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ON THE INITIAL VALUE PROBLEM FOR THE NON-STATIONARY NONLINEAR HIROTA EQUATION WITH A LOWER-ORDER TERM

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Abstract. In this paper, we establish that the inverse scattering method (ISM) is an extremely efficient approach for solving the initial value problem associated with the Hirota equation, an equation modified by an additional term and time-varying coefficients, in the framework of rapidly decreasing functions. This study focuses on investigating how the scattering data of the Dirac operator evolve with time when the potential is defined as the solution to the initial value problem for the time-dependent Hirota equation, which is a nonlinear evolution equation with variable coefficients including an additional term in the class of rapidly decreasing functions. In addition, this work also investigates the solvability of the Cauchy problem for both the complex modified Korteweg–de Vries (cmKdV) equation and the nonlinear Schrödinger (NLS) equation. In each case, the equations are modified by adding an additional term and endowed with coefficients that vary with time. A detailed algorithmic procedure for solving these problems is presented, accompanied by several examples illustrating the application of the proposed method.

1. Introduction

A solitary wave is a traveling wave represented by a single peak or trough that propagates in isolation, maintaining its shape, size, and speed. The Korteweg–de Vries equation is widely recognized as one of the most essential and celebrated partial differential equations, serving as a primary tool for modeling and analyzing the propagation of solitary waves in shallow water environments:

$$q_t - 6qq_x + q_{xxx} = 0.$$

In 1967, researchers C. Gardner, J. Greene, M. Kruskal, and R. Miura [12] succeeded in solving this equation by applying the inverse scattering method (ISM) to the Sturm-Liouville operator defined on the entire real line.

Zakharov and Shabat demonstrated the universality of the ISM in 1972 by extending it to solve the Cauchy problem for the nonlinear Schrödinger (NLS) equation [38]. This work laid the foundation for the study of other equations.

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In 1973, Hirota [15] considered the following equation:

$$iq_t + 3i\zeta|q|^2 q_x + \lambda q_{xx} + i\mu q_{xxx} + \eta|q|^2 q = 0, \tag{1.1}$$

where $i^2 = -1$, ζ, λ, μ, $\eta \in \mathbb{R}$ and these parameters satisfy the condition $\zeta\lambda = \mu\eta$. In his paper [15], Hirota applied the method named after him [16] to obtain N-soliton solutions of this equation. Equation (1.1) can be rewritten as

$$iq_t + \chi\left(q_{xx} + 2q|q|^2\right) + i\delta\left(6|q|^2 q_x + q_{xxx}\right) = 0, \tag{1.2}$$

where $\zeta = 2\delta$, $\lambda = \chi$, $\mu = \delta$, $\eta = 2\chi$. It is important to note that when $\chi = 1$ and $\delta = 0$, equation (1.2) simplifies to an NLS equation, whereas when $\chi = 0$ and $\delta = 1$, it becomes a complex modified Korteweg-de Vries (cmKdV) equation, demonstrating how the equation transforms under these particular conditions. In 1991, Fukumoto and Miyazaki [11] demonstrated the relevance of Hirota's equation (1.2) in modeling the motion of vortex filaments in three-dimensional incompressible Euler fluid dynamics.

The remarkable growth in research on optical solitons has been driven by their significant impact on optical fibers, which are effectively used for transmitting information over intercontinental distances. In optical communication systems, the transmission line includes optical fibers and amplifiers, which introduce interference, noise, and signal distortions. Thus, a more precise approach to this research area is critical for advancing this field. Hirota's equation, [2, 6, 10, 35], which moderately differs from the standard nonlinear Schrödinger equation, is used to study solitons in the context of optical wave propagation in fibers. As the Hirota equation is a universal tool for modeling nonlinear processes in various fields of science and technology, many effective solution methods and algorithms have been proposed, some of which are described in detail in [3, 7, 8, 17, 22, 23, 27, 29, 33, 39].

A number of efficient techniques and robust algorithms have been established to address different kinds of nonlinear equations, with some of these approaches detailed in [4, 5, 13, 14, 18, 19, 20, 24, 25, 28, 30, 31, 34]. In many real physical scenarios, differential equations with variable coefficients often provide a more accurate and meaningful representation than equations with constant coefficients, especially when taking into account inhomogeneous properties of the medium. From a practical standpoint, equations featuring variable coefficients demonstrate greater accuracy [1, 17, 26, 32, 36, 37].

2. Formulation of the problem

In this study, we concentrate on the non-stationary Hirota equation with a lower-order term. More precisely, we examine the following equation:

$$iq_t + \alpha(t)\big(q_{xx} + 2q|q|^2\big) + i\beta(t)\big(6|q|^2 q_x + q_{xxx}\big) + \vartheta(t) q_x = 0. \tag{2.1}$$

The functions $\alpha(t)$, $\beta(t)$, and $\vartheta(t)$ are of class C^1, ensuring their derivatives exist and remain continuous throughout the domain of consideration. Let us analyze equation (2.1) in conjunction with the given initial condition, which is expressed as

$$q(x, 0) = q_0(x), \tag{2.2}$$

where $q_0(x)$ represents the prescribed initial state of the function, serving as the foundation for investigating the evolution of solutions over time within the framework of the problem.

The constraints detailed hereafter are rigorously valid:

1)

$$e^{\varepsilon|x|}|q_0(x)| \leq \frac{C}{(1+|x|)^{1+\zeta}}, \quad \zeta > 0, \tag{2.3}$$

where $\epsilon > 0$ and $C > 0$ is a constant;

2) the operator

$$\mathrm{L}(0) = i \begin{pmatrix} \frac{d}{dx} & -q_0(x) \\ -q_0^*(x) & -\frac{d}{dx} \end{pmatrix}$$

is characterized by precisely N distinct eigenvalues, labeled $\rho_1(0), \rho_2(0), \dots, \rho_N(0)$, each associated with respective multiplicities $k_1(0), k_2(0), \dots, k_N(0)$, all situated entirely within the upper half-plane $\mathbb{C}^+ = \{\rho \in \mathbb{C} : \mathrm{Im}(\rho) > 0\}$.

The solution $q(x,t)$ will be sought from the following specified class of functions:

$$M = \left\{ q(x,t) : \sum_{j=0}^{3} \left| \frac{\partial^j q(x,t)}{\partial x^j} \right| e^{\varepsilon|x|} \leq \frac{C}{(1+|x|)^{1+\zeta}}, \ \varepsilon > 0, \ C > 0, \ \zeta > 0 \right\}. \tag{2.4}$$

3. Preliminaries

Our investigation centers on the Dirac system, formulated as the coupled equations below:

$$\begin{cases} y_{1x} + i\rho v_1 = q(x) y_2, \\ y_{2x} - i\rho y_2 = -q^*(x) y_1, \end{cases} \tag{3.1}$$

This system is defined on the entire real line, that is, for all values of x in the region $(-\infty, \infty)$ where the functions y_1, y_2, and $q(x)$, together with their respective derivatives, describe the behavior of the system at every point in this infinite region. In this context, $q(x)$ is a complex-valued function that fulfills a specific boundedness condition for a certain positive parameter ε, ensuring that

$$e^{\varepsilon|x|}|q(x)| \leq \frac{C}{(1+|x|)^{1+\zeta}}, \quad \zeta > 0, \tag{3.2}$$

where C is a constant, and the decay rate governed by ζ guarantees that $q(x)$ vanishes sufficiently fast at infinity, which is crucial for the well-posedness and asymptotic analysis of the problem. By defining the operator

$$\mathrm{L} = i \begin{pmatrix} \frac{d}{dx} & -q(x) \\ -q^*(x) & -\frac{d}{dx} \end{pmatrix},$$

we can express the system in the compact eigenvalue form $\mathrm{L}\, Y = \rho Y$, where the vector $Y = (y_1, y_2)^T$ represents the solution components.

