Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

A CUBATURE FORMULA FOR A CLASS OF VECTOR POTENTIALS WITH SINGULAR KERNELS

ELNUR H. KHALILOV AND VAFA O. SAFAROVA

Abstract. In this work, a cubature formula is constructed for a class of vector potentials with singular kernels, and error estimates for the constructed cubature formulas are provided.

1. Introduction

It is known (see [2, pp. 153-154]) that the internal and external electric boundary value problems, as well as the internal and external magnetic boundary value problems, are reduced to a system of integral equations dependent on vector potentials:

$$(Ff)(x) = 2\int_{\Omega} div_x \left\{ \Phi_k(x,\, y) \left[n(y), f(y) \right] \right\} d\Omega_y, \quad x = (x_1, x_2, x_3) \in \Omega, \quad (1.1)$$

$$(Gg)(x) = 2\int_{\Omega} \left(n(x), rot_x \left\{ \Phi_k(x,\, y)\ g(y) \right\} \right) d\Omega_y, \quad x = (x_1, x_2, x_3) \in \Omega, \quad (1.2)$$

$$(K\lambda)(x) = 2\int_{\Omega} \left[n(x), grad_x \left\{ \Phi_k(x,\, y)\ \lambda(y) \right\} \right] d\Omega_y, \quad x = (x_1, x_2, x_3) \in \Omega, \quad (1.3)$$

and

$$(T\mu)(x) = -2\int_{\Omega} \left[n(x), \left[n(x), rot_x \left\{ \Phi_k(x,\, y)\ \mu(y)\, n(y) \right\} \right] \right] d\Omega_y, \\ x = (x_1,\, x_2,\, x_3) \in \Omega, \quad (1.4)$$

where $\Omega \subset R^3$ is the Lyapunov surface, $n(x) = (n_1(x), n_2(x), n_3(x))$ is the outward unit normal at point $x \in \Omega$, $f(x) = (f_1(x), f_2(x), f_3(x))$ and $g(x) = (g_1(x), g_2(x), g_3(x))$ are vector functions continuous on the surface Ω, while $\lambda(x)$ and $\mu(x)$ are scalar functions continuous on the surface Ω. The notation $[a, b]$ denotes the cross product of vectors a and b, and (a, b) −denotes the dot product;

$$\Phi_k(x,\, y) = \frac{\exp(ik\,|x-y|)}{4\pi\,|x-y|}, \quad x, y \in R^3, \quad x \neq y,$$

2010 *Mathematics Subject Classification.* 65R20, 45E05.

Key words and phrases. electric boundary value problems, magnetic boundary value problems, vector potentials, cubature formula.

is the fundamental solution of the Helmholtz equation $\Delta u + k^2 u = 0$, Δ is the Laplace operator, and k is the wave number, where $Im\, k \geq 0$.

Since in many cases it is impossible to find exact solutions to the integral equations, the study of approximate solutions to these integral equations becomes of interest. To find an approximate solution, it is first necessary to construct cubature formulas for the integrals involved in these equations. It is worth noting that the counterexample constructed by A.M. Lyapunov (see [3, pp. 89–90]) demonstrates that for single-layer and double-layer potentials with continuous density, the derivative, generally speaking, does not exist. Consequently, the operators F, G, K and T are not defined in the space of continuous functions on the surface Ω. It is worth noting that in [1], by considering the derivative of the double-layer potential as a hypersingular integral, a cubature formula for the normal derivative of the double-layer potential was constructed. It should be noted that the cubature formula constructed in [1] is not practical in the sense that its coefficients are singular integrals. However, in [6], the existence of the normal derivative of the acoustic double-layer potential in the sense of the Cauchy principal value is demonstrated, and a formula for computing the normal derivative of the acoustic double-layer potential is provided. Furthermore, in [5], a cubature formula for the normal derivative of the acoustic single-layer potential is constructed, while in [7], a cubature formula for the normal derivative of the acoustic double-layer potential is developed. In the present work, the convergence of integrals (1.1) – (1.4) in the sense of the Cauchy principal value is proven, and cubature formulas for these integrals are constructed.

2. On the convergence of integrals (1.1) – (1.4)

Let us introduce the modulus of continuity of the function $f \in C(\Omega)$:

$$\omega(f,\delta) = \delta \sup_{\tau \geq \delta} \frac{\bar{\omega}(f,\tau)}{\tau}, \quad \delta > 0,$$

where

$$\bar{\omega}(f,\tau) = \max_{\substack{|x-y|\leq\tau, \\ x,y\in\Omega}} |f(x) - f(y)|,$$

$$|f(x) - f(y)| = \sqrt{(f_1(x) - f_1(y))^2 + (f_2(x) - f_2(y))^2 + (f_3(x) - f_3(y))^2},$$

and $C(\Omega)$ denotes the space of all continuous functions on the surface Ω with the norm $\|f\|_\infty = \max_{x\in\Omega} |f(x)|$.

Theorem 2.1. *Let $\Omega \subset R^3$ be a Lyapunov surface with exponent $0 < \alpha \leq 1$ and*

$$\int_0^{diam\,\Omega} \frac{\omega(f,t)}{t} dt < +\infty.$$

Then the integral (1.1) *exists in the sense of the Cauchy principal value, and*

$$\sup_{x\in\Omega} |\,(Ff)\,(x)\,| \le M^{1}\left(\|f\|_\infty + \int\limits_0^{diam\;\Omega} \frac{\omega\,(f,t)}{t}dt\right).$$

Proof: It is obvious that

$$(Ff)\,(x) = 2\int\limits_\Omega \Big(\frac{\partial\Phi_k\,(x,\,y)}{\partial x_1}\,(n_2\,(y)\,f_3\,(y) - n_3\,(y)\,f_2\,(y)) +$$

$$+\frac{\partial\Phi_k\,(x,\,y)}{\partial x_2}\,(n_3\,(y)\,f_1\,(y) - n_1\,(y)\,f_3\,(y)) +$$

$$+\frac{\partial\Phi_k\,(x,\,y)}{\partial x_3}\,(n_1\,(y)\,f_2\,(y) - n_2\,(y)\,f_1\,(y))\Big)d\Omega_y.$$

First, we prove that under the conditions of the theorem, the integral

$$F_1\,(x) = 2\int\limits_\Omega \frac{\partial\Phi_k\,(x,\,y)}{\partial x_1}\,(n_2\,(y)\,f_3\,(y) - n_3\,(y)\,f_2\,(y))\,d\Omega_y$$

exists in the sense of the Cauchy principal value.

