\|^2 + \frac{\mu}{2}\|P_N u\|^2. \tag{2.10}$$

Next we utilize the Poincaré type inequality (1.7) and get

$$(\frac{\varepsilon}{2} - r_2 - \tau\varepsilon^2 - \frac{1}{d})\|u\|^2 \le (\frac{\varepsilon}{2} - r_2 - \tau\varepsilon^2 - \frac{1}{d})\|P_N u\|^2 + |\frac{\varepsilon}{2} - r_2 - \tau\varepsilon^2 - \frac{1}{d}|\lambda_{N+1}^{-1}\|\partial_x u\|^2.$$

By using the last estimate we deduce form (2.10) that if

$$\left|\frac{\varepsilon}{2} - r_2 - \tau\varepsilon^2 - \frac{1}{d}\right|\lambda_{N+1}^{-1} \le \frac{1}{4} \quad \text{and} \quad \mu \ge 2r_2 + 2\tau\varepsilon^2 + \frac{2}{d} - \varepsilon, \tag{2.11}$$

then

$$\Phi_\varepsilon(t) \ge \frac{\tau}{4}\|\partial_t u\|^2 + \frac{1}{4}\|\partial_x u\|^2 + (\frac{d}{4} + \frac{\varepsilon}{2b})\|v\|^2. \tag{2.12}$$

Adding to the left-hand side of (2.9), $\delta\Phi_\varepsilon(t) - \delta\Phi_\varepsilon(t)$ with some $\delta \in (0,\varepsilon)$ and utilising the inequalities

$$\delta\varepsilon\tau|(\partial_t u, u)| \le \frac{\delta\varepsilon\tau}{2}\|\partial_t u\|^2 + \frac{\delta\varepsilon\tau}{2\lambda_1}\|\partial_x u\|^2,$$

$$\delta|(u,v)| \le \frac{\delta}{2\lambda_1}\|\partial_x u\|^2 + \frac{\delta}{2}\|v\|^2.$$

we can rewrite it in the following form.

$$\frac{d}{dt}\Phi_\varepsilon(t)+\delta\Phi_\varepsilon(t)+(1-\varepsilon\tau-\frac{\delta\varepsilon\tau}{2})\|\partial_t u\|^2+(\varepsilon-\frac{\delta}{2}-\frac{\delta\varepsilon\tau}{2\lambda_1}-\frac{\delta}{2\lambda_1})\|\partial_x u\|^2$$
$$+\varepsilon\left[(f(u),u)-(\mathcal{F}(u),1)\right]+(\varepsilon-\delta)(\mathcal{F}(u),1)-\frac{1}{2}(1+b)^2\|u\|^2$$
$$+\left[\frac{d\varepsilon}{b}-\frac{\delta}{2}-\delta(\frac{d}{2}+\frac{\varepsilon}{2b})\right]\|v\|^2+\frac{1}{2}\|\partial_t v\|^2+\mu(\varepsilon-\frac{\delta}{2})\|P_N u\|^2\le 0. \quad (2.13)$$

If $\varepsilon=\min\left\{\frac{1}{2\tau},\frac{b}{2d}\right\}$ and $\delta>0$ is small enough we obtain from (2.13) that

$$\frac{d}{dt}\Phi_\varepsilon(t)+\delta\Phi_\varepsilon(t)+\frac{1}{4\tau}\|\partial_x u\|^2+\frac{1}{2\tau}\left[(f(u),u)-(\mathcal{F}(u),1)\right]$$
$$+(\frac{1}{2\tau}-\delta)(\mathcal{F}(u),1)-\frac{1}{2}(1+b)^2\|u\|^2+\mu(\frac{1}{2\tau}-\frac{\delta}{2})\|P_N u\|^2\le 0.$$

According to the conditions (1.2) and (1.3)

$$\frac{1}{2\tau}\left[(f(u),u)-(\mathcal{F}(u),1)\right]+(\frac{1}{2\tau}-\delta)(\mathcal{F}(u),1)\ge-\frac{1}{2\tau}(r_1+r_2)\|u\|^2.$$

Thus we have

$$\frac{d}{dt}\Phi_\varepsilon(t)+\delta\Phi_\varepsilon(t)+\frac{1}{4\tau}\|\partial_x u\|^2-\frac{1}{2\tau}\left[r_1+r_2+\tau(1+b)^2\right]\|u\|^2+\mu(\frac{1}{2\tau}-\frac{\delta}{2})\|P_N u\|^2\le 0.$$