The direct and inverse scattering problems for the operator L have been extensively investigated in previous works, including [21], [9], among others. Assuming condition (3.2), the system (3.1) admits Jost solutions exhibiting asymptotic behavior at infinity for $\mathrm{Im}(\rho) > -\varepsilon/2$:

$$\begin{cases} \Upsilon(x,\rho) = e^{-i\rho x}\left[\begin{pmatrix}1\\0\end{pmatrix} + o(1)\right], \\ \bar{\Upsilon}(x,\rho) = e^{i\rho x}\left[\begin{pmatrix}0\\-1\end{pmatrix} + o(1)\right], \end{cases} \quad \text{as} \quad x \to -\infty,$$
$$\begin{cases} \Psi(x,\rho) = e^{i\rho x}\left[\begin{pmatrix}0\\1\end{pmatrix} + o(1)\right], \\ \bar{\Psi}(x,\rho) = e^{-i\rho x}\left[\begin{pmatrix}1\\0\end{pmatrix} + o(1)\right], \end{cases} \quad \text{as} \quad x \to +\infty, \tag{3.3}$$

Note that here the notation bar does not mean complex conjugation, for this we will use the notation $*$.

Under condition (3.2), such solutions exist and are uniquely determined by the asymptotics (3.3). Within the region defined by $\mathrm{Im}\rho > -\varepsilon/2$, it is readily apparent that the pairs of vector functions $\{\Upsilon(x,\rho), \bar{\Upsilon}(x,\rho)\}$ and $\{\Psi(x,\rho), \bar{\Psi}(x,\rho)\}$ form complete sets of fundamental solutions. Consequently, in this strip, the following relations hold:

$$\Upsilon(x,\rho) = a(\rho)\bar{\Psi}(x,\rho) + b(\rho)\Psi(x,\rho), \quad \bar{\Upsilon}(x,\rho) = -\bar{a}(\rho)\Psi(x,\rho) + \bar{b}(\rho)\bar{\Psi}(x,\rho), \tag{3.4}$$

where

$$\Upsilon(x,\rho) = (\upsilon_1(x,\rho), \upsilon_2(x,\rho))^T, \quad \Psi = (\psi_1(x,\rho), \psi_2(x,\rho))^T,$$
$$a(\rho) = W\{\Upsilon(x,\rho), \Psi(x,\rho)\} \equiv \upsilon_1(x,\rho)\psi_2(x,\rho) - \upsilon_2(x,\rho)\psi_1(x,\rho),$$
$$a(\rho)\bar{a}(\rho) + b(\rho)\bar{b}(\rho) = 1, \qquad \bar{b}(\rho) = b^*(\rho^*), \quad N = \bar{N}, \quad \bar{\rho}_n = \rho_n^*$$
$$\bar{\Psi}(x,\rho) = \begin{pmatrix} \psi_2^*(x,\rho^*) \\ -\psi_1^*(x,\rho^*) \end{pmatrix}, \quad \bar{\Upsilon}(x,\rho) = \begin{pmatrix} \upsilon_2^*(x,\rho^*) \\ -\upsilon_1^*(x,\rho^*) \end{pmatrix}, \quad \bar{a}(\rho) = a^*(\rho^*).$$

The function $a(\rho)$ (and, by a similar reasoning, $\bar{a}(\rho)$) is characterized by having only a finite number of zeros in the complex plane. More specifically, there exist zeros ρ_n located in the open upper half of the complex plane, that is, $\rho_n \in \mathbb{C}^+$, where each zero is counted with a multiplicity k_n for $n = 1, 2, \ldots, N$. In a similar fashion, the function $\bar{a}(\rho)$ exhibits zeros $\bar{\rho}_n$ in the open lower half-plane, $\bar{\rho}_n \in \mathbb{C}^- = \{\rho \in \mathbb{C} : \mathrm{Im}(\rho) < 0\}$, with each of these zeros occurring with a multiplicity $\bar{k}_n$ for $n = 1, 2, \ldots, \bar{N}$. The eigenvalues of the operator L in the upper (lower) half-plane are represented by the values ρ_n $(\bar{\rho}_n)$ where the functions $a(\rho)$ $(\bar{a}(\rho))$ vanish.

There exist sequences of numbers $\{\varkappa_0^n, \varkappa_1^n, \ldots, \varkappa_{k_n-1}^n\}$ and $\{\bar{\varkappa}_0^n, \bar{\varkappa}_1^n, \ldots, \bar{\varkappa}_{k_n-1}^n\}$ satisfying:

$$\frac{1}{j!}\left(\frac{d}{d\rho}\right)^j \Upsilon(x,\rho)\Big|_{\rho=\rho_n} = \sum_{l=0}^{j} \varkappa_{j-l}^n \frac{1}{l!}\left(\frac{d}{d\rho}\right)^l \Psi(x,\rho)\Big|_{\rho=\rho_n}, \tag{3.5}$$
$$\frac{1}{j!}\left(\frac{d}{d\rho}\right)^j \Psi(x,\rho)\Big|_{\rho=\rho_n} = \sum_{l=0}^{j} \bar{\varkappa}_{j-l}^n \frac{1}{l!}\left(\frac{d}{d\rho}\right)^l \Upsilon(x,\rho)\Big|_{\rho=\rho_n},$$

where $\varkappa_0^n \neq 0$ and $\bar{\varkappa}_0^n \neq 0$. The sequence $\{\varkappa_0^n, \varkappa_1^n, \ldots, \varkappa_{k_n-1}^n\}$ is called the normalization chain of the operator L, associated with the eigenvalues $\{\rho_n\}_{n=1}^N$, while $\{\bar{\varkappa}_0^n, \bar{\varkappa}_1^n, \ldots, \bar{\varkappa}_{k_n-1}^n\}$ is the conjugate chain.

The lack of spectral singularities in the operator L indicates that the functions $a(\rho)$ and $\bar{a}(\rho)$ have no real zeros.

The Jost solutions $\Psi(x,\rho)$ and $\bar{\Psi}(x,\rho)$, which are central to the spectral analysis of the operator under consideration, admit integral representations of a certain structure that essentially reflects the interaction between the spatial variable x and the spectral parameter ρ. In particular, for x in the domain and ρ in the corresponding domains of the complex plane, these functions can be expressed as:

$$\begin{aligned}\Psi(x,\rho) &= \begin{pmatrix}0\\1\end{pmatrix} e^{i\rho x} + \int_x^\infty \Omega(x,s)e^{i\rho s}ds,\\ \bar{\Psi}(x,\rho) &= \begin{pmatrix}1\\0\end{pmatrix} e^{-i\rho x} + \int_x^\infty \bar{\Omega}(x,s)e^{-i\rho s}ds,\end{aligned} \tag{3.6}$$

where

$$\Omega(x,s) = \begin{pmatrix}\omega_1(x,s) & \omega_2(x,s)\end{pmatrix}^T, \quad \bar{\Omega}(x,s) = \begin{pmatrix}\bar{\omega}_1(x,s) & \bar{\omega}_2(x,s)\end{pmatrix}^T,$$
$$q(x) = -2\omega_1(x,x).$$