Let us denote by $d > 0$ the radius of the standard sphere for Ω (see [9, pp. 400]) and let $\Omega_\varepsilon\,(x) = \{y \in \Omega :\ |x-y| < \varepsilon\}$, where $x \in \Omega$ and $\varepsilon > 0$. It is known that for each $x \in \Omega$ the set $\Omega_d\,(x)$ is projected uniquely onto the set $\Pi_d(x)$ lying in the tangent plane $\Gamma(x)$ to Ω at the point x. On the piece $\Omega_d(x)$ we choose a local rectangular coordinate system (u, v, w), with the origin at the point x, where the w-axis is directed along the normal $n\,(x)$, and the u- and v-axes lie in the tangent plane $\Gamma\,(x)$. Then in these coordinates the neighborhood $\Omega_d(x)$ can be defined by the equation $w = \psi\,(u,\,v)\,,\ (u,\,v) \in \Pi_d(x)$, where

$$\psi \in H_{1,\alpha}\,(\Pi_d\,(x)) \ \text{ and } \psi\,(0,\,0) = 0,\quad \frac{\partial\psi\,(0,\,0)}{\partial u} = 0,\quad \frac{\partial\psi\,(0,\,0)}{\partial v} = 0. \tag{2.1}$$

Here $H_{1,\alpha}\,(\Pi_d\,(x))$ denotes the linear space of all continuously differentiable functions ψ on $\Pi_d\,(x)$, whose $grad\psi$ satisfies the Holder condition with the exponent $0 < \alpha \le 1$, i.e.,

$$|\,grad\psi\,(u_1,v_1) - grad\,\psi\,(u_2,v_2)\,| \le M_\psi\left(\sqrt{(u_1-u_2)^2 + (v_1-v_2)^2}\right)^\alpha,$$

$$\forall\,(u_1,v_1)\,,(u_2,v_2) \in \Pi_d(x),$$

where M_ψ is a positive constant depending on ψ, and not from (u_1,v_1) or (u_2,v_2). Moreover, let $\Gamma_d\,(x)$ be the part of the tangent plane $\Gamma\,(x)$ at the point $x \in \Omega$, enclosed inside a sphere of radius d with center at the point x. Obviously, if a point $\tilde{y} \in \Gamma\,(x)$ is a projection of a point $y \in \Omega$, then

$$|x-\tilde{y}| \le |x-y| \le C_1\,|x-\tilde{y}|\,,\quad mes\Omega_d\,(x) \le C_2\,mes\Gamma_d\,(x)\,, \tag{2.2}$$

where C_1 and C_2 are positive constants depending only on Ω (if Ω is a sphere, then $C_1 = \sqrt{2}$ and $C_2 = 2$).

[1]From here on we will denote by M positive constants that are different in different inequalities.

It is not difficult to calculate that

$$F_1(x) = F_{1,1}(x) + F_{1,2}(x) + F_{1,3}(x) + F_{1,4}(x), \tag{2.3}$$

where

$$F_{1,1}(x) = 2\int_{\Omega} \frac{(ik\,|x-y|\exp(ik\,|x-y|) - (\exp(ik\,|x-y|) - 1))\,(x_1 - y_1)}{4\pi\,|x-y|^3} \times$$

$$\times (n_2(y)\,f_3(y) - n_3(y)\,f_2(y))\,d\Omega_y,$$

$$F_{1,2}(x) = \int_{\Omega \backslash \Omega_d(x)} \frac{y_1 - x_1}{4\pi\,|x-y|^3}\,(n_2(y)\,f_3(y) - n_3(y)\,f_2(y))\,d\Omega_y,$$

$$F_{1,3}(x) = \int_{\Omega_d(x)} \frac{y_1 - x_1}{4\pi\,|x-y|^3}\Big((n_2(y)\,f_3(y) - n_3(y)\,f_2(y)) -$$

$$-(n_2(x)\,f_3(x) - n_3(x)\,f_2(x))\Big)d\Omega_y$$

and

$$F_{1,4}(x) = (n_2(x)\,f_3(x) - n_3(x)\,f_2(x))\int_{\Omega_d(x)} \frac{y_1 - x_1}{4\pi\,|x-y|^3}\,d\Omega_y.$$

Taking into account the inequality

$$|\exp(ik\,|x-y|) - 1\,| \le |k|\,|x-y|,$$

we obtain that

$$\left|\frac{(ik\,|x-y|\exp(ik\,|x-y|) - (\exp(ik\,|x-y|) - 1))\,(x_1 - y_1)}{4\pi\,|x-y|^3}\right| \le \frac{M}{|x-y|}. \tag{2.4}$$

This means that the integral $F_{1,1}(x)$ converges as improper and

$$|\,F_{1,1}(x)| \le M\,\|f\|_{\infty}, \quad \forall x \in \Omega.$$

As can be seen, the integral $F_{1,2}(x)$ exists as a proper integral, and therefore

$$|\,F_{1,2}(x)| \le M\,\|f\|_{\infty}, \quad \forall x \in \Omega.$$

Taking into account the inequality (see [9, p. 400])

$$|n(y) - n(x)| \le M\,|y-x|^{\alpha}, \quad \forall x, y \in \Omega,$$

we obtain that

$$|(n_2(y)\,f_3(y) - n_3(y)\,f_2(y)) - (n_2(x)\,f_3(x) - n_3(x)\,f_2(x))| \le$$

$$\le M\left(|x-y|^{\alpha}\,\|f\|_{\infty} + \omega(f, |x-y|)\right). \tag{2.5}$$

Then, moving on to the double integral, we have

$$|F_{1,3}(x)| \le M\left(\|f\|_{\infty}\int_{\Omega_d(x)} \frac{d\Omega_y}{|x-y|^{2-\alpha}} + \int_{\Omega_d(x)} \frac{\omega(f, |x-y|)}{|x-y|^2}\,d\Omega_y\right) \le$$

$$\le M\left(\|f\|_{\infty} + \int_0^d \frac{\omega(f, t)}{t}\,dt\right) < +\infty, \quad \forall x \in \Omega.$$

It remains to prove that the integral $F_{1,4}(x)$ exists in the sense of the Cauchy principal value. Let $d_0 = d/(2C_1)$. Obviously,

$$O_{d_0}(x) = \left\{ (u, v, 0) \mid \sqrt{u^2+v^2} < d_0 \right\} \subset \Pi_d(x).$$

Then, according to the formula for reducing a surface integral to an iterated one, we obtain

$$\int_{\Omega_d(x)} \frac{y_1 - x_1}{|x-y|^3} d\Omega_y = \int_{\Pi_d(x)\backslash O_{d_0}(x)} \frac{u}{\left(\sqrt{u^2+v^2+\psi^2(u,v)}\right)^3} \times$$

$$\times \sqrt{1 + \left(\frac{\partial\psi(u,v)}{\partial u}\right)^2 + \left(\frac{\partial\psi(u,v)}{\partial v}\right)^2} du\, dv +$$

$$+ \int_{O_{d_0}(x)} \frac{u}{\left(\sqrt{u^2+v^2+\psi^2(u,v)}\right)^3} \times$$

$$\times \left(\sqrt{1 + \left(\frac{\partial\psi(u,v)}{\partial u}\right)^2 + \left(\frac{\partial\psi(u,v)}{\partial v}\right)^2} - 1 \right) du\, dv +$$

$$+ \int_{O_{d_0}(x)} \frac{u}{\left(\sqrt{u^2+v^2}\right)^3} du\, dv +$$

$$\int_{O_{d_0}(x)} u \left(\frac{1}{\left(\sqrt{u^2+v^2+\psi^2(u,v)}\right)^3} - \frac{1}{\left(\sqrt{u^2+v^2}\right)^3} \right) du\, dv.$$