Finally employing the inequality (1.7) we obtain

$$\frac{d}{dt}\Phi_\varepsilon(t)+\delta\Phi_\varepsilon(t)+\left[\frac{1}{4\tau}-\frac{1}{2\tau\lambda_{N+1}}(r_1+r_2+\tau(1+b)^2)\right]\|\partial_x u\|^2$$
$$+\left[\frac{\mu}{4\tau}-\frac{1}{2\tau}(r_1+r_2+\tau(1+b)^2)\right]\|P_N u\|^2\le 0.$$

Thus, if N and μ are so large that

$$\lambda_{N+1}\ge 2(r_1+r_2+\tau(1+b)^2), \quad (2.14)$$

and

$$\mu\ge 2(r_1+r_2+\tau(1+b)^2), \quad (2.15)$$

then

$$\frac{d}{dt}\Phi_\varepsilon(t)+\delta\Phi_\varepsilon(t)\le 0. \quad (2.16)$$

This inequality (2.16) implies that

$$\Phi_\varepsilon(t)\le e^{-\delta t}\Phi_\varepsilon(0).$$

Thanks to uniform estimates of $\|u_t(t)\|^2$, $\|u_x(t)\|^2$, $\|v(t)\|^2$ we can use the standard Faedo-Galerkin method to prove global existence and uniqueness of solution of the problem (see, e.g., [12], [16]). So we proved the following theorem.

Theorem 2.1. *If μ and N are so large that the conditions* (2.14) *and* (2.15) *are satisfied then for given $u_0\in H_0^1(0,L)$ and $u_1\in L^2(0,L)$ then the problem has a unique weak solution for which the following estimate holds true*

$$\frac{\tau}{4}\|\partial_t u(t)\|^2+\frac{1}{4}\|\partial_x u(t)\|^2+\frac{b}{8\tau b}\|v(t)\|^2\le\Phi_\varepsilon(0)e^{-\delta t},\ t>0. \quad (2.17)$$

2.1. Stabilization of strong solutions. In this section we prove stabilization to the zero stationary state of the strong solution of the problem.

Definition 2.1. A pair of functions $[u, v]$ is called a strong solution of the problem (2.2) -(2.5) if $u \in C(0,T; H^2(0,L)\cap H_0^1(0,L))$ such that $v, \partial_t u \in C(0,T; H_0^1(0;L))$, $\partial_t v \in L^2(0,T; H_0^1(0,l))$, $\forall T \in (0,\infty)$ and the system (2.2),(2.3) is satisfied the sense of distributions.

The main result of this section is the following theorem.

Theorem 2.2. *Suppose the initial data satisfy the conditions*

$$u_0 \in H^2(0,L) \cap H_0^1(0,L),\ \ u_1, v_0 \in H_0^1(0,L), \tag{2.18}$$

and the nonlinear term $f(u)$ satisfies the conditions of the Theorem 2.1. Then

$$\|\partial_{xt}u(t)\|^2 + \|\partial_x^2 u(t)\|^2 + \|\partial_x v(t)\|^2 \le C_0 e^{-r_0 t},\ \ t > t_0. \tag{2.19}$$

where $C_0 > 0$ depends on $\|u_0'\|, \|u_1'\|, \|v_0'\|$ and the parameters of the system.

Proof. First we multiply the equation (2.2) by $\partial_t\partial_x^2 u(t)$ and integrate over the interval $(0,L)$

$$\frac{d}{dt}\left[\frac{\tau}{2}\|\partial_{xt}u(t)\|^2 + \frac{1}{2}\|\partial_x^2 u(t)\|^2 + \frac{\mu}{2}\sum_{k=1}^N \lambda_k (u(t), w_k)^2\right] \\ + (f'(u(t))\partial_x u(t), \partial_{xt}u(t)) + \|\partial_{xt}u(t)\|^2 + (\partial_x v(t), \partial_{xt}u(t)) = 0 \tag{2.20}$$