Here the kernels $\Omega(x,s)$ and $\bar{\Omega}(x,s)$ satisfy certain analyticity and damping conditions dictated by the constraints of the problem. These representations are not only fundamental for deriving asymptotic expansions like $|x|\to\infty$ or $|\rho|\to\infty$, but also play a decisive role in establishing the existence and uniqueness of solutions of related inverse spectral problems. The functions $\Omega(x,s)$ and $\bar{\Omega}(x,s)$, which are defined for $y>x$, are governed by the system of equations

$$\begin{aligned}\bar{\Omega}(x,y) + \begin{pmatrix}0\\1\end{pmatrix}\Lambda(x+y) + \int_x^\infty \bar{\Omega}(x,s)\Lambda(s+y)\,ds = 0,\\ \Omega(x,y) - \begin{pmatrix}1\\0\end{pmatrix}\Lambda^*(x+y) - \int_x^\infty \Omega(x,s)\Lambda^*(s+y)\,ds = 0.\end{aligned} \tag{3.7}$$

which is recognized as the system of Gelfand-Levitan-Marchenko (sGLM) integral equations, playing a fundamental role in inverse scattering theory by providing a framework for reconstructing potentials from scattering data and analyzing the spectral properties of associated differential operators. Here

$$\begin{aligned}\Lambda(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{b(\rho)}{a(\rho)}e^{i\rho x}\,d\rho - i\sum_{n=1}^{N}\sum_{j=0}^{k_n-1}\frac{\varkappa^n_{k_n-1-j}}{j!}\left(\frac{d^j f(x,\rho)}{d\rho^j}\right)\Big|_{\rho=\rho_n},\\ \Lambda^*(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\bar{b}(\rho)}{a(\rho)}e^{-i\rho x}\,d\rho + i\sum_{n=1}^{N}\sum_{j=0}^{k_n-1}\frac{\bar{\varkappa}^n_{k_n-1-j}}{j!}\left(\frac{d^j \bar{f}(x,\rho)}{d\rho^j}\right)\Big|_{\rho=\rho_n}.\end{aligned} \tag{3.8}$$
$$f(x,\rho) = \frac{(\rho-\rho_n)^{k_n}}{a(\rho)}e^{i\rho x}, \qquad \bar{f}(x,\rho) = \frac{(\rho-\rho_n)^{k_n}}{a(\rho)}e^{-i\rho x}$$

The function $s^+(\rho) = \dfrac{b(\rho)}{a(\rho)}$ is called the scattering function of the operator L.

The collection of elements

$$\left\{s^+(\rho),\, \rho_n,\, \varkappa^n_j,\, j=0,1,\dots,k_n-1,\, n=\overline{1,N}\right\}$$

is referred to as the scattering data for the system of equations (3.1), representing a fundamental set of parameters that encapsulate essential information about the interaction of waves with a potential, thereby playing a crucial role in the inverse scattering transform and the analysis of soliton solutions.

Definition 1. We say that the function $s^+(\rho)$ is a function of type (S) in the strip $|\mathrm{Im}\ \rho| < \varepsilon/2$ if:

1) $s^+(\rho)$ is a meromorphic function in the strip $|\mathrm{Im}\ \rho| < \varepsilon/2$ and for each $\zeta < \varepsilon/2$,

$$s^+(\rho) = O\left(\frac{1}{\rho}\right), \quad |\rho| \to \infty,$$

uniformly in the strip $|\mathrm{Im}\ \rho| \leq \zeta$;

2) $s^+(\rho)$ has no non-real poles in the strip $|\mathrm{Im}\ \rho| < \varepsilon/2$ and

$$1 + s^+(\rho)\bar{s}(\rho) \neq 0,$$

where $\bar{s}(\rho) = \dfrac{\bar{b}(\rho)}{\bar{a}(\rho)}$.

The theorem presented below is valid [21].

Theorem 3.1. *For the function $s^+(\rho)$, defined in the strip $|\mathrm{Im}\ \rho| < \varepsilon/2$, the values $\rho_1, \rho_2, \ldots, \rho_N$ residing in the upper half-plane $\mathrm{Im}\ \rho > 0$, and their associated sequences $\varkappa_0^n, \varkappa_1^n, \ldots, \varkappa_{k_n-1}^n$ (where $\varkappa_0^n \neq 0$ for all $n = 1, 2, \ldots, N$), to constitute valid scattering data for an operator of type (3.1) with a complex-valued potential adhering to (3.2), the following criteria must fulfill:*

1) the function $s^+(\rho)$ is a function of type (S) in the strip $|\mathrm{Im}\ \rho| < \varepsilon/2$;

2) the functions $F(x)$ and $\bar{F}(x)$, constructed from the above quantities, according to formula (3.8) satisfy the condition

$$\left|\Lambda(x)\right| e^{\varepsilon x} \leq \frac{C_1}{(1+x)^{1+\zeta}}, \quad \text{for } x \geq a, \quad (a > -\infty),$$

$$\left|\bar{\Lambda}(x)\right| e^{-\varepsilon x} \leq \frac{C_2}{(1+x)^{1+\zeta}}, \quad \text{for } x \leq a, \quad (a < +\infty),$$

for all $a \in \mathbb{R}$.

Theorem 3.2 ([9], §6.2)**.** *The scattering data of the operator* L *allows for the unique reconstruction of the operator* L.

4. Time Evolution

Suppose that the potential $q(x,t)$ in the eigenvalue system

$$L(t)\Upsilon = \rho\Upsilon, \tag{4.1}$$

satisfies equation (2.1), which governs its evolution, and note that this equation (2.1) admits a Lax pair representation characterized by the differential operators

$$\Gamma = \alpha(t)\begin{pmatrix} i\left(|q|^2 - 2\rho^2\right) & iq_x + 2\rho q \\ iq_x^* - 2\rho q^* & i\left(-|q|^2 + 2\rho^2\right) \end{pmatrix}, \tag{4.2}$$

$$\Pi = \beta(t)\begin{pmatrix} -4i\rho^3 - 2i\rho|q|^2 - q^*q_x + q_x^*q & 4\rho^2 q + 2i\rho q_x - q_{xx} - 2q|q|^2 \\ -4\rho^2 q^* + 2i\rho q_x^* + q_{xx}^* + 2q^*|q|^2 & 4i\rho^3 - 2i\rho|q|^2 + q^*q_x - q_x^*q \end{pmatrix}. \tag{4.3}$$

providing a powerful analytical framework for studying the integrability of the system and facilitating the application of the inverse scattering method to analyze its solution structure. As a result, this equation can be reformulated in the Lax representation as

$$\mathrm{L_t(t)} + \big[\mathrm{L(t)}, \Gamma + \Pi\big] = \mathrm{G}, \tag{4.4}$$

where

$$\mathrm{G} = \begin{pmatrix} 0 & G_1 \\ G_2 & 0 \end{pmatrix},$$

with $G_1 = \vartheta(t) q_x,\ G_2 = \vartheta(t) q_x^*$ and $\big[\mathrm{L(t)}, \Gamma\big] = \mathrm{L(t)}\,\Gamma - \mathrm{A}\mathrm{L(t)}$ denotes the commutator of the operators $\mathrm{L(t)}$ and Γ .