As can be seen, the first term of the integral in the last equality exists as a proper one. Moreover, taking into account the inequalities (see [9, pp. 402])

$$\left|\frac{\partial\psi(u,v)}{\partial u}\right| \le M\left(\sqrt{u^2+v^2}\right)^\alpha, \left|\frac{\partial\psi(u,v)}{\partial v}\right| \le M\left(\sqrt{u^2+v^2}\right)^\alpha, \qquad (2.6)$$

we find

$$\left| \sqrt{1 + \left(\frac{\partial\psi(u,v)}{\partial u}\right)^2 + \left(\frac{\partial\psi(u,v)}{\partial v}\right)^2} - 1 \right| \le M\left(u^2+v^2\right)^\alpha.$$

Then we obtain the following estimate for the second term of the integral:

$$\left| \int_{O_{d_0}(x)} \frac{u\left(\sqrt{1 + \left(\frac{\partial\psi(u,v)}{\partial u}\right)^2 + \left(\frac{\partial\psi(u,v)}{\partial v}\right)^2} - 1 \right)}{\left(\sqrt{u^2+v^2+\psi^2(u,v)}\right)^3} du\, dv \right| \le M.$$

Moving to the polar coordinate system

$$\begin{cases} u = r\cos\tau, \\ v = r\sin\tau, \end{cases} \qquad 0 \le \tau \le 2\pi,$$

we obtain that the third term of the integral is equal to zero:

$$\int\limits_{O_{d_0}(x)} \frac{u}{\left(\sqrt{u^2+v^2}\right)^3} du\,dv = \lim_{\varepsilon\to+0} \int\limits_{O_{d_0}(x)\backslash O_\varepsilon(x)} \frac{u}{\left(\sqrt{u^2+v^2}\right)^3} du\,dv =$$

$$= \lim_{\varepsilon\to+0} \int\limits_{\varepsilon}^{d_0} \int\limits_{0}^{2\pi} \frac{\cos\varphi}{r}\, d\varphi\, dr = 0. \tag{2.7}$$

Since there exists a point $(\theta_1 u\ ,\ \theta_2 v)$ such that

$$\psi\left(u,\,v\right) - \psi\left(0,\,0\right)\ = \frac{\partial\psi\left(\theta_1 u\ ,\ \theta_2 v\right)}{\partial u} u + \frac{\partial\psi\left(\theta_1 u\ ,\ \theta_2 v\right)}{\partial v} v,$$

where $0 < \theta_1 < 1$ and $0 < \theta_2 < 1$, then taking into account (2.1) and (2.6), we find that

$$\left|\,\psi\left(u,\,v\right)\right| = \left|\,\psi\left(u,\,v\right)\ - \psi\left(0,\,0\right)\right| \le M\,\left(\sqrt{u^2+v^2}\right)^{1+\alpha}.$$

Therefore,

$$\left| \frac{1}{\left(\sqrt{u^2+v^2+\psi^2\left(u,\,v\right)}\right)^3} - \frac{1}{\left(\sqrt{u^2+v^2}\right)^3} \right| \le M\, \frac{1}{\left(\sqrt{u^2+v^2}\right)^{3-2\alpha}},$$

$$\forall\left(u,\,v\right) \in \Pi_d\left(x'\right) \backslash \left(0,0\right).$$

Then, moving on to the repeated integral, for the last term of the integral we obtain the following estimate:

$$\left| \int\limits_{O_{d_0}(x)} u \left(\frac{1}{\left(\sqrt{u^2+v^2+\psi^2\left(u,\,v\right)}\right)^3} - \frac{1}{\left(\sqrt{u^2+v^2}\right)^3} \right) du\,dv \right| \le M.$$

As a result, we obtain that the integral $F_{1,4}\left(x\right)$ exists in the sense of the Cauchy principal value and

$$\left|F_{1,4}\left(x\right)\right| \le M\left\|f\right\|_\infty, \forall x\in\Omega.$$

As a result, taking into account the obtained estimates for the expressions $F_{1,1}\left(x\right)$, $F_{1,2}\left(x\right)$, $F_{1,3}\left(x\right)$ and $F_{1,4}\left(x\right)$ in equality (2.3), we obtain that the integral

$$F_1\left(x\right) = 2\int\limits_{\Omega} \frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1} \left(n_2\left(y\right) f_3\left(y\right) - n_3\left(y\right) f_2\left(y\right)\right) d\Omega_y$$

exists in the sense of the Cauchy principal value, and

$$\sup_{x\in\Omega}\left|\,F_1\left(x\right)\right| \le M\left(\left\|f\right\|_\infty + \int\limits_0^{diam\ \Omega} \frac{\omega\left(f,t\right)}{t} dt \right).$$

In a similar way, it can be shown that the integrals

$$F_2(x) = 2\int\limits_{\Omega} \frac{\partial \Phi_k(x, y)}{\partial x_2} \left(n_3(y) f_1(y) - n_1(y) f_3(y)\right) d\Omega_y$$

and

$$F_3(x) = 2\int\limits_{\Omega} \frac{\partial \Phi_k(x, y)}{\partial x_3} \left(n_1(y) f_2(y) - n_2(y) f_1(y)\right) d\Omega_y$$

exist in the Cauchy principal value sense, with

$$\sup_{x\in\Omega} |\, F_2(x)\, | \le M \left(\|f\|_\infty + \int\limits_0^{diam\ \Omega} \frac{\omega(f,t)}{t} dt \right)$$

and

$$\sup_{x\in\Omega} |\, F_3(x)\, | \le M \left(\|f\|_\infty + \int\limits_0^{diam\ \Omega} \frac{\omega(f,t)}{t} dt \right).$$

This completes the proof of the theorem.

