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ABSENCE OF GLOBAL SOLUTIONS OF A SYSTEM OF HIGHER ORDER SEMILINEAR EVOLUTION EQUATIONS WITH A SINGULAR POTENTIAL IN EXTERIOR DOMAIN

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Abstract. The system of evolution equations of higher order with respect to time is considered in the domain $Q'_R = \{x : |x| > R\} \times (0; +\infty)$, which includes a biharmonic operator with respect to spatial variables and a singular potential. The absence of global solutions is treated. Using the test function method, a critical exponent of the absence of global solution is found. This work is a generalization of a previously published work [6], where the case of the first order with respect to time was studied.

1. Introduction

Let $R > 0$, $B_R = \{x : |x| < R\}$, $B'_R = \{x : |x| > R\}$, $\partial B_R = \{x : |x| = R\}$, $B_{R_1,R_2} = \{x : R_1 < |x| < R_2\}$, $\bar{B}'_R = R^n \backslash B_R, Q_R = B_R \times (0; +\infty), Q'_R = B'_R \times (0; +\infty)$, $x = (x_1, ..., x_n) \in R^n$, $r = |x| = \sqrt{x_1^2 + ... + x_n^2}$.

We consider the following system of equations

$$\begin{cases} \frac{\partial^k u_1}{\partial t^k} + \Delta^2 u_1 - \frac{C_1}{|x|^4} u_1 = |x|^{\sigma_1} |u_2|^{q_1} \\ \frac{\partial^k u_2}{\partial t^k} + \Delta^2 u_2 - \frac{C_2}{|x|^4} u_2 = |x|^{\sigma_2} |u_1|^{q_2} \end{cases} \tag{1.1}$$

in the domain Q'_R with an initial condition

$$\left. \frac{\partial^{k-1} u_i}{\partial t^{k-1}} \right|_{t=0} = u_{i0}^{k-1}(x) \tag{1.2}$$

and the boundary conditions

$$\int_{\partial B_R} u_i(x,t) ds \geq 0, \quad \int_{\partial B_R} \Delta u_i(x,t) ds \leq 0, \tag{1.3}$$

where $n > 4, q_i > 1,\ k \geq 1,\ \sigma_i \in R, 0 \leq C_i < \left(\frac{n(n-4)}{4}\right)^2$, $u_{i0}^{k-1}(x) \in C(\bar{B}'_R)$, $u_{i0}^{k-1} \geq 0$, $\Delta^2 u = \Delta(\Delta u)$, $\Delta u = \sum\limits_{k=1}^{n} \frac{\partial^2 u}{\partial x_k^2}$, $i = 1, 2$.

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We will deal with the absence of global solutions. By global solution of the problem (1.1)-(1.3), we mean a pair of functions (u_1, u_2) which satisfy the relations $u_1(x,t), u_2(x,t) \in C^{4,k}_{x,t}(Q'_R) \bigcap C^{3,k-1}_{x,t}(\bar{B}'_R \times [0,+\infty))$, the system (1.1) at every point of Q'_R, the initial condition (1.2) and the boundary conditions (1.3).

The problems of absence of global solutions for different classes of differential equations and inequalities are important in theory and applications, that's why they are constantly attracting the attention of mathematicians. A lot of research has been dedicated to these problems (see [26]).

In 1966, Fujita in his famous work [16] considered the following problem:

$$\frac{\partial u}{\partial t} = \Delta u + u^q, \qquad (x,t) \in R^n \times \ (0,+\infty), \tag{1.4}$$

$$u\mid_{t=0} = u_0(x) \geq 0, \qquad x \in R^n, \tag{1.5}$$

where $q > 1$, $u_0(x)$ is a continuous bounded function. As is known, for every $q > 1$ there exists $T = T(u_0)$ such that the problem (1.4), (1.5) has a classical solution in the cylinder $R^n \times (0,T)$. The question is: what happens when $T = +\infty$? Fujita proved that if $1 < q < q_{cr} = 1 + \frac{2}{n}$, then for every u_0 there exists $T_0 = T_0(u_0) < \infty$ such that $\overline{\lim_{t \to T_0 - 0}} \sup_{R^n} u(x,t) = +\infty$. He also proved that if $q > q_{cr}$, then for sufficiently small u_0 the global solution does exist, i.e. $T = +\infty$. Later, Hayakawa [19] and Kobayashi [21] showed that in case $q = 1 + \frac{2}{n}$ the global solutions also do not exist. So there is a value of the exponent of nonlinearity such that the global solution may exist or not depending on this value. In mathematical literature, such a value is referred to as a "critical exponent of nonlinearity".

Fujita's famous work mentioned above marked the beginning of a significant development in the field of how the size of nonlinearity affects the existence or nonexistence of a global solution. Various generalizations of Fujita's work have been made. For example, R^n was replaced by different bounded or unbounded domains, the Laplace operator was replaced by other differential operators, the nonlinearity in the equation (1.4) was replaced by the one of a different type (for more detail, see [7, 13, 26, 28]).

Another generalization of Fujita's result is to explore the existence of global positive solutions to the system of semilinear Fujita-type equations. For example, M. Escobedo and M. A. Herrero[14] considered the following initial value problem:

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u + v^{q_1} \\ \frac{\partial v}{\partial t} = \Delta v + u^{q_2} \end{cases} (x,t) \in R^n \times \ (0,+\infty),$$

$$u|_{t=0} = u_0(x),\ v|_{t=0} = v_0(x), x \in R^n,$$

where $q_1 > 0$, $q_2 > 0$, $u_0(x), v_0(x)$ are continuous, bounded and non-negative functions. It was proved that if $q_1 q_2 > 1$, then for $\max\left(\frac{q_1+1}{q_1q_2-1}, \frac{q_2+1}{q_1q_2-1}\right) \geq \frac{n}{2}$ there is no non-negative global solution for any non-trivial initial data, and for $\max\left(\frac{q_1+1}{q_1q_2-1}, \frac{q_2+1}{q_1q_2-1}\right) < \frac{n}{2}$ there exist both non-trivial global solutions and non-global solutions. Besides, in case $q_1 q_2 < 1$ all non-negative solutions are global. The absence of global solutions to the system of nonlinear Fujita-type equations has been also considered in, e.g., [2, 3, 6, 11, 15, 27].

In this work, we study the system of equations of higher order with respect to time variable including a biharmonic operator with respect to spatial variables and a singular potential. The problems considered in this work have been earlier studied in [4, 5, 8, 9, 17, 22, 23, 25, 29].

Note that the semilinear stationary and non-stationary equations with a biharmonic operator in principal part have also been considered by many authors (see, e.g., [4, 6, 12, 24, 30, 31, 32]).

As for the existence or non-existence of solutions to the higher order (with respect to time) evolution inequalities, a number of works have been done in this field. For example, El Hamidi and Laptev [18] treated the non-existence of weak solutions to higher order evolution inequalities of the following form:

$$\begin{cases} \frac{\partial^k u}{\partial t^k} - \Delta u + \frac{C}{|x|^2} u \geq |u|^q & in\ R^n \times (0,\infty), \\ \frac{\partial^{k-1} u}{\partial t^{k-1}}(0,x) \geq 0 & in\ R^n, \end{cases}$$

where $n > 2, q > 1,\ C \geq -\left(\frac{n-2}{2}\right)^2$.