Utilizing the inequalities

$$|(f'(u(t))\partial_x u(t), \partial_{xt}u(t))| \le \frac{1}{4}\|\partial_{xt}u(t)\|^2 + \|(f'(u(t))\partial_x u(t)\|^2$$

and

$$|(\partial_x v(t), \partial_{xt}u(t))| \le \frac{1}{4}\|\partial_{xt}u(t)\|^2 + \|\partial_x v(t)\|^2$$

we obtain form (2.20) the estimate

$$\frac{d}{dt}\left[\tau\|\partial_{xt}u(t)\|^2 + \|\partial_x^2 u(t)\|^2 + \mu\sum_{k=1}^N \lambda_k (u(t), w_k)^2\right] + \|\partial_{xt}u(t)\|^2 \\ \le 2\|f'(u(t))\partial_x u(t)\|^2 + 2\|\partial_x v(t)\|^2. \tag{2.21}$$

Next we multiply the equation (2.2) by $-\varepsilon\partial_x^2 u(t)$ with some $\varepsilon > 0$ that will be determined below:

$$\frac{d}{dt}\left[-\varepsilon\tau(\partial_t u(t), \partial_x^2 u(t)) + \frac{\varepsilon}{2}\|\partial_x u(t)\|^2\right] - \varepsilon\tau\|\partial_{xt}u(t)\|^2 + \varepsilon\|\partial_x^2 u(t)\|^2 \\ + \varepsilon(f'(u(t)), |\partial_x u(t)|^2) + \varepsilon(\partial_x v(t), \partial_x u(t)) = -\varepsilon\mu\sum_{k=1}^N \lambda_k (u(t), w_k)^2. \tag{2.22}$$

□

Adding (2.21) and (2.22) we obtin

$$\frac{d}{dt}L_\varepsilon(t)+(1-\varepsilon\tau)\|\partial_{xt}u(t)\|^2+\varepsilon\|\partial_x^2u(t)\|^2\le 2\|f'(u(t))\partial_xu(t)\|^2+2\|\partial_xv(t)\|^2$$
$$-\varepsilon(f'(u(t)),|\partial_xu(t)|^2)-\varepsilon(\partial_xv(t),\partial_xu(t))-\varepsilon\mu\sum_{k=1}^N\lambda_k(u(t),w_k)^2, \quad (2.23)$$

where

$$L_\varepsilon(t):=\tau\|\partial_{xt}u(t)\|^2+\|\partial_x^2u(t)\|^2+\mu\sum_{k=1}^N\lambda_k(u(t),w_k)^2$$
$$+\frac{\varepsilon}{2}\|\partial_xu(t)\|^2-\varepsilon\tau(\partial_tu(t),\partial_x^2u(t)).$$

Utilizing the inequality

$$\varepsilon\tau|(\partial_tu(t),\partial_x^2u(t))|\le\frac{\tau}{2}\|\partial_{xt}u(t)\|+\frac{\varepsilon^2\tau}{2\lambda_1}\|\partial_x^2u(t)\|$$

and choosing

$$\varepsilon\le\sqrt{\frac{\lambda_1}{\tau}} \quad (2.24)$$

we deduce that

$$L_\varepsilon(t)\ge\frac{\tau}{2}\|\partial_{xt}u(t)\|^2+\frac{1}{2}\|\partial_x^2u(t)\|^2+\mu\sum_{k=1}^N\lambda_k(u(t),w_k)^2+\frac{\varepsilon}{2}\|\partial_xu(t)\|^2. \quad (2.25)$$

By using the fact that $f'(s)$ is continuous on $\mathbb{R}$, the Sobolev inequality

$$\|u\|_{L^\infty(0,L)}\le C_0\|\partial_xu\|, \quad u\in H_0^1(0;L), \quad (2.26)$$

and the estimate (2.17), we can find $t_1>0$ and $M_0>0$ such that $\|f'(u)\|_{L^\infty(0,L)}\le M_0,\ \forall t\ge t_1$. Therefore by choosing $\varepsilon=\varepsilon_0:=\min\left\{\frac{1}{2\tau},\sqrt{\frac{\lambda_1}{\tau}}\right\}$ we can get from (2.23) the following inequality

$$\frac{d}{dt}L_{\varepsilon_0}(t)+\frac{1}{2}\|\partial_{xt}u(t)\|^2+\varepsilon_0\|\partial_x^2u(t)\|^2+\varepsilon_0\mu\sum_{k=1}^N\lambda_k(u(t),w_k)^2$$
$$\le(2M_0^2+\varepsilon_0M_0)\|\partial_xu(t)\|^2+\|\partial_xv(t)\|^2. \quad (2.27)$$