It is not difficult to calculate that

$$(Gg)(x) = 2\int\limits_{\Omega} \left(n_1(x) \left(\frac{\partial \Phi_k(x, y)}{\partial x_2} g_3(y) - \frac{\partial \Phi_k(x, y)}{\partial x_3} g_2(y) \right) + \right.$$
$$+ n_2(x) \left(\frac{\partial \Phi_k(x, y)}{\partial x_3} g_1(y) - \frac{\partial \Phi_k(x, y)}{\partial x_1} g_3(y) \right) +$$
$$\left. + n_3(x) \left(\frac{\partial \Phi_k(x, y)}{\partial x_1} g_2(y) - \frac{\partial \Phi_k(x, y)}{\partial x_2} g_1(y) \right) \right) d\Omega_y,$$

$$(K\lambda)(x) = 2\int\limits_{\Omega} \left(\left(\frac{\partial \Phi_k(x, y)}{\partial x_3} n_2(x) - \frac{\partial \Phi_k(x, y)}{\partial x_2} n_3(x) \right) e_1 + \right.$$
$$+ \left(\frac{\partial \Phi_k(x, y)}{\partial x_1} n_3(x) - \frac{\partial \Phi_k(x, y)}{\partial x_3} n_1(x) \right) e_2 +$$
$$\left. + \left(\frac{\partial \Phi_k(x, y)}{\partial x_2} n_1(x) - \frac{\partial \Phi_k(x, y)}{\partial x_1} n_2(x) \right) e_3 \right) \lambda(y)\, d\Omega_y$$

and

$$(T\mu)(x) = -2\int\limits_{\Omega} \left(V_1(x,y)\, e_1 + V_2(x,y)\, e_2 + V_3(x,y)\, e_3 \right) \mu(y)\, d\Omega_y,$$

where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$,

$$V_1(x,y) = n_1(x)\, n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} n_1(y) - n_1(x)\, n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_1} n_3(y) -$$
$$- (n_2(x))^2 \frac{\partial \Phi_k(x, y)}{\partial x_2} n_3(y) + (n_2(x))^2 \frac{\partial \Phi_k(x, y)}{\partial x_3} n_2(y) -$$

$$
\begin{aligned}
&-\left(n_3(x)\right)^2 \frac{\partial \Phi_k(x, y)}{\partial x_2} n_3(y)+\left(n_3(x)\right)^2 \frac{\partial \Phi_k(x, y)}{\partial x_3} n_2(y)+\\
&+n_1(x) n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_1} n_2(y)-n_1(x) n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} n_1(y),
\end{aligned}
$$

$$
\begin{aligned}
V_2(x, y)= & n_2(x) n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_1} n_2(y)-n_2(x) n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} n_1(y)- \\
& -\left(n_3(x)\right)^2 \frac{\partial \Phi_k(x, y)}{\partial x_3} n_1(y)+\left(n_3(x)\right)^2 \frac{\partial \Phi_k(x, y)}{\partial x_1} n_3(y)- \\
& -\left(n_1(x)\right)^2 \frac{\partial \Phi_k(x, y)}{\partial x_3} n_1(y)+\left(n_1(x)\right)^2 \frac{\partial \Phi_k(x, y)}{\partial x_1} n_3(y)+ \\
& +n_1(x) n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} n_3(y)-n_1(x) n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} n_2(y)
\end{aligned}
$$

and

$$
\begin{aligned}
V_3(x, y)= & n_1(x) n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} n_3(y)-n_1(x) n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} n_2(y)- \\
& -\left(n_1(x)\right)^2 \frac{\partial \Phi_k(x, y)}{\partial x_1} n_2(y)+\left(n_1(x)\right)^2 \frac{\partial \Phi_k(x, y)}{\partial x_2} n_1(y)- \\
& -\left(n_2(x)\right)^2 \frac{\partial \Phi_k(x, y)}{\partial x_1} n_2(y)+\left(n_2(x)\right)^2 \frac{\partial \Phi_k(x, y)}{\partial x_2} n_1(y)+ \\
& +n_2(x) n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} n_1(y)-n_2(x) n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_1} n_3(y) .
\end{aligned}
$$

Then, proceeding in exactly the same way as in the proof of Theorem 2.1, we can prove the validity of the following theorems.

Theorem 2.2. *Let $\Omega \subset R^3$ be a Lyapunov surface with exponent $0<\alpha \leq 1$ and*

$$
\int_0^{diam\, \Omega} \frac{\omega(g, t)}{t} dt<+\infty .
$$

Then the integral (1.2) *exists in the sense of the Cauchy principal value, and*

$$
\sup_{x \in \Omega}|(G g)(x)| \leq M\left(\|g\|_{\infty}+\int_0^{diam\, \Omega} \frac{\omega(g, t)}{t} dt\right) .
$$

Theorem 2.3. *Let $\Omega \subset R^3$ be a Lyapunov surface with exponent $0<\alpha \leq 1$ and*

$$
\int_0^{diam\, \Omega} \frac{\omega(\lambda, t)}{t} dt<+\infty .
$$

Then the integral (1.3) *exists in the sense of the Cauchy principal value, and*

$$
\sup_{x \in \Omega}|(K \lambda)(x)| \leq M\left(\|\lambda\|_{\infty}+\int_0^{diam\, \Omega} \frac{\omega(\lambda, t)}{t} dt\right) .
$$

Theorem 2.4. *Let* $\Omega \subset R^3$ *be a Lyapunov surface with exponent* $0 < \alpha \leq 1$ *and*

$$\int\limits_0^{diam\,\Omega} \frac{\omega(\mu, t)}{t} dt < +\infty.$$

Then the integral (1.4) *exists in the sense of the Cauchy principal value, and*

$$\sup_{x \in \Omega} |(T\mu)(x)| \leq M \left(\|\mu\|_\infty + \int\limits_0^{diam\,\Omega} \frac{\omega(\mu, t)}{t} dt \right).$$

3. Cubature formula for integrals (1.1) − (1.4)

We partition Ω into "regular" elementary parts: $\Omega = \bigcup\limits_{l=1}^{N} \Omega_l$. By a "regular" elementary part we mean a set of points subordinate to the following requirements:

(1) for each $l \in \{1, 2, ..., N\}$ the elementary part Ω_l is closed and the set $\overset{0}{\Omega}_l$ of its interior points with respect to Ω is not empty; moreover, $mes\,\overset{0}{\Omega}_l = mes\Omega_l$ and $\overset{0}{\Omega}_l \bigcap \overset{0}{\Omega}_j = \emptyset$ for $j \in \{1, 2, ...N\}$, $j \neq l$;

(2) for each $l \in \{1, 2, ..., N\}$ the elementary part Ω_l is a connected piece of the surface Ω and the boundary of the elementary part Ω_l is a continuous curve;

(3) for each $l \in \{1, 2, ..., N\}$ there exists a so-called supporting point $x(l) = (x_1(l), x_2(l), x_3(l)) \in \Omega_l$ such that

(3.1) $r_l(N) \sim R_l(N)$ (the expression $r_l(N) \sim R_l(N)$ means that $r_l(N)$ and $R_l(N)$ are equivalent, i.e., there exist numbers $C_1 > 0$ and $C_2 < +\infty$ such that $C_1 \leq \frac{r_l(N)}{R_l(N)} \leq C_2$ for any N), where $r_l(N) = \min\limits_{x \in \partial\Omega_l} |x - x(l)|$ and $R_l(N) = \max\limits_{x \in \partial\Omega_l} |x - x(l)|$;

(3.2) $R_l(N) \leq \frac{d}{2}$;

(3.3) $r_j(N) \sim r_l(N)$ for each $j \in \{1, 2, ..., N\}$.