In [10], Caristi considered the evolution inequality of the form

$$\frac{\partial^k u}{\partial t^k} - |x|^\sigma \Delta^m u \geq |u|^q \text{ in } R^n \times (0,\infty),$$

where $n > 2, m$ is a positive integer, $q > 1,\ \sigma \leq 2m$.

In [1, 20], the existence and nonexistence of global solutions to the evolution inequality

$$\frac{\partial^k u}{\partial t^k} - \Delta(|u|^{m-1} u) + \frac{C}{|x|^2}(x \cdot \nabla(|u|^{m-1} u) \geq |x|^\alpha |u|^q$$

have been studied in $\Omega \times (0,\infty)$, where $\Omega \subset R^n$ is an exterior of a ball or a semiball, $k \geq 1,\ n \geq 2, q > m \geq 1,\ C, \sigma \in R$.

Our aim in this work is to generalize the results of [6], where the case $k = 1$ is considered, and to find an exact critical exponent of the absence of global solution for $k \geq 1$ using also the test function method developed by Mitidieri and Pohozaev in [25, 26].

2. Main result and its proof

For simplicity, we will use the following notations:

$$D_i = \sqrt{(n-2)^2 + C_i}, \quad \lambda_i^{\pm} = \sqrt{\left(\frac{n-2}{2}\right)^2 + 1 \pm D_i},$$

$$\mu_i = \frac{1}{2}\left(1 + \frac{D_i - \lambda_i^+}{\lambda_i^-}\right), \bar{\mu}_i = \frac{1}{2}\left(1 - \frac{D_i - \lambda_i^+}{\lambda_i^-}\right),$$

$$\alpha_i = \frac{\lambda_i^- + \sigma_i + \frac{n-4}{2}}{\lambda_j^- + \frac{n-4}{2}}, \beta_i = \frac{\lambda_i^- + \sigma_i + \frac{n+4}{2} + \frac{4}{k}}{\lambda_j^- + \frac{n-4}{2}}, \gamma_i = \frac{4 - \frac{4}{k}}{\lambda_i^- + \frac{n-4}{2}},$$

$$\theta_i = \frac{\sigma_i + 4 + q_i(\sigma_j + 4)}{q_1 q_2 - 1} - \lambda_i^- - \frac{n-4}{2} - \frac{4}{k},\ \ i, j = 1, 2,\ i \neq j.$$

Consider the functions

$$\xi_i(x) = \mu_i |x|^{-\frac{n-4}{2} + \lambda_i^-} + \bar{\mu}_i |x|^{-\frac{n-4}{2} - \lambda_i^-} - |x|^{-\frac{n-4}{2} - \lambda_i^+},\ \ i = 1, 2.$$

It is easy to see that the function $\xi_i(x)$ is a solution of the equation

$$\Delta^2 u - \frac{C_i}{|x|^4} u = 0 \tag{2.1}$$

in $R^n \backslash \{0\}$ and

$$\xi_i = 0, \quad \frac{\partial \xi_i}{\partial r} = D_i \geq 0, \Delta \xi_i = 0, \ \frac{\partial (\Delta \xi_i)}{\partial r} = D_i \leq 0 \tag{2.2}$$

for $|x| = 1$.

Our main result is the following theorem:

Theorem 2.1. *Let* $n > 4, q_i > 1, \ k \geq 1, \ \sigma_i \in R, 0 \leq C_i < \left(\frac{n(n-4)}{4}\right)^2, \ i = 1, 2$ *and the following conditions be satisfied:*

(1) $q_1 > \max(1, \alpha_1), q_2 > \max(1, \alpha_2)$ *and* $\max(\theta_1, \theta_2) \geq 0$,
(2) $k = 1; \alpha_2 > 1, \ q_2 \leq \alpha_2, (q_1, q_2) \in (1, \beta_1] \times (1, \alpha_2] \backslash \{\beta_1, \alpha_2\}$ *and* $\alpha_1 > 1, \ q_1 \leq \alpha_1, (q_1, q_2) \in (1, \alpha_1] \times (1, \beta_2] \backslash \{\alpha_1, \beta_2\}$,
(3) $k > 1; \alpha_2 > 1, \ q_2 \leq \alpha_2, q_1 > 1, \ q_2 \in \left[\max\left(1, \frac{1}{\gamma_2}\left(1 - \frac{\beta_1}{q_1}\right)\right), \ \alpha_2\right] \backslash \{1\}$ *for* $\alpha_2\gamma_2 \geq 1$, *and* $q_1 \in \left(1, \frac{\beta_1}{1 - \alpha_2\gamma_2}\right), \ q_2 \in \left[\max\left(1, \frac{1}{\gamma_2}\left(1 - \frac{\beta_1}{q_1}\right)\right), \ \alpha_2\right] \backslash \{1\}$ *for* $\alpha_2\gamma_2 < 1$,
(4) $k > 1; \alpha_1 > 1, \ q_1 \leq \alpha_1 \ q_2 > 1, \ q_1 \in \left[\max\left(1, \frac{1}{\gamma_1}\left(1 - \frac{\beta_2}{q_2}\right)\right), \ \alpha_1\right] \backslash \{1\}$ *for* $\alpha_1\gamma_1 \geq 1$, *and* $q_2 \in \left(1, \frac{\beta_2}{1 - \alpha_1\gamma_1}\right), \ q_1 \in \left[\max\left(1, \frac{1}{\gamma_1}(1 - \frac{\beta_2}{q_2}\right), \ \alpha_1\right] \backslash \{1\}$ *for* $\alpha_1\gamma_1 < 1$.
If $(u_1(x,t), u_2(x,t))$ *is a solution of the problem* (1.1)-(1.3), *then* $u_1 \equiv 0, u_2 \equiv 0$.