By differentiating equation (4.1) with respect to t, substituting (4.4) we derive the relation

$$\left(L(t) - \rho I\right)\left(\Upsilon_t(x,\rho,t) - (\Gamma + \Pi)\Upsilon(x,\rho,t)\right) = -G\Upsilon(x,\rho,t). \tag{4.5}$$

Applying the method of variation of parameters, we seek a solution to equation (4.5) in the form

$$\Upsilon_t(x,\rho,t) - (\Gamma + \Pi)\Upsilon(x,\rho,t) = C(x,\rho,t)\Psi(x,\rho) + D(x,\rho,t)\Upsilon(x,\rho),$$

where $C(x,\rho,t)$ and $D(x,\rho,t)$ are undetermined coefficients. In order to determine the functions $C(x,\rho,t)$ and $D(x,\rho,t)$, we derive the following system of equations which, once solved under the appropriate conditions, allows for the determination of these unknown functions by using the relationships between the components of the system:

$$\begin{cases} C'(x,\rho,t)\psi_1(x,\rho,t) + D'(x,\rho,t)\upsilon_1(x,\rho,t) = iG_1\upsilon_2(x,\rho,t), \\ C'(x,\rho,t)\psi_2(x,\rho,t) + D'(x,\rho,t)\upsilon_2(x,\rho,t) = -iG_2\upsilon_1(x,\rho,t). \end{cases} \tag{4.6}$$

This notation will be used consistently in all subsequent discussions. By solving the system of equations (4.6), we arrive at the following result, which provides the expressions for the unknown functions involved in the system

$$\begin{aligned} C'(x,\rho,t) &= -\frac{i}{a(\rho,t)}\tilde{\Upsilon}^T(x,\rho,t)\,\mathrm{G}\,\Upsilon(x,\rho,t), \\ D'(x,\rho,t) &= \frac{i}{a(\rho,t)}\Psi^T(x,\rho,t)\,\mathrm{G}^T\,\tilde{\Upsilon}(x,\rho,t), \end{aligned} \tag{4.7}$$

where $\tilde{\Upsilon}(x,\rho,t) = \big(\upsilon_2(x,\rho,t) \quad \upsilon_1(x,\rho,t)\big)^T$. According to formulas (4.2), (4.3), and (4.5), as the spatial variable x tends to negative infinity ($x \to -\infty$) the following limit characteristics of the functions $C(x,\rho,t)$ and $D(x,\rho,t)$ are observed. The function $C(x,\rho,t)$ exhibits a decreasing character, asymptotically tending to zero, which is mathematically expressed by the limit relation

$$C(x,\rho,t) \to 0.$$

Meanwhile, the function $D(x,\rho,t)$ tends to a nontrivial expression determined by the structure of the operator and the parameters of the system: its limit takes the form

$$D(x,\rho,t) \to 2i\rho^2\left(\alpha(t) + 2\rho\beta(t)\right),$$

where the coefficients $\alpha(t)$ and $\beta(t)$, depending on the time variable t, reflect the dynamic properties of the model under consideration.

Therefore, from (4.7) we can determine

$$C(x,\rho,t) = -ia^{-1}(\rho,t)\int_{-\infty}^{x} \tilde{\Upsilon}^T(z,\rho,t)\,\mathrm{G}\,\Upsilon(z,\rho,t)dz,$$

$$D(x,\rho,t) = ia^{-1}(\rho,t)\int_{-\infty}^{x} \Psi^T(z,\rho,t)\,\mathrm{G}^T\,\tilde{\Upsilon}(z,\rho,t)dz + 2i\rho^2\big(\alpha(t)+2\rho\beta(t)\big).$$

Thus, the equality (4.5) has the form

$$\begin{aligned}\Upsilon_t(x,\rho,t) &- (\Gamma+\Pi)\Upsilon(x,\rho,t)\\ = -\frac{i}{a(\rho,t)}(\rho,t)&\int_{-\infty}^{x} \tilde{\Upsilon}^T(z,\rho,t)\,\mathrm{G}\,\Upsilon(z,\rho,t)dz\cdot\Psi(x,\rho,t)\\ +\frac{i}{a(\rho,t)}&\int_{-\infty}^{x} \Psi^T(z,\rho,t)\,\mathrm{G}^T\,\tilde{\Upsilon}(z,\rho,t)dz\cdot\Upsilon(x,\rho,t)\\ &+2i\rho^2\big(\alpha(t)+2\rho\beta(t)\big)\Upsilon(x,\rho,t).\end{aligned} \tag{4.8}$$

Based on the expression provided in (3.4), the equality (4.8) can be rewritten in the following form, which allows for a more detailed representation and analysis of the relationships between the variables involved, thereby offering a clearer understanding of how they interact under the given conditions:

$$a_t(\rho,t)\bar{\Psi}(x,\rho,t) + b_t(\rho,t)\Psi(x,\rho,t) - (\Gamma+\Pi)\left(a(\rho,t)\bar{\Psi}(x,\rho,t)+b(\rho,t)\Psi(x,\rho,t)\right)$$

$$= -ia^{-1}(\rho,t)\int_{-\infty}^{x} \tilde{\Upsilon}^T(z,\rho,t)\,\mathrm{G}\,\Upsilon(z,\rho,t)dz\cdot\Psi(x,\rho,t)$$

$$+\left(ia^{-1}(\rho,t)\int_{-\infty}^{x} \Psi^T(z,\rho,t)\,\mathrm{G}^T\,\tilde{\Upsilon}(z,\rho,t)dz\right)\left(a(\rho,t)\bar{\psi}(x,\rho,t)+b(\rho,t)\psi(x,\rho,t)\right)$$

$$+2i\rho^2\big(\alpha(t)+2\rho\beta(t)\big)\big(a(\rho,t)\bar{\psi}(x,\rho,t)+b(\rho,t)\psi(x,\rho,t)\big).$$

By taking the limit as $x\to+\infty$ in the preceding equality and using the conditions specified in (4.2) and (4.3), we obtain

$$a_t(\rho,t) = i\int_{-\infty}^{\infty} \Psi^T(z,\rho,t)\,\mathrm{G}^T\,\tilde{\Upsilon}(z,\rho,t))dz, \tag{4.9}$$

$$b_t(\rho,t) = -ia^{-1}(\rho,t)\int_{-\infty}^{\infty} \tilde{\Upsilon}^T(z,\rho,t)\,\mathrm{G}\,\Upsilon(z,\rho,t)dz$$

$$+ib(\rho,t)a^{-1}(\rho,t)\int_{-\infty}^{\infty} \Psi^T(z,\rho,t)\,\mathrm{G}^T\,\tilde{\Upsilon}(z,\rho,t)dz + 4i\rho^2 b(\rho,t)\big(\alpha(t)+2\rho\beta(t)\big).$$

Given that $s^+(\rho,t) = \dfrac{b(\rho,t)}{a(\rho,t)}$, we can derive the following from the previous expressions:

$$\begin{aligned}\frac{ds^+(\rho,t)}{dt} = 4i\rho^2\big(\alpha(t)+2\rho\beta(t)\big)s^+(\rho,t)&\\ -\frac{i}{a^2(\rho,t)}\int_{-\infty}^{\infty} \tilde{\Upsilon}^T(z,\rho,t)\,\mathrm{G}\,\Upsilon(z,\rho,t)dz.&\end{aligned} \tag{4.10}$$

Lemma 4.1. *The vector-functions*

$$\Upsilon(x,\rho,t)=\begin{pmatrix}\upsilon_1(x,\rho,t)\\ \upsilon_2(x,\rho,t)\end{pmatrix}\quad \textit{and}\quad \Psi(x,\rho,t)=\begin{pmatrix}\psi_1(x,\rho,t)\\ \psi_2(x,\rho,t)\end{pmatrix},$$

composed of their scalar components υ_1, υ_2 and ψ_1, ψ_2, act as fundamental solutions for the dynamical system defined by (4.1), *with their components fulfilling the identities below:*

$$\int_{-\infty}^{\infty}\Psi^T(x,\rho,t)\,\mathrm{G}^T\,\tilde{\Upsilon}(x,\rho,t)dx=0, \tag{4.11}$$

$$\int_{-\infty}^{\infty}\tilde{\Upsilon}^T(x,\rho,t)\,\mathrm{G}\,\Upsilon(x,\rho,t)dx=-2i\rho\vartheta(t)a(\rho,t)b(\rho,t). \tag{4.12}$$