Obviously, $r(N) \sim R(N)$ and $\lim\limits_{N \to \infty} r(N) = \lim\limits_{N \to \infty} R(N) = 0$, where $R(N) = \max\limits_{l=\overline{1,N}} R_l(N)$, $r(N) = \min\limits_{l=\overline{1,N}} r_l(N)$.

The following lemmas are true.

Lemma 3.1. *([8]). There exist constants* $C_0' > 0$ *and* $C_1' > 0$ *not depending on* N *such that, for all* $l, j \in \{1, 2, ..., N\}$, $j \neq l$, *and all* $y \in \Omega_j$, *the following inequalities hold:*

$$C_0' |y - x(l)| \leq |x(j) - x(l)| \leq C_1' |y - x(l)|,$$

where the $x(l)$, $l \in \{1, 2, ..., N\}$, *are supporting points.*

Lemma 3.2. *([8]). For a partition* $\Omega = \bigcup\limits_{l=1}^{N} \Omega_l$ *of the surface* Ω *into regular elementary parts, the following relation holds:* $R(N) \sim \frac{1}{\sqrt{N}}$.

It is obvious that there exists a natural number N_0 such that

$$(R(N))^{\frac{1}{1+\alpha}} \le \min\{1, d/2\}, \forall N > N_0.$$

Let us put

$$P_l = \left\{ j \mid 1 \le j \le N, \ |x(l) - x(j)| \le (R(N))^{\frac{1}{1+\alpha}} \right\},$$

$$Q_l = \left\{ j \mid 1 \le j \le N, \ |x(l) - x(j)| > (R(N))^{\frac{1}{1+\alpha}} \right\}.$$

Theorem 3.1. *Let $\Omega \subset R^3$ be a Lyapunov surface with exponent $0 < \alpha \le 1$ and*

$$\int_0^{diam\,\Omega} \frac{\omega(f,t)}{t} dt < +\infty.$$

Then the sequence

$$\left(F^N f\right)(x(l)) = 2 \sum_{j \in Q_l} div_x \left\{ \Phi_k(x, y) \left[n(y), f(y) \right] \right\} \big|_{x=x(l),\, y=x(j)} \cdot mes\Omega_j$$

converges to $F(x(l))$ at $N \to \infty$, and

$$\max_{l=\overline{1,N}} \left| (Ff)(x(l)) - \left(F^N f\right)(x(l)) \right| \le$$

$$\le M \left(\|f\|_\infty N^{-\frac{\alpha}{2(1+\alpha)}} + \int_0^{N^{-\frac{1}{2(1+\alpha)}}} \frac{\omega(f,t)}{t} dt + N^{-\frac{1}{2(1+\alpha)}} \int_{N^{-\frac{1}{2(1+\alpha)}}}^{diam\Omega} \frac{\omega(f,t)}{t^2} dt \right). \quad (3.1)$$

Proof. It is obvious that

$$\left(F^N f\right)(x(l)) =$$

$$= 2 \sum_{j \in Q_l} \frac{\partial \Phi_k(x(l), x(j))}{\partial x_1(l)} \left(n_2(x(j)) f_3(x(j)) - n_3(x(j)) f_2(x(j)) \right) mes\Omega_j +$$

$$+ 2 \sum_{j \in Q_l} \frac{\partial \Phi_k(x(l), x(j))}{\partial x_2(l)} \left(n_3(x(j)) f_1(x(j)) - n_1(x(j)) f_3(x(j)) \right) mes\Omega_j +$$

$$+ 2 \sum_{j \in Q_l} \frac{\partial \Phi_k(x(l), x(j))}{\partial x_3(l)} \left(n_1(x(j)) f_2(x(j)) - n_2(x(j)) f_1(x(j)) \right) mes\Omega_j.$$

Then, as we can see, it is sufficient to prove that the sequence

$$F_1^N(x(l)) =$$

$$= 2 \sum_{j \in Q_l} \frac{\partial \Phi_k(x(l), x(j))}{\partial x_1(l)} \left(n_2(x(j)) f_3(x(j)) - n_3(x(j)) f_2(x(j)) \right) mes\Omega_j$$

converges to $F_1(x(l))$ at $N \to \infty$. It is obvious that

$$F_1(x(l)) - F_1^N(x(l)) = h_1^N(x(l)) + h_2^N(x(l)),$$

where

$$h_1^N\left(x(l)\right)=2\int\limits_{\bigcup\limits_{j\in P_l}\Omega_j}\frac{\partial\Phi_k\left(x(l),y\right)}{\partial x_1(l)}\left(n_2(y)f_3(y)-n_3(y)f_2(y)\right)d\Omega_y$$

and

$$h_2^N\left(x(l)\right)=2\sum_{j\in Q_i}\int\limits_{\Omega_j}\left(\frac{\partial\Phi_k\left(x(l),y\right)}{\partial x_1(l)}\left(n_2(y)f_3(y)-n_3(y)f_2(y)\right)-\right.$$

$$\left.-\frac{\partial\Phi_k\left(x(l),x(j)\right)}{\partial x_1(l)}\left(n_2(x(j))f_3(x(j))-n_3(x(j))f_2(x(j))\right)\right)d\Omega_y.$$

It is not difficult to calculate that $h_1^N\left(x\left(l\right)\right)=h_{11}^N\left(x\left(l\right)\right)+h_{12}^N\left(x\left(l\right)\right)+h_{13}^N\left(x\left(l\right)\right)$, where

$$h_{11}^N\left(x\left(l\right)\right)=\frac{1}{2\pi}\int\limits_{\bigcup\limits_{j\in P_l}\Omega_j}\left(n_2\left(y\right)f_3\left(y\right)-n_3\left(y\right)f_2\left(y\right)\right)\times$$

$$\times\frac{\left(ik\left|x\left(l\right)-y\right|\exp\left(ik\left|x\left(l\right)-y\right|\right)-\left(\exp\left(ik\left|x\left(l\right)-y\right|\right)-1\right)\right)\left(x_1\left(l\right)-y_1\right)}{\left|x\left(l\right)-y\right|^3}d\Omega_y,$$

$$h_{12}^N\left(x\left(l\right)\right)=\frac{1}{2\pi}\int\limits_{\bigcup\limits_{j\in P_l}\Omega_j}\frac{y_1-x_1\left(l\right)}{\left|x\left(l\right)-y\right|^3}\times$$

$$\times\left(\left(n_2\left(y\right)f_3\left(y\right)-n_3\left(y\right)f_2\left(y\right)\right)-\left(n_2\left(x\left(j\right)\right)f_3\left(x\left(j\right)\right)-n_3\left(x\left(j\right)\right)f_2\left(x\left(j\right)\right)\right)\right)d\Omega_y$$

and

$$h_{13}^N\left(x\left(l\right)\right)=\frac{n_2\left(x\left(j\right)\right)f_3\left(x\left(j\right)\right)-n_3\left(x\left(j\right)\right)f_2\left(x\left(j\right)\right)}{2\pi}\int\limits_{\bigcup\limits_{j\in P_l}\Omega_j}\frac{y_1-x_1\left(l\right)}{\left|x\left(l\right)-y\right|^3}d\Omega_y.$$