Proof. Let $(u_1(x,t), u_2(x,t))$ be the solution of the problem (1.1)-(1.3). Consider the following functions:

$$\varphi(x) = \begin{cases} 1, & if \ \ 1 \leq |x| \leq \rho, \\ \left(\frac{1}{2}\cos\pi\left(\frac{|x|}{\rho} - 1\right) + \frac{1}{2}\right)^{\kappa}, & if \ \ \rho \leq |x| \leq 2\rho, \\ 0, & if \ \ |x| \geq 2\rho, \end{cases}$$

$$T_\rho(t) = \begin{cases} 1, & if \ \ 1 \leq t \leq \rho^{\frac{4}{k}}, \\ \left(\frac{1}{2}\cos\pi\left(\rho^{-\frac{4}{k}}t - 1\right) + \frac{1}{2}\right)^{\gamma}, & if \ \ \rho^{\frac{4}{k}} \leq t \leq 2\rho^{\frac{4}{k}}, \\ 0, & if \ \ t \geq 2\rho^{\frac{4}{k}}, \end{cases}$$

where κ, γ are sufficiently large positive constants, and κ is such that for $|x| = 2\rho$

$$\varphi = \frac{\partial \varphi}{\partial r} = \frac{\partial^2 \varphi}{\partial r^2} = \frac{\partial^3 \varphi}{\partial r^3} = 0. \tag{2.3}$$

For simplicity, assume $R = 1$. Let's multiply the first equation by $\psi_1(x,t) = T_\rho(t)\xi_1(x)\varphi(x)$, and the second one by $\psi_2(x,t) = T_\rho(t)\xi_2(x)\varphi(x)$, and then integrate over Q_1'. After integration by parts, we obtain:

$$\iint_{Q_1'} |x|^{\sigma_i} |u_j|^{q_i} T_\rho(t)\xi_i(x)\varphi(x)dxdt = (-1)^k \iint_{Q_1'} u_i\xi_i\varphi\frac{d^kT_\rho}{dt^k}dxdt +$$

$$+\iint_{Q_1'} u_i T_\rho \Delta^2(\xi_i\varphi)dxdt - \iint_{Q_1'} \frac{C_i}{|x|^4} u_i T_\rho \xi_i \varphi dxdt - \int_{B_1'} u_{i0}^{k-1}(x)\xi_i(x)\varphi(x)dx+$$

$$+\int_0^{2\rho^{\frac{4}{k}}} T_\rho(t) \left[\int_{\partial B_{1,2\rho}} \frac{\partial(\Delta u_i)}{\partial \nu}\xi_i\varphi ds - \int_{\partial B_{1,2\rho}} \Delta u_i \frac{\partial(\xi_i\varphi)}{\partial\nu} ds+\right.$$

$$\left.+\int_{\partial B_{1,2\rho}} \frac{\partial u_i}{\partial \nu}\Delta(\xi_i\varphi)ds - \int_{\partial B_{1,2\rho}} u_i \frac{\partial}{\partial\nu}\Delta(\xi_i\varphi)ds\right] dt\,, \qquad (2.4)$$

where ν is an outward unit normal vector to $\partial B_{1,2\rho}$, $i,j=1,2$, $i\neq j$.

To avoid repetition, in what follows we will assume $i,j=1,2$, $i\neq j$, and we will use the same constant C in all formulas, although in reality C is different in different formulas.

Using (2.2), (2.3), let's estimate the integrals inside square brackets in (2.4):

$$\int_{\partial B_{1,2\rho}} \frac{\partial(\Delta u_i)}{\partial \nu}\xi_i\varphi ds = 0,$$

$$-\int_{\partial B_{1,2\rho}} \Delta u_i \frac{\partial(\xi_i\varphi)}{\partial\nu} ds = -\int_{|x|=1} \Delta u_i \frac{\partial(\xi_i\varphi)}{\partial\nu} ds =$$

$$= \int_{|x|=1} \Delta u_i \left(\frac{\partial\xi_i}{\partial r}\varphi + \xi_i\frac{\partial\varphi}{\partial r}\right) ds = \int_{|x|=1} \Delta u_i \frac{\partial\xi_i}{\partial r} ds \leq 0,$$

$$\int_{\partial B_{1,2\rho}} \frac{\partial u_i}{\partial \nu}\Delta(\xi_i\varphi)ds = \int_{\partial B_{1,2\rho}} \frac{\partial u_i}{\partial \nu}(\Delta\xi_i\varphi + 2(\nabla\xi_i,\nabla\varphi) + \xi_i\Delta\varphi)ds =$$

$$= -\int_{|x|=1} \frac{\partial u_i}{\partial r}\Delta\xi_i ds = 0,$$

$$-\int_{\partial B_{1,2\rho}} u_i \frac{\partial}{\partial\nu}(\Delta(\xi_i\varphi))ds = -\int_{|x|=1} u_i \frac{\partial}{\partial\nu}(\Delta(\xi_i\varphi))ds =$$

$$= \int_{|x|=1} u_i \frac{\partial(\Delta\xi_i)}{\partial r} ds \leq 0.$$

Since

$$\int_{B_1'} u_{i0}^{k-1}(x)\xi_i(x)\varphi(x)dx \geq 0 \text{ and } \int_0^\infty T_\rho(t)dt \geq 0,$$

taking into account that ξ_i are the solutions of (2.1) and using the estimates obtained from (2.4), we obtain

$$\iint_{Q_1'} |x|^{\sigma_i}|u_j|^{q_i} T_\rho(t)\xi_i(x)\varphi(x)dxdt \leq (-1)^k \iint_{Q_1'} u_i\xi_i\varphi\frac{d^kT_\rho}{dt^k}dxdt+$$

$$+\iint_{Q_1'} u_i T_\rho \Delta^2(\xi_i\varphi)dxdt - \iint_{Q_1'} \frac{C_i}{|x|^4} u_i T_\rho \xi_i\varphi dxdt =$$

$$= (-1)^k \iint_{Q_1'} u_i\xi_i\varphi\frac{d^kT_\rho}{dt^k}dxdt + \iint_{Q_1'} u_i T_\rho\varphi\left(\Delta^2\xi_i - \frac{C_i}{|x|^4}\xi_i\right)dxdt+$$

$$+\iint_{Q_1'} u_i T_\rho\left[4(\nabla(\Delta\xi_i),\nabla\varphi) + 4(\nabla\xi_i,\nabla(\Delta\varphi)+\right.$$

$$+2\Delta\xi_i\Delta\varphi+\ 4\sum_{k,m=1}^{n}\frac{\partial^2\xi_i}{\partial x_k\partial x_m}\frac{\partial^2\varphi}{\partial x_k\partial x_m}\Bigg]dxdt\le$$

$$\le(-1)^k\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}}\int_{B_1'}u_i\xi_i\varphi\frac{d^kT_\rho}{dt^k}dxdt+\int_0^{2\rho^{\frac{4}{k}}}\int_{B_{\rho,2\rho}}u_iT_\rho H_i(\xi_i,\varphi)dxdt, \qquad (2.5)$$

where $H(\xi_i,\varphi)$ means the expression inside square brackets, i.e.

$$H(\xi_i,\varphi)=4(\nabla(\Delta\xi_i),\nabla\varphi)+4(\nabla\xi_i,\nabla(\Delta\varphi)+2\Delta\xi_i\Delta\varphi+4\sum_{l,s=1}^{n}\frac{\partial^2\xi_i}{\partial x_l\partial x_s}\frac{\partial^2\varphi}{\partial x_l\partial x_s}.$$