Proof. To calculate the integral in the left part of the identity (4.11), let us use the formulas (2.4), (3.3), (3.4) and (4.1) and obtain the following result, which will help to simplify further calculations:

$$\int_{-\infty}^{\infty}\Psi^T(x,\rho,t)\,\mathrm{G}\,\Upsilon(x,\rho,t)dx$$

$$=\vartheta(t)\int_{-\infty}^{\infty}\Big(q_x^*(x,t)\upsilon_1(x,\rho,t)\psi_1(x,\rho,t)+q_x(x,t)\upsilon_2(x,\rho,t)\psi_2(x,\rho,t)\Big)dx$$

$$=\vartheta(t)\bigg(\int_{-\infty}^{\infty}\upsilon_1(x,\rho,t)\psi_1(x,\rho,t)dq^*(x,t)+\int_{-\infty}^{\infty}\upsilon_2(x,\rho,t)\psi_2(x,\rho,t)dq(x,t)\bigg)$$

$$=\lim_{d\to\infty}\vartheta(t)\Big(q^*(x,t)\upsilon_1(x,\rho,t)\psi_1(x,\rho,t)+q(x,t)\upsilon_2(x,\rho,t)\psi_2(x,\rho,t)\Big)\Big|_{-d}^{d}$$

$$-\vartheta(t)\int_{-\infty}^{\infty}q^*(x,t)\big(\upsilon_1'(x,\rho,t)\psi_1(x,\rho,t)+\upsilon_1(x,\rho,t)\psi_1'(x,\rho,t)\big)dx$$

$$-\vartheta(t)\int_{-\infty}^{\infty}q(x,t)\big(\upsilon_2'(x,\rho,t)\psi_2(x,\rho,t)+\upsilon_2(x,\rho,t)\psi_2'(x,\rho,t)\big)dx$$

$$=\lim_{d\to\infty}\vartheta(t)\big[-i\rho\big(\upsilon_1(x,\rho,t)\psi_2(x,\rho,t)+\upsilon_1(x,\rho,t)\psi_1(x,\rho,t)\big)\big]\Big|_{-d}^{d}=0.$$

In this context, the prime symbol is used to indicate differentiation with respect to the variable x, and in a similar manner, the following integral is computed by applying the same methodological approach, ensuring consistency in the analytical framework and facilitating the derivation of relevant expressions:

$$\int_{-\infty}^{\infty}\tilde{\Upsilon}^T(x,\rho,t)\,\mathrm{G}\,\Upsilon(x,\rho,t)dx=\vartheta(t)\int_{-\infty}^{\infty}\Big(q_x^*(x,t)\upsilon_1^2(x,\rho,t)+q_x(x,t)\upsilon_2^2(x,\rho,t)\Big)dx$$

$$=\lim_{d\to\infty}\vartheta(t)\,\big(q^*(x,t)\upsilon_1^2(x,\rho,t)+q(x,t)\upsilon_2^2(x,\rho,t)\big)\Big|_{-d}^{d}$$

$$-2\vartheta(t)\int_{-\infty}^{\infty}\Big(q^*(x,t)\upsilon_1(x,\rho,t)\upsilon_1'(x,\rho,t)+q(x,t)\upsilon_2(x,\rho,t)\upsilon_2'(x,\rho,t)\Big)dx$$

$$=-2\vartheta(t)\int_{-\infty}^{\infty}\Big[\upsilon_1'(x,\rho,t)\big(-\upsilon_2(x,\rho,t)+i\rho\upsilon_2(x,\rho,t)\big)$$

$$+\upsilon_2'(x,\rho,t)\big(-\upsilon_1(x,\rho,t)+i\rho\upsilon_1(x,\rho,t)\big)\Big]dx=-2i\rho\vartheta(t)a(\rho,t)b(\rho,t).$$

□

Corollary. Utilizing Lemma 4.1 and formula (4.9) produces the following identity:

$$\frac{\partial a(\rho,t)}{\partial t}=0.$$

The time-independence of $a(\rho,t)$ further implies that the associated spectral data – including bound state eigenvalues $\rho_n(t)$ and their multiplicities $k_n(t)$ – remain stationary over time, preserving the intrinsic connection between the scattering data and the potential's geometry, i.e.

$$k_n(t)=k_n(0),\quad \rho_n(t)=\rho_n(0),\quad n=\overline{1,N}. \tag{4.13}$$

Combining equations (4.10) and (4.12), the temporal derivative of $s^+(\rho,t)$ is derived as

$$\frac{ds^+(\rho,t)}{dt}=2i\rho\left(2\rho\alpha(t)+4\rho^2\beta(t)+i\vartheta(t)\right)s^+(\rho,t). \tag{4.14}$$

Now we investigate the evolution of the normalizing chain $\left\{\varkappa_0^n,\varkappa_1^n,\ldots,\varkappa_{k_n-1}^n\right\}$ corresponding to the eigenvalue ρ_n, for $n=\overline{1,N}$. Subsequently, we re-express equation (4.8) in the restructured form:

$$\begin{aligned}\Upsilon_t(x,\rho,t)-(\Gamma+\Pi)\Upsilon(x,\rho,t)&=2i\rho^2\big(\alpha(t)+2\rho\beta(t)\big)\Upsilon(x,\rho,t)\\ &-\frac{i}{a(\rho,t)}\bigg(\int_{-\infty}^{x}\tilde{\Upsilon}^T(z,\rho,t)\,\mathrm{G}\,\Upsilon(z,\rho,t)dz\cdot\Psi(x,\rho,t)\\ &-\int_{-\infty}^{x}\Psi^T(x,\rho,t)\,\mathrm{G}^T\,\tilde{\Upsilon}(z,\rho,t)dz\cdot\Upsilon(z,\rho,t)\bigg).\end{aligned} \tag{4.15}$$

To initiate, we evaluate the subsequent integral by applying expressions (2.4) and (3.1)-(3.4):

$$\int_{-\infty}^{x}\Psi^T(z,\rho,t)\,\mathrm{G}^T\,\tilde{\Upsilon}(z,\rho,t)dz$$

$$\begin{aligned}&=\vartheta(t)\int_{-\infty}^{x}\big(q_z^*(z,t)\upsilon_1(z,\rho,t)\psi_1(z,\rho,t)+q_z(z,t)\upsilon_2(z,\rho,t)\psi_2(z,\rho,t)\big)dz\\ &=\vartheta(t)\bigg(\int_{-\infty}^{x}\upsilon_1(z,\rho,t)\psi_1(z,\rho,t)dq^*(z,t)+\int_{-\infty}^{x}\upsilon_2(z,\rho,t)\psi_2(z,\rho,t)dq(z,t)\bigg)\\ &=\vartheta(t)\lim_{d\to\infty}\Big(q^*(z,t)\upsilon_1(z,\rho,t)\psi_1(z,\rho,t)+q(z,t)\upsilon_2(z,\rho,t)\psi_2(z,\rho,t)\Big)\Big|_{-d}^{x}\\ &-\vartheta(t)\int_{-\infty}^{x}q^*(z,t)\Big(\upsilon_1'(z,\rho,t)\psi_1(z,\rho,t)+\upsilon_1(z,\rho,t)\psi_1'(z,\rho,t)\Big)dz\\ &-\vartheta(t)\int_{-\infty}^{x}q(z,t)\Big(\upsilon_2'(z,\rho,t)\psi_2(z,\rho,t)+\upsilon_2(z,\rho,t)\psi_2'(z,\rho,t)\Big)dz\\ &=\vartheta(t)\Big(q^*(x,t)\upsilon_1(x,\rho,t)\psi_1(x,\rho,t)+q(x,t)\upsilon_2(x,\rho,t)\psi_2(x,\rho,t)\Big)\\ &-i\rho\vartheta(t)\int_{-\infty}^{x}\Big(\upsilon_1(z,\rho,t)\psi_2(z,\rho,t)+\upsilon_2(z,\rho,t)\psi_1(z,\rho,t)\Big)'dz\\ &=\vartheta(t)\Big(q^*(x,t)\upsilon_1(x,\rho,t)\psi_1(x,\rho,t)+q(x,t)\upsilon_2(x,\rho,t)\psi_2(x,\rho,t)\Big)\\ &-2i\rho\vartheta(t)\upsilon_2(x,\rho,t)\psi_1(x,\rho,t).\end{aligned}$$