Next, taking the inner product of (2.3) with $-\partial_x^2v(t)$ in $L^2(0,L)$ we get

$$\frac{1}{2}\frac{d}{dt}\|\partial_xv(t)\|^2+d\|\partial_xv(t)\|^2=b(\partial_xu(t),\partial_xv(t))\le\frac{d}{2}\|\partial_xv(t)\|^2+\frac{b^2}{2d}\|\partial_xu(t)\|^2.$$

Hence

$$\frac{d}{dt}\|\partial_xv(t)\|^2+d\|\partial_xv(t)\|^2\le\frac{b^2}{d}\|\partial_xu(t)\|^2.$$

Finally we multiply the last inequality by $\dfrac{2}{d}$ and add to (2.27):

$$\frac{d}{dt}\left[L_{\varepsilon_0}(t)+\frac{2}{d}\|\partial_x v(t)\|^2\right]+\frac{1}{2}\|\partial_{xt}u(t)\|^2+\varepsilon_0\|\partial_x^2 u(t)\|^2$$
$$+\varepsilon_0\mu\sum_{k=1}^N\lambda_k(u(t),w_k)^2+\|\partial_x v(t)\|^2\leq(2M_0^2+\varepsilon_0 M_0+\frac{2b^2}{d^2})\|\partial_x u(t)\|^2. \quad (2.28)$$

Utilizing the inequality (1.7) we have

$$(2M_0^2+\varepsilon_0 M_0+\frac{2b^2}{d^2})\|\partial_x u(t)\|^2\leq(2M_0^2+\varepsilon_0 M_0+\frac{2b^2}{d^2})\sum_{k=1}^N\lambda_k(u(t),w_k)^2$$
$$+\lambda_{N+1}^{-1}(2M_0^2+\varepsilon_0 M_0+\frac{2b^2}{d^2})\|\partial_x^2 u(t)\|^2. \quad (2.29)$$

Due to the last inequality we can choose

$$\mu\geq\frac{2}{\varepsilon_0}(2M_0^2+\varepsilon_0 M_0+\frac{2b^2}{d^2}) \text{ and N so large that } \lambda_{N+1}\geq\frac{2}{\varepsilon_0}(2M_0^2+\varepsilon_0 M_0+\frac{2b^2}{d^2}) \quad (2.30)$$

and deduce from (2.28) the inequality

$$\frac{d}{dt}\left[L_{\varepsilon_0}(t)+\frac{2}{d}\|\partial_x v(t)\|^2\right]+\frac{1}{2}\|\partial_{xt}u(t)\|^2+\frac{\varepsilon_0}{2}\|\partial_x^2 u(t)\|^2$$
$$+\|\partial_x v(t)\|^2+\frac{1}{2}\varepsilon_0\mu\sum_{k=1}^N\lambda_k(u(t),w_k)^2\leq 0.$$

Adding to the left hand side of the last inequality the expression $\gamma\left[L_{\varepsilon_0}(t)+\frac{2}{d}\|\partial_x v(t)\|^2\right]-\gamma\left[L_{\varepsilon_0}(t)+\frac{2}{d}\|\partial_x v(t)\|^2\right]$ with some $\gamma>0$ and using the inequality

$$\gamma\varepsilon_0\tau|(\partial_t u(t),\partial_x^2 u(t))|\leq\frac{\gamma\varepsilon_0\tau}{2\lambda_1}\|\partial_x^2 u(t)\|^2+\frac{1}{2}\gamma\varepsilon_0\tau\|\partial_{tx}u(t)\|^2$$

we get

$$\frac{d}{dt}\left[L_{\varepsilon_0}(t)+\frac{2}{d}\|\partial_x v(t)\|^2\right]+\gamma\left[L_{\varepsilon_0}(t)+\frac{2}{d}\|\partial_x v(t)\|^2\right]$$
$$+(\frac{1}{2}-\gamma\tau-\frac{1}{2}\gamma\varepsilon_0\tau)\|\partial_{xt}u(t)\|^2+(\frac{\varepsilon_0}{2}-\gamma-\frac{\gamma\varepsilon_0}{2\lambda_1}-\frac{\gamma\varepsilon_0\tau}{2\lambda_1})\|\partial_x^2 u(t)\|^2$$
$$+(1-\frac{2\gamma}{d})\|\partial_x v(t)\|^2+\mu(\frac{\varepsilon_0}{2}-\gamma)\sum_{k=1}^N\lambda_k(u(t),w_k)^2\leq 0. \quad (2.31)$$