Now, Using Hölder's inequality, let's estimate the right-hand side of (2.5). Then we obtain

$$\iint_{Q_1'}|x|^{\sigma_i}|u_j|^{q_i}T_\rho\xi_i\varphi dxdt\le$$

$$\le\left(\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}}\int_{B_1'}|x|^{\sigma_j}|u_i|^{q_j}T_\rho\xi_j\varphi dxdt\right)^{\frac{1}{q_j}}\left(\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}}\int_{B_1'}\frac{\left|\frac{d^kT_\rho}{dt^k}\right|^{q_j'}\xi_i^{q_j'}}{T_\rho^{q_j'-1}|x|^{\sigma_j(q_j'-1)}\xi_j^{q_j'-1}}dxdt\right)^{\frac{1}{q_j'}}+$$

$$+\left(\int_0^{2\rho^{\frac{4}{k}}}\int_{B_{\rho,2\rho}}|x|^{\sigma_j}|u_i|^{q_j}T_\rho\xi_j\varphi dxdt\right)^{\frac{1}{q_j}}\times$$

$$\times\left(\int_0^{2\rho^{\frac{4}{k}}}\int_{B_{\rho,2\rho}}\frac{|H_i(\xi_i,\varphi)|^{q_j'}T_\rho}{|x|^{\sigma_j(q_j'-1)}\xi_j^{q_j'-1}\varphi^{q_j'-1}}dxdt\right)^{\frac{1}{q_j'}}, \qquad (2.6)$$

where $\frac{1}{q_j}+\frac{1}{q_j'}=1$.
Denote the second integral in the first term by I_i, and the second one in the second term by J_i. From (2.6) we obtain:

$$\iint_{Q_1'}|x|^{\sigma_1}|u_2|^{q_1}T_\rho\xi_1\varphi dxdt\le$$

$$\le\left(\iint_{Q_1'}|x|^{\sigma_2}|u_1|^{q_2}T_\rho\xi_2\varphi dxdt\right)^{\frac{1}{q_2}}\left[I_1^{\frac{1}{q_2'}}+J_1^{\frac{1}{q_2'}}\right], \qquad (2.7)$$

$$\iint_{Q_1'}|x|^{\sigma_2}|u_1|^{q_2}T_\rho\xi_2\varphi dxdt\le$$

$$\le\left(\iint_{Q_1'}|x|^{\sigma_1}|u_2|^{q_1}T_\rho\xi_1\varphi dxdt\right)^{\frac{1}{q_1}}\left[I_2^{\frac{1}{q_1'}}+J_2^{\frac{1}{q_1'}}\right]. \qquad (2.8)$$

Considering these inequalities and using (2.6) again, we obtain:

$$\iint_{Q_1'}|x|^{\sigma_1}|u_2|^{q_1}T_\rho\xi_1\varphi dxdt\le\left[\left(\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}}\int_{B_1'}|x|^{\sigma_1}|u_2|^{q_1}T_\rho\xi_1\varphi dxdt\right)^{\frac{1}{q_1}}I_2^{\frac{1}{q_1'}}+\right.$$

$$+\left(\int_0^{2\rho^{\frac{4}{k}}}\int_{B_{\rho,2\rho}}|x|^{\sigma_1}|u_2|^{q_1}T_\rho\xi_1\varphi dxdt\right)^{\frac{1}{q_1}}J_2^{\frac{1}{q_1'}}\Bigg]^{\frac{1}{q_2}}\left[I_1^{\frac{1}{q_2'}}+J_1^{\frac{1}{q_2'}}\right], \tag{2.9}$$

$$\iint_{Q_1'}|x|^{\sigma_2}|u_1|^{q_2}T_\rho\xi_2\varphi dxdt\le\Bigg[\left(\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}}\int_{B_1'}|x|^{\sigma_2}|u_1|^{q_2}T_\rho\xi_2\varphi dxdt\right)^{\frac{1}{q_2}}I_1^{\frac{1}{q_2'}}+$$

$$+\left(\int_0^{2\rho^{\frac{4}{k}}}\int_{B_{\rho,2\rho}}|x|^{\sigma_2}|u_1|^{q_2}T_\rho\xi_2\varphi dxdt\right)^{\frac{1}{q_2}}J_1^{\frac{1}{q_2'}}\Bigg]^{\frac{1}{q_1}}\left[I_2^{\frac{1}{q_1'}}+J_2^{\frac{1}{q_1'}}\right]. \tag{2.10}$$

Substituting (2.8) in (2.7) and (2.7) in (2.8), we have

$$\iint_{Q_1'}|x|^{\sigma_1}|u_2|^{q_1}T_\rho\xi_1\varphi dxdt\le$$

$$\le\left(\iint_{Q_1'}|x|^{\sigma_1}|u_2|^{q_1}T_\rho\xi_1\varphi dxdt\right)^{\frac{1}{q_1q_2}}\left[I_1^{\frac{1}{q_2'}}+J_1^{\frac{1}{q_2'}}\right]\left[I_2^{\frac{1}{q_1'}}+J_2^{\frac{1}{q_1'}}\right]^{\frac{1}{q_2}}\times$$

$$\times\iint_{Q_1'}|x|^{\sigma_2}|u_1|^{q_2}T_\rho\xi_2\varphi dxdt\le$$

$$\le\left(\iint_{Q_1'}|x|^{\sigma_2}|u_1|^{q_2}T_\rho\xi_2\varphi dxdt\right)^{\frac{1}{q_1q_2}}\left[I_2^{\frac{1}{q_1'}}+J_2^{\frac{1}{q_1'}}\right]\left[I_1^{\frac{1}{q_2'}}+J_1^{\frac{1}{q_2'}}\right]^{\frac{1}{q_1}}.$$

Then

$$\iint_{Q_1'}|x|^{\sigma_1}|u_2|^{q_1}T_\rho\xi_1\varphi dxdt\le\left[I_1^{\frac{1}{q_2'}}+J_1^{\frac{1}{q_2'}}\right]^{\frac{q_1q_2}{q_1q_2-1}}\left[I_2^{\frac{1}{q_1'}}+J_2^{\frac{1}{q_1'}}\right]^{\frac{q_1}{q_1q_2-1}}, \tag{2.11}$$

$$\iint_{Q_1'}|x|^{\sigma_2}|u_1|^{q_2}T_\rho\xi_2\varphi dxdt\le\left[I_2^{\frac{1}{q_1'}}+J_2^{\frac{1}{q_1'}}\right]^{\frac{q_1q_2}{q_1q_2-1}}\left[I_1^{\frac{1}{q_2'}}+J_1^{\frac{1}{q_2'}}\right]^{\frac{q_2}{q_1q_2-1}}. \tag{2.12}$$