Let $y\in\partial\left(\bigcup\limits_{j\in P_l}\Omega_j\right)$. Obviously, there exist natural numbers $p\in P_l$ and $m\in Q_l$ such that $y\in\partial\Omega_p$ and $y\in\partial\Omega_m$. From here we have

$$\left|x\left(l\right)-y\right|\le\left|x\left(l\right)-x\left(p\right)\right|+\left|x\left(p\right)-y\right|\le\left(R\left(N\right)\right)^{\frac{1}{1+\alpha}}+R\left(N\right)$$

and

$$\left|x\left(l\right)-y\right|\ge\left|x\left(l\right)-x\left(m\right)\right|-\left|x\left(m\right)-y\right|>\left(R\left(N\right)\right)^{\frac{1}{1+\alpha}}-R\left(N\right),$$

therefore,

$$\left(R\left(N\right)\right)^{\frac{1}{1+\alpha}}-R\left(N\right)<\left|x\left(l\right)-y\right|\le$$

$$\le\left(R\left(N\right)\right)^{\frac{1}{1+\alpha}}+R\left(N\right),\forall y\in\partial\left(\bigcup_{j\in P_l}\Omega_j\right). \tag{3.2}$$

Then, taking into account inequality (2.4) and moving on to the double integral, we obtain that

$$\left|h_{11}^N(x(l))\right|\le M\int\limits_{\bigcup\limits_{j\in P_l}\Omega_j}\frac{\left\|f_3\right\|_\infty+\left\|f_2\right\|_\infty}{\left|x(l)-y\right|}d\Omega_y\le$$

$$\le M \|f\|_\infty \int_0^{2\pi} d\varphi \int_0^{(R(N))^{\frac{1}{1+\alpha}}} dt = 2\pi M \|f\|_\infty (R(N))^{\frac{1}{1+\alpha}},$$

and taking into account inequality (2.5), we obtain that

$$\left|h_{12}^N(x(l))\right| \le M \int_{\bigcup_{j\in P_l}\Omega_j} \left(\frac{\|f\|_\infty}{|x(l)-y|^{2-\alpha}} + \frac{\omega(f,|x(l)-y|)}{|x(l)-y|^2} \right) d\Omega_y \le$$

$$\le M \int_0^{2\pi} d\varphi \int_0^{(R(N))^{\frac{1}{1+\alpha}}} \left(\frac{\|f\|_\infty}{t^{2-\alpha}} + \frac{\omega(f,t)}{t^2} \right) t dt =$$

$$= 2\pi M \left(\frac{\|f\|_\infty}{\alpha} (R(N))^{\frac{\alpha}{1+\alpha}} + \int_0^{(R(N))^{\frac{1}{1+\alpha}}} \frac{\omega(f,t)}{t} dt \right).$$

To evaluate expressions $h_{13}^N(x(l))$ on a piece $\Omega_d(x(l))$ we choose a local rectangular coordinate system $(u,\, v,\, w)$ with the origin at the point $x(l)$, where the axis w is directed along the normal $n(x(l))$, and the axes u and v will lie on the tangent plane $\Gamma(x(l))$. By Π_l^* we denote the projection of the set $\bigcup_{j\in P_l}\Omega_j$ onto the tangent plane $\Gamma(x(l))$. Let

$$d_l = \min_{\tilde{y}\in\partial\Pi_l^*} |x(l)-\tilde{y}|,\, O_{d_l}(x(l)) = \left\{ u^2+v^2<d_l \right\} \subset \Gamma(x(l)).$$

Then, according to the formula for reducing a surface integral to an iterated integral, we obtain that

$$\int_{\bigcup_{j\in P_l}\Omega_j} \frac{y_1-x_1(l)}{|x(l)-y|^3} d\Omega_y = \int_{O_{dl}(x(l))} \frac{u}{\left(\sqrt{u^2+v^2}\right)^3} dudv +$$

$$+ \int_{O_{d_l}(x(l))} \frac{u\left(\sqrt{1+\left(\frac{\partial\psi}{\partial u}\right)^2+\left(\frac{\partial\psi}{\partial v}\right)^2}-1\right)}{\left(\sqrt{u^2+v^2+\psi^2(u,\, v)}\right)^3} dudv +$$

$$+ \int_{O_{d_l}(x(l))} u \left(\frac{1}{\left(\sqrt{u^2+v^2+\psi^2(u,v)}\right)^3} - \frac{1}{\left(\sqrt{u^2+v^2}\right)^3} \right) dudv +$$

$$+ \int_{\Pi_l^*\backslash O_{d_l}(x(l))} \frac{u\sqrt{1+\left(\frac{\partial\psi}{\partial u}\right)^2+\left(\frac{\partial\psi}{\partial v}\right)^2}}{\left(\sqrt{u^2+v^2+\psi^2(u,\, v)}\right)^3} dudv. \tag{3.3}$$

From (2.7) it is obvious that the first integral on the right-hand side of equality (3.3) exists in the sense of the Cauchy principal value and is equal to zero. Moreover, taking into account inequalities (2.1) and (2.2), we have

$$\left| \int\limits_{O_{d_l}(x(l))} \frac{u\left(\sqrt{1+\left(\frac{\partial\psi}{\partial u}\right)^2+\left(\frac{\partial\psi}{\partial v}\right)^2}-1\right)}{\left(\sqrt{u^2+v^2+\psi^2(u,\,v)}\right)^3}\,dudv \right| \leq M\ (R(N))^{\frac{2\alpha}{1+\alpha}},$$

$$\left| \int\limits_{O_{d_l}(x(l))} u\left(\frac{1}{\left(\sqrt{u^2+v^2+\psi^2(u,v)}\right)^3}-\frac{1}{\left(\sqrt{u^2+v^2}\right)^3}\right)dudv \right| \leq M\ (R(N))^{\frac{2\alpha}{1+\alpha}}.$$