By choosing $\gamma=\min\left\{\dfrac{1}{\tau(2+\varepsilon_0)},\dfrac{d}{2},\dfrac{\varepsilon_0}{2},\varepsilon_0\lambda_1(2\lambda_1+\varepsilon_0+\varepsilon_0\tau)^{-1}\right\}$ we infer form (2.31) the inequality

$$\frac{d}{dt}\left[L_{\varepsilon_0}(t)+\frac{2}{d}\|\partial_x v(t)\|^2\right]+\gamma\left[L_{\varepsilon_0}(t)+\frac{2}{d}\|\partial_x v(t)\|^2\right]\leq 0,$$

which implies the desired estimate (2.19).

3. Stabilization employing finite volume elements feedback control

In this section we consider the following feedback control problem

$$\begin{cases} \tau\partial_t^2 u - \partial_x^2 u + \partial_t u + f(u) + v = -\mu \sum\limits_{k=1}^{N} \overline{u}_k \chi_{J_k}(x), \quad x \in (0,L),\ t > 0, \\ \partial_t v + dv - bu = 0,\ x \in (0,L),\ t > 0, \\ \partial_x u(0,t) = \partial_x u(L,t) = 0,\ t > 0, \\ u(x,0 = u_0(x), \quad \partial_t u(x,0) = u_1(x), \quad x \in (0,L). \end{cases} \tag{3.1}$$

Here $J_k := \left[(k-1)\frac{L}{N}, k\frac{L}{N}\right)$, for $k = 1, 2, \cdots N-1$ and $J_N = [\frac{N-1}{N}L, L]$, $\overline{\phi}_k := \frac{1}{|J_k|}\int\limits_{J_k} \phi(x)dx$, and $\chi_{J_k}(x)$ is the characteristic function of the interval J_k. In what follows we will need the following lemma.

Lemma 3.1. *(see* [3]*) Let $\phi \in H^1(0,L)$. Then*

$$\|\phi - \sum_{k=1}^{N} \overline{\phi}_k \chi_{J_k}(\cdot)\| \le h\|\phi_x\|, \tag{3.2}$$

and

$$\|\phi\|^2 \le h\sum_{k=1}^{N} \overline{\phi}_k^2 + \left(\frac{h}{2\pi}\right)^2 \|\phi_x\|^2, \tag{3.3}$$

where $h := \frac{L}{N}$.

By employing this lemma, we proved the following theorem:

Theorem 3.1. *Suppose that the nonlinear term $f(\cdot)$ satisfies the conditions* (1.2) *and* (1.3)*, the parameter μ is large enough and h is small enough such that*

$$\mu \ge 4\tau B(\tau) \qquad \text{and} \qquad \frac{1}{4\tau} \ge B(\tau)\frac{h^2}{4\pi^2} \tag{3.4}$$

where

$$B(\tau) := \frac{1}{2\tau}(r_1 + r_2) + \frac{(1+b)^2}{4} + \frac{b\tau}{d}.$$

Then each solution of the problem (3.1) *satisfies the following decay estimate:*

$$\|\partial_t u(t)\|^2 + \|\partial_x u(t)\|^2 \le K(\|u_1\|^2 + \|\partial_x u_0\|^2)e^{-\frac{t}{4\tau}}, \tag{3.5}$$

where K is some positive constant depending on parameters of the system.

Proof. Taking the $L^2(0,L)$ inner product of first equation in (3.1) with $\partial_t u + \varepsilon u$ obtain the equality

$$\frac{d}{dt}\left[E_\varepsilon(t) + \frac{1}{2}h\mu\sum_{k=1}^{N} \overline{u}_k^2\right] + (1-\varepsilon\tau)\|\partial_t u\|^2 + \varepsilon\|\partial_x u\|^2 + \varepsilon(f(u),u) - (u,\partial_t v)$$
$$+ \varepsilon\mu h\sum_{k=1}^{N} \overline{u}_k^2 = 0, \tag{3.6}$$

where $E_\varepsilon(t)$ is defined in (2.7).