Making the changes

$$t=\rho^{\frac{4}{k}}\tau,\quad \tau=\rho s,\quad x=\rho y,\quad \tilde{T}(\tau)=T_\rho(\rho^{\frac{4}{k}}\tau),$$
$$\tilde{\xi}_i(y)=\xi_i(\rho y),\quad \tilde{\varphi}(y)=\varphi(\rho y),$$

let's estimate the right-hand sides of (2.11) and (2.12).
Let's first estimate the integrals $I_i,\ i=1,2$:

$$I_i=\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}}\int_{B_1'}\frac{\left|\frac{d^kT_\rho}{dt^k}\right|^{q_j'}\xi_i^{q_j'}\varphi}{T_\rho^{q_j'-1}|x|^{\sigma_j(q_j'-1)}\xi_j^{q_j'-1}}dxdt\le$$

$$\le\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}}\frac{\left|\frac{d^kT_\rho}{dt^k}\right|^{q_j'}}{T_\rho^{q_j'-1}}dt\int_{B_1'}\frac{\xi_i^{q_j'}}{|x|^{\sigma_j(q_j'-1)}\xi_j^{q_j'-1}}dx\le$$

$$\leq C\rho^{-4q'_j+\frac{4}{k}}\int_1^2\frac{\left|\frac{d^k\tilde{T}}{d\tau^k}\right|^{q'_j}}{\tilde{T}_\rho^{q'_j-1}}d\tau\int_{B'_1}\frac{\xi_i^{q'_j}}{|x|^{\sigma_j(q'_j-1)}\,\xi_j^{q'_j-1}}dx\leq$$

$$\leq C\rho^{-4q'_j+\frac{4}{k}}\tilde{I}_j(\tilde{T})\int_{B'_1}\frac{\xi_i^{q'_j}}{|x|^{\sigma_j(q'_j-1)}\,\xi_j^{q'_j-1}}dx, \tag{2.13}$$

where $\tilde{I}_j(\tilde{T})=\int_1^2\frac{\left|\frac{d^k\tilde{T}}{d\tau^k}\right|^{q'_j}}{\tilde{T}_\rho^{q'_j-1}}d\tau$.

Estimating the last integral in (2.13) as in [6], we obtain:

$$\int_{B'_1}\frac{\xi_i^{q'_j}}{|x|^{\sigma_j(q'_j-1)}\,\xi_j^{q'_j-1}}dx\leq C\int_1^{2\rho}r^{\lambda_i^-q'_j-\lambda_j^-(q'_j-1)-\sigma_j(q'_j-1)+\frac{n+4}{2}-1}dr=$$

$$=C\int_1^{2\rho}r^{\frac{q'_j}{q_j}(\lambda_i^-q_j-\lambda_j^--\sigma_j+\frac{n+4}{2}(q_j-1))-1}dr\leq C\begin{cases}\rho^{\frac{q'_j}{q_j}\eta_i}, & if\quad \eta_i>0\\ \ln(2\rho), & if\quad \eta_i=0\\ 1\,, & if\quad \eta_i<0,\end{cases} \tag{2.14}$$

where $\eta_i=\lambda_i^-q_j-\lambda_j^--\sigma_j+\frac{n+4}{2}(q_j-1)$.

By (2.14), from (2.13) we have

$$I_i\leq C\begin{cases}\tilde{I}_j(\tilde{T})\rho^{-4q'_j+\frac{4}{k}+\frac{q'_j}{q_j}\eta_i}, & if\quad \eta_i>0\\ \tilde{I}_j(\tilde{T})\ln(2\rho)\rho^{-4q'_j+\frac{4}{k}}, & if\quad \eta_i=0\\ \tilde{I}_j(\tilde{T})\,\rho^{-4q'_j+\frac{4}{k}}\,, & if\quad \eta_i<0\,.\end{cases} \tag{2.15}$$

Using (2.13) and the estimates for each term $H_i(\xi_i,\varphi)$ obtained in [6], let's estimate $J_i,\ i=1,2$.

$$J_i=\int_0^{2\rho^{\frac{4}{k}}}\int_{B_{\rho,2\rho}}\frac{|H_i(\xi_i,\varphi)|^{q'_j}\,T_\rho}{|x|^{\sigma_j(q'_j-1)}\,\xi_j^{q'_j-1}\,\varphi^{q'_j-1}}dxdt\leq$$

$$\leq\int_0^{2\rho^{\frac{4}{k}}}T_\rho dt\int_{B_{\rho,2\rho}}\frac{|H_i(\xi_i,\varphi)|^{q'_j}}{|x|^{\sigma_j(q'_j-1)}\,\xi_j^{q'_j-1}\,\varphi^{q'_j-1}}dx\leq$$

$$\leq C\rho^{\left(-\frac{n-4}{2}+\lambda_i^--4\right)q'_j-\sigma_j(q'_j-1)-\left(-\frac{n-4}{2}+\lambda_j^-\right)(q'_j-1)+n+\frac{4}{k}}\times$$

$$\times\int_1^2\frac{\left(\left|\frac{d^3\tilde{\varphi}}{ds^3}\right|+\left|\frac{d\tilde{\varphi}}{ds^2}\right|+\left|\frac{d\tilde{\varphi}}{ds}\right|\right)^{q'_j}}{s^{\sigma_j(q'_j-1)}\tilde{\varphi}^{q'_j-1}}ds\leq$$

$$\leq C\rho^{-4q'_j+\frac{4}{k}+\lambda_i^-q'_j-\lambda_j^-(q'_j-1)-\sigma_j(q'_j-1)+\frac{n+4}{2}}\tilde{J}_j(\tilde{\varphi})=C\rho^{-4q'_j+\frac{4}{k}+\frac{q'_j}{q_j}\eta_i}\tilde{J}_j(\tilde{\varphi}), \tag{2.16}$$

where $\tilde{J}_j(\tilde{\varphi})$ means the last integral.

Now, using (2.15), (2.16), let's estimate the right-hand sides of (2.11), (2.12). As is known, the integrals $\tilde{I}_j(\tilde{T}),\tilde{J}_j(\tilde{\varphi})$ are bounded for large values of k and γ (see [26]). Depending on the sign of η_i, let's consider different cases.

I. Assume $\alpha_1>1,\alpha_2>1$ and consider the following cases:

(a) $\eta_1 \le 0,\ \eta_2 \le 0$. This means $q_1 \le \alpha_1,\ q_2 \le \alpha_2$. Then, in view of (2.15), (2.16), from (2.11), (2.12) we obtain

$$\int_{Q_1'} \int |x|^{\sigma_i} |u_j|^{q_i} T_\rho \xi_i \varphi dx dt \le$$

$$\le C\rho^{-\frac{4q_i}{q_1q_2-1}(q_j+1)+\frac{4}{k}} \left[f_i^{\frac{1}{q_j'}} \tilde{I}_i^{\frac{1}{q_j'}} + \tilde{J}_i^{\frac{1}{q_j'}} \right]^{\frac{q_1q_2}{q_1q_2-1}} \left[f_j^{\frac{1}{q_i'}} \tilde{I}_j^{\frac{1}{q_i'}} + \tilde{J}_j^{\frac{1}{q_i'}} \right]^{\frac{q_i}{q_1q_2-1}} \le$$

$$\le C\rho^{-\frac{q_1q_2(4-\frac{4}{k})+4q_i+\frac{4}{k}}{q_1q_2-1}} \left[f_i^{\frac{1}{q_j'}} \tilde{I}_i^{\frac{1}{q_j'}} + \tilde{I}_i^{\frac{1}{q_j'}} \right]^{\frac{q_1q_2}{q_1q_2-1}} \left[f_j^{\frac{1}{q_i'}} \tilde{I}_j^{\frac{1}{q_i'}} + \tilde{I}_j^{\frac{1}{q_i'}} \right]^{\frac{q_i}{q_1q_2-1}},$$

where

$$f_i(\rho) = \begin{cases} 1, & if\ \eta_i < 0, \\ \ln(2\rho), & if\ \eta_i = 0. \end{cases}$$

Making ρ tend to $+\infty$, we have

$$\int_{Q_1'} \int |x|^{\sigma_i} |u_j|^{q_i} T_\rho \xi_i \varphi dx dt \le 0.$$

Then $u_1 \equiv 0,\ u_2 \equiv 0$.