Let us estimate the last integral in equality (3.3). First of all, there is a point $\tilde{y}_* \in \Pi_l^*$ such that $d_l = |x(l)-\tilde{y}_*|$. Let $y_* \in \partial\left(\bigcup\limits_{j\in P_l}\Omega_j\right)$ denote the preimage of the point $\tilde{y}_*$, and let $\gamma(a\,,b)$ denote the angle between the vectors a and b. Applying inequality (3.2), we obtain that

$$d_l = |x(l)-y_*|\,\cos\gamma\,(y_*-x(l)\ ,\,\tilde{y}_*-x(l)) =$$
$$= |x(l)-y_*|\,\sqrt{1-\cos^2\gamma\,(y_*-x(l)\ ,\,n(x(l)))} \geq$$
$$\geq |x(l)-y_*|\,\sqrt{1-M^2\,|x(l)-y_*|^{2\alpha}} \geq$$
$$\geq \left((R(N))^{\frac{1}{1+\alpha}}-R(N)\right)\sqrt{1-M^2\left((R(N))^{\frac{1}{1+\alpha}}+R(N)\right)^{2\alpha}} \geq$$
$$\geq \left((R(N))^{\frac{1}{1+\alpha}}-R(N)\right)\sqrt{1-M^2\left(2\,(R(N))^{\frac{1}{1+\alpha}}\right)^{2\alpha}} =$$
$$= \left((R(N))^{\frac{1}{1+\alpha}}-R(N)\right)\sqrt{\left(1-2^\alpha M\,(R(N))^{\frac{\alpha}{1+\alpha}}\right)\left(1+2^\alpha M\,(R(N))^{\frac{\alpha}{1+\alpha}}\right)} \geq$$
$$\geq \left((R(N))^{\frac{1}{1+\alpha}}-R(N)\right)\left(1-2^\alpha M\,(R(N))^{\frac{\alpha}{1+\alpha}}\right) \geq$$
$$\geq (R(N))^{\frac{1}{1+\alpha}}-(1+2^\alpha M)\ R(N).$$

Then

$$\left| \int\limits_{\Pi_l^*\setminus O_{d_l}(x(l))} \frac{u}{\left(\sqrt{u^2+v^2+\psi^2(u,\,v)}\right)^3}\sqrt{1+\left(\frac{\partial\psi}{\partial u}\right)^2+\left(\frac{\partial\psi}{\partial v}\right)^2}\,dudv \right| \leq$$

$$\leq M \int\limits_{(R(N))^{\frac{1}{1+\alpha}}-(1+2^\alpha M)\,R(N)}^{(R(N))^{\frac{1}{1+\alpha}}+R(N)} \frac{dt}{t} \leq$$

$$\leq M\,\frac{(2+2^\alpha M)\ R(N)}{(R(N))^{\frac{1}{1+\alpha}}-R(N)\ (1+2^\alpha M)} \leq M\ (R(N))^{\frac{\alpha}{1+\alpha}}.$$

As a result, we have

$$\left|h_{13}^{N}\left(x\left(l\right)\right)\right|\leq M\left\|f\right\|_{\infty}\left(R\left(N\right)\right)^{\frac{\alpha}{1+\alpha}}.$$

Summing up the obtained estimates for the expressions $h_{11}^{N}\left(x\left(l\right)\right)$, $h_{12}^{N}\left(x\left(l\right)\right)$ and $h_{13}^{N}\left(x\left(l\right)\right)$, we obtain

$$\left|h_{1}^{N}\left(x\left(l\right)\right)\right|\leq M\left(\left\|f\right\|_{\infty}\left(R\left(N\right)\right)^{\frac{\alpha}{1+\alpha}}+\int\limits_{0}^{\left(R\left(N\right)\right)^{\frac{1}{1+\alpha}}}\frac{\omega\left(f,t\right)}{t}dt\right).$$

To evaluate the expression $h_{2}^{N}\left(x\left(l\right)\right)$ we use the representation

$$h_{2}^{N}\left(x\left(l\right)\right)=h_{21}^{N}\left(x\left(l\right)\right)+h_{22}^{N}\left(x\left(l\right)\right),$$

where

$$h_{21}^{N}\left(x\left(l\right)\right)=2\sum_{j\in Q_{l}}\int\limits_{\Omega_{j}}\frac{\partial\Phi_{k}\left(x\left(l\right),y\right)}{\partial x_{1}\left(l\right)}\times$$

$$\times\left(\left(n_{2}\left(y\right)f_{3}\left(y\right)-n_{3}\left(y\right)f_{2}\left(y\right)\right)-\left(n_{2}\left(x\left(j\right)\right)f_{3}\left(x\left(j\right)\right)-n_{3}\left(x\left(j\right)\right)f_{2}\left(x\left(j\right)\right)\right)\right)d\Omega_{y}$$

and

$$h_{22}^{N}\left(x\left(l\right)\right)=2\sum_{j\in Q_{l}}\int\limits_{\Omega_{j}}\left(\frac{\partial\Phi_{k}\left(x\left(l\right),y\right)}{\partial x_{1}\left(l\right)}-\frac{\partial\Phi_{k}\left(x\left(l\right),x\left(j\right)\right)}{\partial x_{1}\left(l\right)}\right)\times$$

$$\times\left(n_{2}\left(x\left(j\right)\right)f_{3}\left(x\left(j\right)\right)-n_{3}\left(x\left(j\right)\right)f_{2}\left(x\left(j\right)\right)\right)d\Omega_{y}.$$

It is obvious that the function $\omega\left(f,\delta\right)$ does not decrease, and the function $\omega\left(f,\delta\right)/\delta$ does not increase. Then, taking into account inequality (2.5), we find

$$\left|h_{21}^{N}\left(x\left(l\right)\right)\right|\leq M\sum_{j\in Q_{l}}\int\limits_{\Omega_{j}}\frac{\left|x\left(j\right)-y\right|^{\alpha}\left\|f\right\|_{\infty}+\omega\left(f,\left|x\left(j\right)-y\right|\right)}{\left|x\left(l\right)-y\right|^{2}}d\Omega_{y}\leq$$

$$\leq M\left(\left\|f\right\|_{\infty}\left(R\left(N\right)\right)^{\alpha}\int\limits_{\bigcup_{j\in Q_{l}}\Omega_{j}}\frac{d\Omega_{y}}{\left|x\left(l\right)-y\right|^{2}}+\omega\left(f,R\left(N\right)\right)\int\limits_{\bigcup_{j\in Q_{l}}\Omega_{j}}\frac{d\Omega_{y}}{\left|x\left(l\right)-y\right|^{2}}\right)\leq$$

$$\leq M\left(\left\|f\right\|_{\infty}\left(R\left(N\right)\right)^{\alpha}\int\limits_{\left(R\left(N\right)\right)^{\frac{1}{1+\alpha}}}^{diam\Omega}\frac{dt}{t}+\omega\left(f,R\left(N\right)\right)\int\limits_{\left(R\left(N\right)\right)^{\frac{1}{1+\alpha}}}^{diam\Omega}\frac{dt}{t}\right)\leq$$

$$\leq M\left(\|f\|_{\infty}(R(N))^{\alpha}|\ln R(N)|+\omega\left(f,(R(N))^{\frac{1}{1+\alpha}}\right)\int\limits_{(R(N))^{\frac{1}{1+\alpha}}}^{diam\Omega}\frac{dt}{t}\right)\leq$$

$$\leq M\left(\|f\|_{\infty}(R(N))^{\alpha}|\ln R(N)|+(R(N))^{\frac{1}{1+\alpha}}\int\limits_{(R(N))^{\frac{1}{1+\alpha}}}^{diam\Omega}\frac{\omega(f,t)}{t^{2}}dt\right).$$