By following a similar derivation process, the following integral expression is obtained:

$$\int_{-\infty}^{x} \tilde{\Upsilon}^T(z,\rho,t)\,\mathrm{G}\,\Upsilon(z,\rho,t)dz = \vartheta(t)\Big(q^*(x,t)\upsilon_1^2(x,\rho,t) + q(x,t)\upsilon_2^2(x,\rho,t)\Big)$$
$$-2i\rho\vartheta(t)\upsilon_1(x,\rho,t)\upsilon_2(x,\rho,t).$$

which allows equation (4.15) to be reformulated in an equivalent form, facilitating further analysis and manipulation within the given mathematical framework:

$$\Upsilon_t(x,\rho,t) - (\Gamma+\Pi)\Upsilon(x,\rho,t) = 2i\rho^2\big(\alpha(t)+2\rho\beta(t)\big)\Upsilon(x,\rho,t)$$
$$-\frac{i}{a(\rho,t)}\Big[\Big\{\vartheta(t)\big(q^*(x,t)\upsilon_1^2(x,\rho,t)+q(x,t)\upsilon_2^2(x,\rho,t)\big)$$
$$-2i\rho\vartheta(t)\upsilon_1(x,\rho,t)\upsilon_2(x,\rho,t)\Big\}\Psi(x,\rho,t)$$
$$-\Big\{\vartheta(t)(q^*(x,t)\upsilon_1(x,\rho,t)\psi_1(x,\rho,t)+q(x,t)\upsilon_2(x,\rho,t)\psi_2(x,\rho,t))$$
$$-2i\rho\vartheta(t)\upsilon_2(x,\rho,t)\psi_1(x,\rho,t)\Big\}\Upsilon(x,\rho,t)\Big]$$
$$=-\frac{i}{a(\rho,t)}\vartheta(t)\Big(q^*(x,t)\upsilon_1^2(x,\rho,t)\Psi+q(x,t)\upsilon_2^2(x,\rho,t)\Psi(x,\rho,t)\Big)$$
$$+\frac{i}{a(\rho,t)}\vartheta(t)\Big(q^*(x,t)\upsilon_1(x,\rho,t)\psi_1(x,\rho,t)\Upsilon(x,\rho,t)$$
$$+q(x,t)\upsilon_2(x,\rho,t)\psi_2(x,\rho,t)\Upsilon(x,\rho,t)\Big)$$
$$+\frac{2\rho}{a(\rho,t)}\vartheta(t)\Big(\upsilon_2(x,\rho,t)\psi_1(x,\rho,t)\Upsilon(x,\rho,t)-\upsilon_1(x,\rho,t)\upsilon_2(x,\rho,t)\Psi(x,\rho,t)\Big)$$
$$+2i\rho^2\big(\alpha(t)+2\rho\beta(t)\big)\Upsilon(x,\rho,t) = -i\vartheta(t)\begin{pmatrix} -q(x,t)\upsilon_2(x,\rho,t) \\ q^*(x,t)\upsilon_1(x,\rho,t)\end{pmatrix}$$
$$-2\rho\vartheta(t)\begin{pmatrix} 0 \\ \upsilon_2(x,\rho,t)\end{pmatrix} + 2i\rho^2\big(\alpha(t)+2\rho\beta(t)\big)\Upsilon(x,\rho,t). \tag{4.16}$$

By differentiating the equality (4.16) (k_n-1) times with respect to ρ and evaluating at $\rho=\rho_n$, we get

$$\frac{\partial}{\partial t}\big(\overset{(k_n-1)}{\Upsilon_n}\big)-(\Gamma_0+\Pi_0)\overset{(k_n-1)}{\Upsilon_n}-(k_n-1)(\Gamma_1+\Pi_1)\overset{(k_n-2)}{\Upsilon_n}-\frac{1}{2!}(k_n-1)(k_n-2)(\Gamma_2+\Pi_2)\overset{(k_n-3)}{\Upsilon_n}$$
$$-\frac{1}{3!}(k_n-1)(k_n-2)(k_n-3)\Pi_3\overset{(k_n-4)}{\Upsilon_n} = 2i\rho_n^2\big(2\rho_n\beta(t)+\alpha(t)\big)\overset{(k_n-1)}{\Upsilon_n}$$
$$+4i\rho_n(k_n-1)\big(3\rho_n\beta(t)+\alpha(t)\big)\overset{(k_n-2)}{\Upsilon_n}+2i(k_n-1)(k_n-2)\big(6\rho_n\beta(t)+\alpha(t)\big)\overset{(k_n-3)}{\Upsilon_n}$$
$$+2i(k_n-1)(k_n-2)(k_n-3)\beta(t)\overset{(k_n-4)}{\Upsilon_n}+\vartheta(t)q(x,t)\overset{(k_n-1)}{\Upsilon_n}$$
$$-2\vartheta(t)\begin{pmatrix} 0 \\ \overset{(k_n-1)}{\upsilon_{2n}}\end{pmatrix}-2(k_n-1)\vartheta(t)\begin{pmatrix} 0 \\ \overset{(k_n-2)}{\upsilon_{2n}}\end{pmatrix}, \tag{4.17}$$

where

$$\overset{(r)}{\Upsilon_n}=\frac{\partial^r\Upsilon(x,\rho,t)}{\partial\rho^r}\Big|_{\rho=\rho_n},\ \Gamma_j=\frac{d^j}{d\rho^j}\Gamma\Big|_{\rho=\rho_n},\ \Pi_l=\frac{d^l}{d\rho^l}\Pi\Big|_{\rho=\rho_n},\ \ j=\overline{0,2},\ \ l=\overline{0,3}.$$

By applying the formulas (2.4) and (3.5), we first evaluate the limit of the equality (4.17) as $x \to \infty$, and in this limit, we compare the coefficients of the term $\begin{pmatrix} 0 \\ 1 \end{pmatrix} (ix)^l e^{i\rho_n x}$ for each $l = k_n - 1, k_n - 2, \dots, 0$, ultimately yielding the following result:

$$\frac{\partial \varkappa_0^n(t)}{\partial t} = \left(8i\rho_n^3\beta(t) + 4i\rho_n^2\alpha(t) - 2\rho_n\vartheta(t)\right)\varkappa_0^n(t),$$

$$\frac{\partial \varkappa_1^n(t)}{\partial t} = \left(8i\rho_n^3\beta(t)+4i\rho_n^2\alpha(t)-2\rho_n\vartheta(t)\right)\varkappa_1^n(t)+\left(24i\rho_n^2\beta(t)+8i\rho_n\alpha(t)-2\vartheta(t)\right)\varkappa_0^n(t),$$

$$\frac{\partial \varkappa_2^n(t)}{\partial t} = \left(8i\rho_n^3\beta(t)+4i\rho_n^2\alpha(t)-2\rho_n\vartheta(t)\right)\varkappa_2^n(t)+\left(24i\rho_n^2\beta(t)+8i\rho_n\alpha(t)-2\vartheta(t)\right)\varkappa_1^n(t) + \left(24i\rho_n\beta(t) + 4i\alpha(t)\right)\varkappa_0^n(t),$$

$$\begin{aligned} \frac{\partial \varkappa_l^n(t)}{\partial t} &= \left(8i\rho_n^3\beta(t) + 4i\rho_n^2\alpha(t) - 2\rho_n\vartheta(t)\right)\varkappa_l^n(t) \\ &+ \left(24i\rho_n^2\beta(t) + 8i\rho_n\alpha(t) - 2\vartheta(t)\right)\varkappa_{l-1}^n(t) \\ &+ \left(24i\rho_n\beta(t) + 4i\alpha(t)\right)\varkappa_{l-2}^n(t) + 8i\beta(t)\varkappa_{l-3}^n(t), \quad l = 3, 4, \dots, k_n - 1. \end{aligned} \tag{4.18}$$