(b) Now let $\eta_1 > 0,\ \eta_2 > 0$. This means $q_1 > \alpha_1,\ q_2 > \alpha_2$. Using (2.15), (2.16) again, from (2.11), (2.12) we obtain

$$\iint_{Q_1'} |x|^{\sigma_i} |u_j|^{q_i} T_\rho \xi_i \varphi dx dt \le$$

$$\le C\rho^{\left(-4q_j'+\frac{4}{k}+\frac{q_j'}{q_j}\eta_i\right)\frac{1}{q_j'}\frac{q_iq_j}{q_1q_2-1}} \rho^{\left(-4q_i'+\frac{4}{k}+\frac{q_i'}{q_i}\eta_j\right)\frac{1}{q_i'}\frac{q_i}{q_1q_2-1}} \times$$

$$\times \left[I_j^{\frac{1}{q_j'}}(\tilde{T}) + \tilde{J}_j^{\frac{1}{q_j'}}(\tilde{T}) \right]^{\frac{q_1q_2}{q_1q_2-1}} \left[\tilde{I}_i^{\frac{1}{q_i'}}(\tilde{T}) + \tilde{J}_i^{\frac{1}{q_i'}}(\tilde{T}) \right]^{\frac{q_1}{q_1q_2-1}} \le$$

$$\le C\rho^{\left(-4q_j+\frac{4}{k}\frac{q_j}{q_j'}+\eta_i\right)\frac{q_i}{q_1q_2-1}+\left(-4q_i+\frac{4}{k}\frac{q_i}{q_i'}+\eta_j\right)\frac{1}{q_1q_2-1}} =$$

$$= C\rho^{\frac{1}{q_1q_2-1}\left[-4q_iq_j-4q_i+\frac{4}{k}(q_j-1)q_i+\frac{4}{k}(q_i-1)+\eta_iq_j+\eta_j\right]} =$$

$$= C\rho^{\frac{1}{q_1q_2-1}\left[-4(q_1q_2-1)-4(q_i+1)+\frac{4}{k}(q_1q_2-1)+\eta_iq_j+\eta_j\right]} =$$

$$= C\rho^{\frac{1}{q_1q_2-1}\left[-(4-\frac{4}{k})(q_1q_2-1)-4(q_i+1)+\lambda_i^-q_iq_j-\sigma_jq_i+\frac{n+4}{2}q_iq_j-\lambda_i^--\sigma_i-\frac{n+4}{2}\right]} =$$

$$= C\rho^{-\frac{1}{q_1q_2-1}\left[(4-\frac{4}{k})(q_1q_2-1)+4(q_i+1)-\lambda_i^-(q_1q_2-1)+\sigma_i+\sigma_jq_i-\frac{n+4}{2}(q_iq_j-1)\right]} =$$

$$= C\rho^{-\frac{1}{q_1q_2-1}\left[(4-\frac{4}{k}-\frac{n+4}{2}-\lambda_i^-)(q_1q_2-1)+\sigma_i+4+q_i(\sigma_j+4)\right]} =$$

$$= C\rho^{-\left[\frac{\sigma_i+4+q_i(\sigma_j+4)}{q_1q_2-1}-\lambda_i^--\frac{n-4}{2}-\frac{4}{k}\right]} = C\rho^{-\theta_i}. \tag{2.17}$$

Assume $\max(\theta_1, \theta_2) > 0$. For definiteness, let $\theta_1 > 0$. Then for $i = 1$ it follows from (2.17) that

$$\iint_{Q_1'} |x|^{\sigma_1} |u_2|^{q_1} T_\rho \xi_1 \varphi dx dt \le C\rho^{-\theta_1}.$$

Making ρ tend to $+\infty$, we have

$$\iint_{Q_1'} |x|^{\sigma_1} |u_2|^{q_1} T_\rho \xi_1 dxdt \le 0.$$

So $u_2 \equiv 0$. Then it follows from the second equation of the system that $u_1 \equiv 0$. For $\theta_2 > 0$, we similarly obtain $u_1 \equiv 0, u_2 \equiv 0$. Now let $\max(\theta_1, \theta_2) = 0$. Consider, for example, $\theta_1 = 0$. Then it follows from (2.17) that

$$\iint_{Q_1'} |x|^{\sigma_1} |u_2|^{q_1} T_\rho \xi_1 dxdt \le C.$$

From the properties of integral it follows

$$\int_0^\infty \int_{B_{\rho,2\rho}} |x|^{\sigma_1} |u_2|^{q_1} \xi_1 dxdt \to 0, \tag{2.18}$$

$$\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}} \int_{B_1'} |x|^{\sigma_1} |u_2|^{q_1} \xi_1 dxdt \to 0. \tag{2.19}$$

Then, taking into account (2.18) and (2.19), from (2.9) we obtain

$$\iint_{Q_1'} |x|^{\sigma_1} |u_2|^{q_1} T_\rho \xi_1 \varphi dxdt \le \left[\left(\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}} \int_{B_1'} |x|^{\sigma_1} |u_2|^{q_1} \xi_1 \varphi dxdt \right)^{\frac{1}{q_1}} I_2^{\frac{1}{q_1'}} + \right.$$

$$\left. + \left(\int_0^\infty \int_{B_1'} |x|^{\sigma_1} |u_2|^{q_1} \xi_1 \varphi dxdt \right)^{\frac{1}{q_1}} J_2^{\frac{1}{q_1'}} \right]^{\frac{1}{q_2}} \left[I_1^{\frac{1}{q_2'}} + J_1^{\frac{1}{q_2'}} \right] \le$$

$$\le C\rho^{\left(-4q_1' + \frac{4}{k} + \frac{q_1'}{q_1}\eta_2\right)\frac{1}{q_1' q_2} + \left(-4q_2' + \frac{4}{k} + \frac{q_2'}{q_2}\eta_1\right)\frac{1}{q_2'}} \times$$

$$\times \left[\left(\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}} \int_{B_1'} |x|^{\sigma_1} |u_2|^{q_1} \xi_1 \varphi dxdt \right)^{\frac{1}{q_1'}} \tilde{I}_1^{\frac{1}{q_1'}} + \right.$$

$$\left. + \left(\int_0^\infty \int_{B_{\rho,2\rho}} |x|^{\sigma_1} |u_2|^{q_1} \xi_1 \varphi dxdt \right)^{\frac{1}{q_1}} \tilde{J}_2^{\frac{1}{q_1'}} \right]^{\frac{1}{q_2}} \left[\tilde{I}_2^{\frac{1}{q_2'}} + \tilde{J}_2^{\frac{1}{q_2'}} \right] \le$$