Adding to (3.6) the equality (2.8) we obtain the relation:

$$\frac{d}{dt}Y_\varepsilon(t)+\delta Y_\varepsilon(t)+(1-\varepsilon\tau)\|\partial_t u\|^2+\varepsilon\|\partial_x u\|^2+\varepsilon(f(u),u)+\varepsilon\mu h\sum_{k=1}^N \overline{u}_k^2+\|\partial_t v\|^2$$
$$-(1+b)(\partial_t v,u)+\frac{\varepsilon d}{b}\|v\|^2-\varepsilon(u,v)-\frac{\delta\tau}{2}\|\partial_t u\|^2-\frac{\delta}{2}\|\partial_x u\|^2-\delta(F(u),1)$$
$$-\frac{\delta\varepsilon}{2}\|u\|^2+\left(\frac{d}{2}+\frac{\varepsilon}{2b}\right)\|v(t)\|^2-\delta\varepsilon\tau(\partial_t u,u)-\delta(u,v)-\frac{\delta}{2}\mu h\sum_{k=1}^N\overline{u}_k^2=0. \quad (3.7)$$

Here

$$Y_\varepsilon(t):=E_\varepsilon(t)+\left(\frac{d}{2}+\frac{\varepsilon}{2b}\right)\|v(t)\|^2+\frac{1}{2}h\mu\sum_{k=1}^N\overline{u}_k^2(t).$$

First, by using the condition (1.3), the inequalities

$$\varepsilon\tau|(\partial_t u,u)|\le\frac{\tau}{4}\|\partial_t u\|^2+\varepsilon^2\tau\|u\|^2 \text{ and } |(u,v)|\le\frac{\varepsilon}{2b}\|v\|^2+\frac{b}{2\varepsilon}\|u\|^2$$

and remembering that $\varepsilon=\frac{1}{2\tau}$ we obtain

$$Y_\varepsilon(t)\ge\frac{\tau}{4}\|\partial_t u\|^2+\frac{1}{2}\|\partial_x u\|^2-(r_2+\tau b)\|u\|^2+\frac{d}{2}\|v^2\|+\frac{\mu h}{2}\sum_{k=1}^N\overline{u}_k^2(t).$$

Then we utilize the inequality (3.3) to get the following lower bound for $Y_\varepsilon(t)$:

$$Y_\varepsilon(t)\ge\frac{\tau}{4}\|\partial_t u\|^2+\left[\frac{1}{2}-\left(\frac{h}{2\pi}\right)^2(r_2+\tau b)\right]\|\partial_x u\|^2+\frac{d}{2}\|v\|^2+\left[\frac{\mu}{2}-r_2-\tau\right]\sum_{k=1}^N\overline{u}_k^2(t).$$

Hence if

$$\mu\ge 2(r_2+\tau) \text{ and } h^2\le\frac{\pi^2}{r_2+\tau b}, \quad (3.8)$$

then

$$Y_\varepsilon(t)\ge\frac{\tau}{4}\|\partial_t u\|^2+\frac{1}{4}\|\partial_x u\|^2+\frac{d}{2}\|v\|^2, \quad \forall t>0. \quad (3.9)$$

Next employing the inequalities

$$\varepsilon\tau|\partial_t u,u)|\le\frac{\varepsilon\tau}{4}\|\partial_t u\|^2+\varepsilon\tau\|u\|^2, \quad (\delta+\varepsilon)|(u,v)|\le\frac{\varepsilon d}{b}\|v\|^2+\frac{b(\delta+\varepsilon)^2}{4\varepsilon d}\|u\|^2, \quad (3.10)$$

$$\delta\varepsilon\tau|\partial_t u,u)|\le\frac{\varepsilon\tau}{2}\|\partial_t u\|^2+\frac{1}{2}\delta^2\varepsilon\tau\|u\|^2, \quad (1+b)|(\partial_t v,u)|\le\|\partial_t v\|^2+\frac{1}{4}(1+b)^2\|u\|^2, \quad (3.11)$$