As a result of the previously conducted calculations, we ultimately arrive at the following result, which essentially represents the main result of this study:

Theorem 4.1. *Let $q(x,t)$ be a solution of the problem defined by* (2.1)–(2.4). *Then the scattering data associated with the non-self-adjoint operator* L(t) *with the potential $q(x,t)$ evolve over time according to the differential equations* (4.13), (4.14) *and* (4.18).

Remark 4.1. The behavior of the scattering data for the operator L(t) is entirely determined by the equations obtained in Theorem 4.1, which lays a robust groundwork for using the ISM to solve the problems (2.1)–(2.4). The resulting set $\left\{s^+(\rho,t),\, \rho_j(t),\, \varkappa_0^j(t),\, \varkappa_1^j(t), \dots,\, \varkappa_{k_j-1}^j(t),\; j = \overline{1,N}\right\}$ satisfies the conditions of Theorem 3.1, so according to Theorem 3.1 and 3.2, the potential $q(x,t)$ in the operator L(t) is uniquely determined and is a solution of problem (2.1)–(2.4).

5. Algorithm and example

Suppose we are provided with an initial function $q_0(x)$ that satisfies the condition specified in (2.3), which ensures that the function adheres to the necessary criteria for the problem under consideration. In this case, the solution of the problem (2.1)-(2.4) is found using the following **algorithm**:

- after resolving the direct problem while utilizing the prescribed initial function $q_0(x)$, we subsequently determine the complete set of scattering data associated with the operator L(0), which is given by the collection of values:

$$\left\{s^+(\rho,0),\quad \rho_j(0), \varkappa_0^j(0), \varkappa_1^j(0), \dots, \varkappa_{k_j-1}^j(0),\quad j = \overline{1,N}\right\};$$

- by utilizing the equations outlined in Theorem 4.1, we obtain the scattering data for $t > 0$, offering a comprehensive description of the system's evolution and the associated wave dynamics:

$$\left\{s^+(\rho,t),\quad \rho_j(t),\, \varkappa_0^j(t),\, \varkappa_1^j(t), \dots,\, \varkappa_{k_j-1}^j(t),\quad j = \overline{1,N}\right\};$$

- by leveraging the scattering data obtained for $t>0$, we proceed to solve the sGLM integral equations, ultimately determining the unique solution $q(x,t)$, as established in Theorem 3.2.

Example 1. Let us examine the following problem, which involves analyzing the given conditions and deriving relevant solutions:

$$\begin{gathered} iq_t+q_{xx}+2q|q|^2+it\left(q_{xxx}+6|q|^2q_x\right)+i\sigma tq_x=0, \\ q(x,0)=-\frac{2id_0\left(\varkappa_0^0\right)^*e^{-2ic_0x}}{\left|\varkappa_0^0\right|\cosh 2(d_0x-\eta_0)}, \end{gathered} \tag{5.1}$$

where $\sigma=const,\quad \rho_0=c_0+id_0,\ (d_0>0),\quad \varkappa_0^0=\left|\varkappa_0^0\right|e^{i\zeta_0},\quad \left|\varkappa_0^0\right|^2=4d_0^2e^{4\eta_0}.$

The scattering data of the operator L(0) may be easily found:

$$s^+(0)=0,\quad \rho(0)=\rho_0=c_0+id_0,\ (d_0>0),\quad \varkappa_0(0)=\varkappa_0^0.$$

By Theorem 4.1, we have

$$\rho_0(t)=\rho_0(0)=c_0+id_0;\qquad \varkappa_0(t)=\varkappa_0^0e^{\gamma(t)},$$

where $\gamma(t)=4i\rho_0^2t+4i\rho_0^3t^2-i\rho_0\sigma t^2$. Therefore,

$$\Lambda(x,t)=-i\varkappa_0(t)e^{i\rho_0x}. \tag{5.2}$$

By solving the system of integral equations given in (3.7), which includes the kernel defined in (5.2), we eventually derive the following explicit expression,

$$\omega_1(x,t)=\frac{i\varkappa_0^*(t)e^{-2i\rho_0^*x}}{1+\frac{|\varkappa_0(t)|^2}{4d_0^2}e^{-4d_0x}},$$

thereby determining the solution to the Cauchy problem (5.1), which gives a comprehensive description of the evolution of the system under the given initial conditions:

$$q(x,t)=-\frac{2i\left(\varkappa_0^0\right)^*e^{-4i(c_0-id_0)^3t^2-4i(c_0-id_0)^2t+(ic_0+d_0)\sigma t^2-2ic_0x-2d_0x}}{1+\frac{\left|\varkappa_0^0\right|^2}{4d_0^2}e^{-4d_0x+8d_0(d_0^2-3c_0^2)t^2-16c_0d_0t+2\sigma d_0t^2}}.$$

Taking into account the expansion $\varkappa_0^0=\left|\varkappa_0^0\right|e^{i\zeta_0}$, the resulting equality can be reformulated as follows:

$$q(x,t)=-\frac{2id_0e^{(-4ic_0^3+12ic_0d_0^2+ic_0\sigma)t^2+4i(d_0^2-c_0^2)t-2ic_0x-i\zeta_0}}{\cosh\left(2d_0x-2\eta_0-4d_0\left(d_0^2-3c_0^2\right)t^2+8c_0d_0t-\sigma d_0t^2\right)}.$$

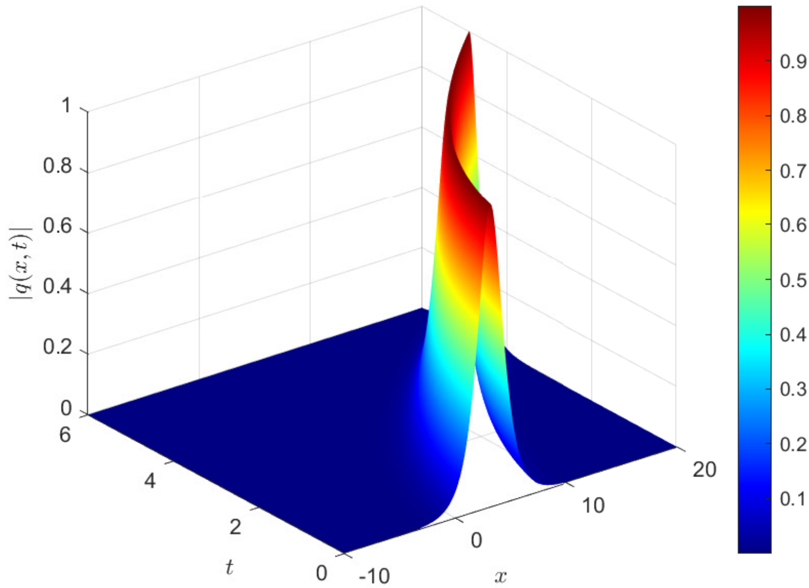

FIGURE 1. Plot of the modulus of the function $q(x,t)$ for the fixed parameter values: $c_0 = -0.3$, $d_0 = 0.5$, $\sigma = 1$, $\varphi = \pi/2$, $\psi = \pi/2$.