$$\le C\rho^{\frac{1}{q_1 q_2}\left(-4q_1(q_2+1) + \frac{4}{k}(q_1 q_2 - 1) + \eta_2 + q_1\eta_1\right)} \times$$

$$\times \left[\left(\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}} \int_{B_1'} |x|^{\sigma_1} |u_2|^{q_1} \xi_1 \varphi dxdt \right)^{\frac{1}{q_1'}} \tilde{I}_1^{\frac{1}{q_1'}} + \right.$$

$$\left. + \left(\int_0^\infty \int_{B_{\rho,2\rho}} |x|^{\sigma_1} |u_2|^{q_1} \xi_1 \varphi dxdt \right)^{\frac{1}{q_1}} \tilde{J}_2^{\frac{1}{q_1'}} \right]^{\frac{1}{q_2}} \left[\tilde{I}_2^{\frac{1}{q_2'}} + \tilde{J}_2^{\frac{1}{q_2'}} \right] \le$$

$$\leq C\rho^{-\frac{q_1q_2-1}{q_1q_2}\theta_1}\left[\left(\left(\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}}\int_{B_1'}|x|^{\sigma_1}|u_2|^{q_1}\xi_1\varphi dxdt\right)^{\frac{1}{q_1}}\tilde{I}_1^{\frac{1}{q_1'}}+\right.\right.$$

$$\left.+\left(\int_0^{\infty}\int_{B_{\rho,2\rho}}|x|^{\sigma_1}|u_2|^{q_1}\xi_1\varphi dxdt\right)^{\frac{1}{q_1}}\tilde{J}_2^{\frac{1}{q_1'}}\right]^{\frac{1}{q_2}}\left[\tilde{I}_2^{\frac{1}{q_2'}}+\tilde{J}_2^{\frac{1}{q_2'}}\right]\leq$$

$$\leq C\left[\left(\left(\int_{\rho^{\frac{4}{k}}}^{2\rho^{\frac{4}{k}}}\int_{B_1'}|x|^{\sigma_1}|u_2|^{q_1}\xi_1\varphi dxdt\right)^{\frac{1}{q_1}}\tilde{I}_1^{\frac{1}{q_1'}}+\right.\right.$$

$$\left.+\left(\int_0^{\infty}\int_{B_{\rho,2\rho}}|x|^{\sigma_1}|u_2|^{q_1}\xi_1\varphi dxdt\right)^{\frac{1}{q_1}}\tilde{J}_2^{\frac{1}{q_1'}}\right]^{\frac{1}{q_2}}\left[\tilde{I}_2^{\frac{1}{q_2'}}+\tilde{J}_2^{\frac{1}{q_2'}}\right]\to 0.$$

So, we again have

$$\iint_{Q_1'}|x|^{\sigma_1}|u_2|^{q_1}\xi_1\varphi dxdt\leq 0.$$

Then $u_2\equiv 0$, and, consequently, $u_1\equiv 0$. If $\theta_2=0$, we can similarly obtain $u_1\equiv 0, u_2\equiv 0$.

(c) Now let's consider the cases where $\eta_i\leq 0,\ \eta_j>0$. Let first $\eta_1\leq 0,\ \eta_2>0$. This means $q_1>\alpha_1,\ q_2\leq\alpha_2$. Then from (2.11), (2.12) it follows that

$$\iint_{Q_1'}|x|^{\sigma_1}|u_2|^{q_1}\xi_1\varphi dxdt\leq$$

$$\leq C\rho^{-\frac{1}{q_1q_2-1}\left[\left(4-\frac{4}{k}\right)(q_1q_2-1)+4(q_1+1)-\eta_2\right]}\times$$

$$\times\left[f_1^{\frac{1}{q_2'}}\tilde{I}_2^{\frac{1}{q_2'}}+\tilde{J}_2^{\frac{1}{q_2'}}\right]^{\frac{q_1q_2}{q_1q_2-1}}\left[\tilde{I}_1^{\frac{1}{q_1'}}+\tilde{J}_1^{\frac{1}{q_1'}}\right]^{\frac{q_1}{q_1q_2-1}}. \tag{2.20}$$

So, if $\eta_1\leq 0, \eta_2<\left(4-\frac{4}{k}\right)(q_1q_2-1)+4(q_1+1)$, then, making ρ tend to $+\infty$, from (2.20) we have

$$\iint_{Q_1'}|x|^{\sigma_1}|u_2|^{q_1}\xi_1\varphi dxdt\leq 0$$

and, consequently, $u_1\equiv 0, u_2\equiv 0$. If $\eta_1<0, \eta_2=\left(4-\frac{4}{k}\right)(q_1q_2-1)+4(q_1+1)$, then from (2.20) it follows that

$$\iint_{Q_1'}|x|^{\sigma_1}|u_2|^{q_1}\xi_1\varphi dxdt\leq C.$$

Then, as in the case (b), we can show that $u_1\equiv 0,\ u_2\equiv 0$.

Note that the conditions $\eta_1<0, 0<\eta_2\leq\left(4-\frac{4}{k}\right)(q_1q_2-1)+4(q_1+1)$ are equivalent to the conditions $q_2<\alpha_2,\ \alpha_1<q_1\leq\gamma_2q_1q_2+\beta_1$, and the conditions $\eta_1=0,\ 0<\eta_2<\left(4-\frac{4}{k}\right)(q_1q_2-1)+4(q_1+1)$ are equivalent to the conditions $q_2=\alpha_2,\ \alpha_1<q_1<\gamma_2q_1q_2+\beta_1$. Since $\gamma_2=0$ for $k=1$, it is obvious that in this case the solution does not exist if $(q_1,q_2)\in(1,\beta_1]\times(1,\alpha_2]\backslash\{\beta_1,\alpha_2\}$. In case $k>1$, let's rewrite the inequality $q_1\leq\gamma_2q_1q_2+\beta_1$ as $q_2\geq\frac{1}{\gamma_2}-\frac{\beta_1}{\gamma_2}\frac{1}{q_1}=\frac{1}{\gamma_2}\left(1-\frac{\beta_1}{q_1}\right)$. Since $q_2>1$, for $q_1\leq\beta_1$ this inequality always holds, and, therefore, for $(q_1,q_2)\in(1,\beta_1]\times(1,\alpha_2]$ the solution also does not exist. If $q_1>\beta_1$, we have to separately consider the cases $\gamma_2<1$, $\gamma_2\geq 1$. In case $\gamma_2\geq 1$, we have $\frac{1}{\gamma_2}\left(1-\frac{\beta_1}{q_1}\right)<1$ for every $q_1>\beta_1$, and therefore there is no solution for $q_1>\alpha_1$, $1<q_2\leq\alpha_2$. If $\gamma_2<1$, then for $1\leq\gamma_2\alpha_2$ the solution does not exist when $\alpha_1<q_1\leq\frac{\beta_1}{1-\gamma_2}$, $1<q_2\leq\alpha_2$ and $q_1>\frac{\beta_1}{1-\gamma_2}$, $\frac{1}{\gamma_2}\left(1-\frac{\beta_1}{q_1}\right)\leq q_2\leq\alpha_2$, and for $1>\gamma_2\alpha_2$ the solution does not exist when $\alpha_1<q_1\leq\frac{\beta_1}{1-\gamma_2}$, $1<q_2\leq\alpha_2$ and $\frac{\beta_1}{1-\gamma_2}<q_1<\frac{\beta_1}{1-\alpha_2\gamma_2}$, $\frac{1}{\gamma_2}\left(1-\frac{\beta_1}{q_1}\right)\leq q_2\leq\alpha_2$.