and the inequality

$$\varepsilon(f(u),u)-\delta(\mathcal{F}(u),1)\ge[\varepsilon r_1+(\varepsilon-\delta)r_2]\,\|u\|^2$$

which follows from (1.2) and (1.3), we derive from (3.7) the following inequality

$$\frac{d}{dt}Y_\varepsilon(t)+\delta Y_\varepsilon(t)+\left(1-\varepsilon\tau-\frac{\delta\tau}{2}-\frac{\delta\varepsilon\tau}{2}\right)\|\partial_t u\|^2+\left(\varepsilon-\frac{\delta}{2}-\frac{\delta\varepsilon}{2\lambda_1}-\frac{\delta}{2\lambda_1}\right)\|\partial_x u\|^2+$$
$$-\left[\varepsilon(r_1+r_2)+\frac{(1+b)^2}{4}+\frac{b}{2\varepsilon d}\right]\|u\|^2+\left[\frac{\varepsilon d}{2b}-\delta(\frac{d}{2}+\frac{\varepsilon}{2b})-\frac{\delta}{2}\right]\|v\|^2$$
$$+\mu h(\varepsilon-\frac{\delta}{2})\sum_{k=1}^N \overline{u}_k^2\leq 0.$$

By choosing $\varepsilon=\frac{1}{2\tau}$ and $\delta<\varepsilon$ small enough we obtain from the last inequality that

$$\frac{d}{dt}Y_\varepsilon(t)+\delta Y_\varepsilon(t)+\frac{1}{4\tau}\|\partial_x u\|^2-B(\tau)\|u\|^2+\frac{\mu h}{4\tau}\sum_{k=1}^N \overline{u}_k^2\leq 0, \tag{3.12}$$

where

$$B(\tau):=\frac{1}{2\tau}(r_1+r_2)+\frac{(1+b)^2}{4}+\frac{b\tau}{d}. \tag{3.13}$$

According to the inequality (3.3)

$$B(\tau)\|u\|^2\leq B(\tau)h\sum_{k=1}^N \overline{u}_k^2+B(\tau)\left(\frac{h}{2\pi}\right)^2\|\partial_x u\|^2,$$

we get from the inequality (3.13) :

$$\frac{d}{dt}Y_\varepsilon(t)+\delta Y_\varepsilon(t)+\left(\frac{1}{4\tau}-B(\tau)\left(\frac{h}{2\pi}\right)^2\right)\|\partial_x u\|^2+\left(\frac{\mu h}{4\tau}-B(\tau)h\right)\sum_{k=1}^N \overline{u}_k^2\leq 0 \tag{3.14}$$

We choose here $\mu\geq 4\tau B(\tau)$ and $\dfrac{1}{4\tau}\geq B(\tau)\dfrac{h^2}{4\pi^2}$ to obtain:

$$\frac{d}{dt}Y_\varepsilon(t)+\frac{1}{4\tau}Y_\varepsilon(t)\leq 0. \tag{3.15}$$

Integrating the last inequality we get

$$\|\partial_t u(t)\|^2+\|\partial_x u(t)\|^2\leq K(\|u_1\|^2+\|u_0'\|^2)e^{-\frac{t}{4\tau}}, \tag{3.16}$$

where K is a positive constant, depending on parameters of the system. □

Remark 3.1. Let us note that the estimates obtained above suffice to guarantee the existence and uniqueness of a unique solution to the problems (1.1) and (3.1) (see, e.g., [15]).

Remark 3.2. It is not difficult to see that the analog of the Theorem 2.1 and Theorem 2.2 hold true also for the system

$$\begin{cases} \tau\partial_t^2 u + \partial_t u - \Delta u + f(u) + v = -\mu \sum_{k=1}^{N} (u, w_k) w_k, \ x \in \Omega, t > 0, \\ \partial_t v + bv - bu = 0, \ x \in G, t > 0, \\ u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x), v(x,0) = v_0(x), \ x \in \Omega, \\ u(x,t) = 0, \ x \in \partial\Omega, t > 0, \end{cases} \tag{3.17}$$

where $\Omega \subset \mathbb{R}^N (N \leq 3)$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, the nonlinear term satisfies conditions (1.2),(1.3) with arbitrary $p \geq 2$ for $G \subset \mathbb{R}^2$ and $p \in [2,3]$ for $\Omega \subset \mathbb{R}^3$.

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