6. Some special cases

In this section, we explore several special cases of equation (2.1), focusing on specific scenarios that allow for a more detailed analysis of the behavior and properties of the system under particular conditions.

1. If we set $\alpha(t) \equiv 0$, $\beta(t) \not\equiv 0$, and $\vartheta(t) \not\equiv 0$ in equation (2.1), we obtain the following modified equation:

$$q_t + \beta(t)\big(6|q|^2 q_x + q_{xxx}\big) - i\vartheta(t) q_x = 0, \tag{6.1}$$

which represents the non-stationary cmKdV equation with a lower-order term, capturing the dynamics of the system under these specific conditions. Thus, by setting $\alpha(t) \equiv 0$ in Theorem 4.1, we obtain the time evolution of the scattering data for the Dirac operator, where the potential is determined by the solution of the problem defined by equation (6.1) and the conditions (2.2)–(2.4).

Example 2. We now examine the problem outlined below:

$$q_t + 3t^2(q_{xxx} + 6|q|^2 q_x) + (2t+1) q_x = 0,$$
$$q(x,0) = -\frac{2id_0 \left(\varkappa_0^0\right)^* e^{-2ic_0 x}}{\left|\varkappa_0^0\right| \cosh 2(d_0 x - \eta_0)},$$

where $\rho_0 = c_0 + id_0$, $(d_0 > 0)$, $\quad \varkappa_0^0 = \left|\varkappa_0^0\right| e^{i\zeta_0}$, $\quad \left|\varkappa_0^0\right|^2 = 4d_0^2 e^{4\eta_0}$.

The solution to this problem is as follows, and it is solved in precisely the same approach as Example 1:

$$q(x,t) = -\frac{2id_0 e^{-8ic_0(c_0^2-3d_0^2)t^3+2ic_0(t^2+t)-i\zeta_0-2ic_0x}}{\cosh\left(2d_0x-2\eta_0-8d_0\left(d_0^2-3c_0^2\right)t^3-2d_0(t^2+t)\right)}.$$

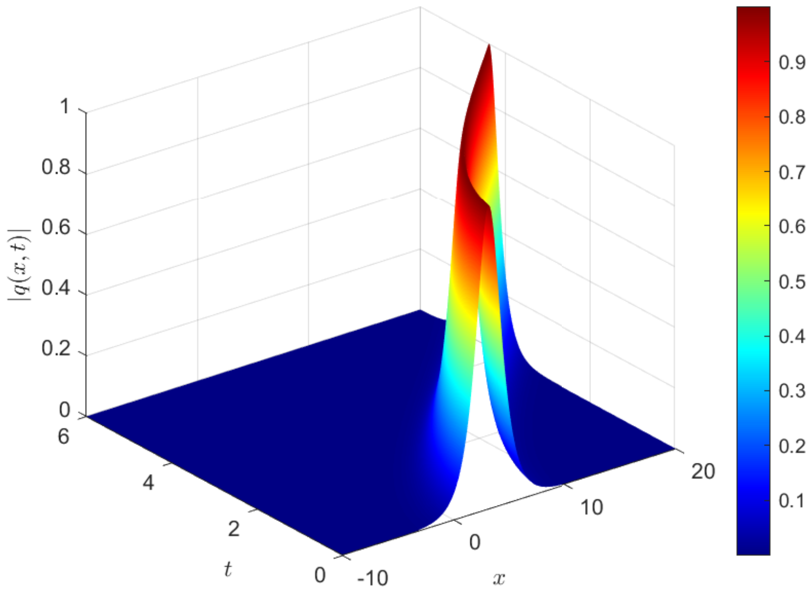

FIGURE 2. Plot of the modulus of the function $q(x,t)$ for the fixed parameter values: $c_0 = -0.3$, $d_0 = 0.5$, $\varphi = \pi/2$, $\psi = \pi/2$.

2. By setting $\beta(t) \equiv 0$ and $\alpha(t) \not\equiv 0$, $\vartheta(t) \not\equiv 0$ in equation (2.1), we derive the following equation:

$$iq_t + \alpha(t)\left(q_{xx} + 2q|q|^2\right) + \vartheta(t)q_x = 0, \tag{6.2}$$

which is a non-stationary NLS equation with a lower-order term. As a result, by setting $\beta(t) \equiv 0$ in Theorem 4.1, we obtain the time evolution of the scattering data for the Dirac operator, with the potential being determined as the solution to equation (6.2) under the conditions given in equations (2.2)–(2.4).

Example 3. Consider the subsequent problem:

$$iq_t + u_{xx} + 2q|q|^2 + 3itq_x = 0,$$

$$q(x,0) = -\frac{2id_0\left(\varkappa_0^0\right)^* e^{-2ic_0x}}{\left|\varkappa_0^0\right|\cosh 2(d_0x-\eta_0)},$$

where $\rho_0 = c_0 + id_0,\ (d_0 > 0),\quad \varkappa_0^0 = \left|\varkappa_0^0\right|e^{i\zeta_0},\quad \left|\varkappa_0^0\right|^2 = 4d_0^2e^{4\eta_0}$.

This example is solved in exactly the same way as Example 1, and the solution to this problem is as follows:

$$q(x,t) = -\frac{2id_0 e^{-4i(c_0^2-d_0^2)t+3ic_0t^2-2ic_0x-i\zeta_0}}{\cosh\left(8c_0d_0t-3d_0t^2+2d_0x-2\eta_0\right)}.$$

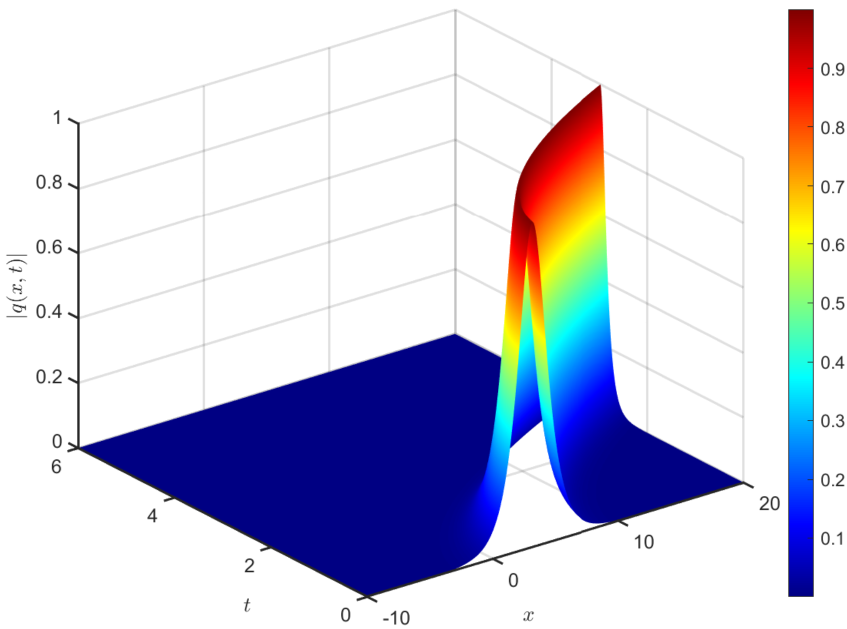

FIGURE 3. Plot of the modulus of the function $q(x,t)$ for the fixed parameter values: $c_0 = -0.3$, $d_0 = 0.5$, $\varphi = \pi/2$, $\psi = \pi/2$.

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