Now let $\eta_1>0$,$\eta_2\leq 0$. Then, similar to the previous case, we come to conclusion that for $\eta_2<0$, $0<\eta_1\leq\left(4-\frac{4}{k}\right)(q_1q_2-1)+4(q_2+1)$ and for $\eta_2=0$, $0<\eta_1<\left(4-\frac{4}{k}\right)(q_1q_2-1)+4(q_2+1)$ the solution is identically zero.

Rewrite the conditions $\eta_2<0$,$0<\eta_1\leq\left(4-\frac{4}{k}\right)(q_1q_2-1)+4(q_2+1)$ in the form $q_1<\alpha_1$, $\alpha_2<q_2\leq\gamma_1q_1q_2+\beta_2$, and the conditions $\eta_2=0$,$0<\eta_1<\left(4-\frac{4}{k}\right)(q_1q_2-1)+4(q_2+1)$ in the form $q_1=\alpha_1,\alpha_2<q_2\leq\gamma_1q_1q_2+\beta_2$. Then, for $k=1$ the solution does not exist if $(q_1,q_2)\in(1,\alpha_1]\times(1,\beta_2]\backslash\{\alpha_1,\beta_2\}$. In case $k>1$, having rewritten the inequality $q_2\leq\gamma_1q_1q_2+\beta_2$ as $q_1\geq\frac{1}{\gamma_1}\left(1-\frac{\beta_2}{q_2}\right)$, we also come to conclusion that the solution does not exist when $(q_1,q_2)\in(1,\alpha_1]\times(1,\beta_2]$. And if $q_2>\beta_2$, then, similar to the previous, we can assert that for $\gamma_1>1$ with $q_2>\alpha_2$, $1<q_1\leq\alpha_1$, and for $\gamma_1<1$ with $\alpha_2<q_2\leq\frac{\beta_2}{1-\gamma_1}$,$1<q_1\leq\alpha_1$ and $q_2>\frac{\beta_2}{1-\gamma_1}$,$\frac{1}{\gamma_1}\left(1-\frac{\beta_2}{q_2}\right)\leq q_1\leq\alpha_1$ in case $1\leq\gamma_1\alpha_1$ and with $\alpha_2<q_2\leq\frac{\beta_2}{1-\gamma_1}$, $1<q_1\leq\alpha_1$ and $\frac{\beta_2}{1-\gamma_1}<q_2<\frac{\beta_2}{1-\alpha_1\gamma_1}$,$\frac{1}{\gamma_1}\left(1-\frac{\beta_2}{q_2}\right)\leq q_1\leq\alpha_1$ in case $1>\gamma_1\alpha_1$, the solution does not exist.

II. Now let $\alpha_1\leq 1$, $\alpha_2>1$. Since $q_1>1$, we have to consider the cases $\eta_1\leq 0$, $\eta_2>0$ and $\eta_1>0$,$\eta_2>0$. For $\eta_1\leq 0$, $\eta_2>0$, similar to the previous case, $u_1\equiv 0$,$u_2\equiv 0$ if $\eta_1<0$, $\eta_2\leq\left(4-\frac{4}{k}\right)(q_1q_2-1)+4\left(q_1+1\right)$ and $\eta_1=0$, $\eta_2<\left(4-\frac{4}{k}\right)(q_1q_2-1)+4\left(q_1+1\right)$.

Similarly we can make a conclusion that for $k=1$ the solution does not exist if $(q_1,q_2)\in(1,\beta_1]\times(1,\alpha_2]\backslash\{\beta_1,\alpha_2\}$, while for $k>1$ the solution does not exist in the following cases:
$q_1>1,q_2\in\left[\max\left(1,\frac{1}{\gamma_2}\left(1-\frac{\beta_1}{q_1}\right)\right),\ \alpha_2\right]\backslash\{1\}$, if $\alpha_2\gamma_2\geq 1$ and $q_1\in\left(1,\frac{\beta_1}{1-\alpha_2\gamma_2}\right)$,
$q_1\in\left(1,\frac{\beta_1}{1-\alpha_2\gamma_2}\right)$, $q_2\in\left[\max\left(1,\frac{1}{\gamma_2}\left(1-\frac{\beta_1}{q_1}\right)\right),\ \alpha_2\right]\backslash\{1\}$, if $\alpha_2\gamma_2<1$.

Now let $\eta_1>0$, $\eta_2>0$. Then, as in case (b), it is easy to see that $u_1\equiv 0$, $u_2\equiv 0$ if $q_1>1,\ q_2>\alpha_2,\ \max\{\theta_1,\theta_2\}\geq 0$.

III. Let $\alpha_1>1,\alpha_2\leq 1$. Then we have to consider the cases $\eta_1>0$, $\eta_2\leq 0$ and $\eta_1>0$, $\eta_2>0$. When $\eta_1>0$, $\eta_2\leq 0$, we similarly come to conclusion that for $k=1$ there is no solution if $(q_1,q_2)\in(1,\alpha_1]\times(1,\beta_2]\backslash\{\alpha_1,\beta_2\}$, while for $k>1$ the solution does not exist in the following cases: $q_2>1$, $q_1\in\left[\max\left(1,\frac{1}{\gamma_1}\left(1-\frac{\beta_2}{q_2}\right)\right),\ \alpha_1\right]\backslash\{1\}$ if $\alpha_1\gamma_1\geq 1$, and $q_2\in\left(1,\frac{\beta_2}{1-\alpha_1\gamma_1}\right)$,
$q_1\in\left[\max\left(1,\frac{1}{\gamma_1}(1-\frac{\beta_2}{q_2}\right),\ \alpha_1\right]\backslash\{1\}$ if $\alpha_1\gamma_1<1$.

When $\eta_1>0$, $\eta_2>0$, then for $q_1>\alpha_1,\ 1<q_2$, $\max\{\theta_1,\theta_2\}\geq 0$ the solution does not exist.

IV. Let $\alpha_1 \leq 1$, $\alpha_2 \leq 1$. Here we have to consider only the case $\eta_1 > 0$, $\eta_2 > 0$. Then $u_1 \equiv 0$, $u_2 \equiv 0$ if $\max\{\theta_1, \theta_2\} \geq 0$.

So the theorem is completely proved.

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