Let $y\in\Omega_j,\ j\in Q_l$. Since

$$\frac{\partial\Phi_k(x(l),y)}{\partial x_1(l)}-\frac{\partial\Phi_k(x(l),x(j))}{\partial x_1(l)}=$$

$$=\left(\frac{1}{4\pi|x(l)-y|^{3}}-\frac{1}{4\pi|x(l)-y(j)|^{3}}\right)\times$$
$$\times(ik\,|x(l)-y(j)|-1)\exp(ik|x(l)-y(j)|)\,(x_1(l)-y_1(j))+$$

$$+\frac{ik(\,|x(l)-y|-|x(l)-x(j)|)\exp(ik|x(l)-y|)\,(x_1(l)-y_1)}{4\pi|x(l)-y|^{3}}+$$

$$+\frac{(\,ik|x(l)-y|-1)(\exp(ik|x(l)-y|)-\exp(ik|x(l)-x(j)|))\,(x_1(l)-y_1)}{4\pi|x(l)-y|^{3}}+$$

$$+\frac{(\,ik|x(l)-y|-1)\exp(ik|x(l)-y|)\,(x_1(j)-y_1)}{4\pi|x(l)-y|^{3}},$$

then taking into account Lemma 3.1, we obtain that

$$\left|\frac{\partial\Phi_k(x(l),y)}{\partial x_1(l)}-\frac{\partial\Phi_k(x(l),x(j))}{\partial x_1(l)}\right|\leq M\,\frac{R(N)}{|x(l)-y|^{3}}.$$

From here we have

$$\left|h_{22}^{N}(x(l))\right|\leq MR(N)\|f\|_{\infty}\int\limits_{\bigcup_{j\in Q_l}\Omega_j}\frac{d\Omega_y}{|x(l)-y|^{3}}\leq$$

$$\leq MR(N)\|f\|_{\infty}\int\limits_{(R(N))^{\frac{1}{1+\alpha}}}^{diam\Omega}\frac{dt}{t^{2}}\leq M(R(N))^{\frac{\alpha}{1+\alpha}}\|f\|_{\infty}.$$

Summing up the obtained estimates for the expressions $h_1^N(x(l))$ and $h_2^N(x(l))$, and taking into account Lemma 3.2, we obtain the validity of estimate (3.1).

Since (see [4, pp. 55])

$$\lim_{\delta\to 0}\left(\int\limits_{0}^{\delta}\frac{\omega(f,t)}{t}dt+\delta\int\limits_{\delta}^{diam\Omega}\frac{\omega(f,t)}{t^{2}}dt\right)=0,$$

it is obvious that

$$\lim_{N\to\infty}\max_{l=\overline{1,N}}\left|\left(Ff\right)\left(x\left(l\right)\right)-\left(F^{N}f\right)\left(x\left(l\right)\right)\right|=0.$$

The theorem is proven.

By proceeding in exactly the same way as in the proof of Theorem 3.1, one can prove the validity of the following theorems.

Theorem 3.2. *Let $\Omega\subset R^3$ be a Lyapunov surface with exponent $0<\alpha\leq 1$ and*

$$\int_0^{diam\,\Omega}\frac{\omega\left(g,t\right)}{t}dt<+\infty.$$

Then the sequence

$$\left(G^{N}g\right)\left(x\right)=2\sum_{j\in Q_l}\left(n\left(x\right),rot_x\left\{\Phi_k\left(x,\,y\right)g\left(y\right)\right\}\right)\Big|_{x=x(l),\,y=x(j)}\cdot mes\Omega_j$$

converges to $\left(Gg\right)\left(x(l)\right)$ at $N\to\infty$, and

$$\max_{l=\overline{1,N}}\Big|\left(Gg\right)\left(x\left(l\right)\right)-\left(G^{N}g\right)\left(x\left(l\right)\right)\Big|\leq$$

$$\leq M\Bigg(\left\|g\right\|_{\infty}N^{-\frac{\alpha}{2(1+\alpha)}}+\int_0^{N^{-\frac{1}{2(1+\alpha)}}}\frac{\omega\left(g,t\right)}{t}dt+N^{-\frac{1}{2(1+\alpha)}}\int_{N^{-\frac{1}{2(1+\alpha)}}}^{diam\Omega}\frac{\omega\left(g,t\right)}{t^2}dt\Bigg).$$

Theorem 3.3. *Let $\Omega\subset R^3$ be a Lyapunov surface with exponent $0<\alpha\leq 1$ and*

$$\int_0^{diam\,\Omega}\frac{\omega\left(\lambda,t\right)}{t}dt<+\infty.$$

Then the sequence

$$\left(K^{N}\lambda\right)\left(x\right)=2\sum_{j\in Q_l}\left[n\left(x\right),grad_x\left\{\Phi_k\left(x,\,y\right)\,\lambda\left(y\right)\right\}\right]\Big|_{x=x(l),\,y=x(j)}\cdot mes\Omega_j$$

converges to $\left(K\lambda\right)\left(x(l)\right)$ at $N\to\infty$, and

$$\max_{l=\overline{1,N}}\Big|\left(K\lambda\right)\left(x\left(l\right)\right)-\left(K^{N}\lambda\right)\left(x\left(l\right)\right)\Big|\leq$$

$$\leq M\Bigg(\left\|\lambda\right\|_{\infty}N^{-\frac{\alpha}{2(1+\alpha)}}+\int_0^{N^{-\frac{1}{2(1+\alpha)}}}\frac{\omega\left(\lambda,t\right)}{t}dt+N^{-\frac{1}{2(1+\alpha)}}\int_{N^{-\frac{1}{2(1+\alpha)}}}^{diam\Omega}\frac{\omega\left(\lambda,t\right)}{t^2}dt\Bigg).$$

Theorem 3.4. *Let $\Omega\subset R^3$ be a Lyapunov surface with exponent $0<\alpha\leq 1$ and*

$$\int_0^{diam\,\Omega}\frac{\omega\left(\mu,t\right)}{t}dt<+\infty.$$

Then the sequence

$$\left(T^N\mu\right)(x) = -2\sum_{j\in Q_l}\left[n\left(x\right),\left[n\left(x\right),rot_x\left\{\Phi_k\left(x,\,y\right)\,\mu\left(y\right)n\left(y\right)\right\}\right]\right]\big|_{x=x(l),\,y=x(j)}\cdot mes\Omega_j$$

converges to $\left(T\mu\right)\left(x(l)\right)$ *at* $N\to\infty$*, and*

$$\max_{l=\overline{1,\,N}}\left|\left(T\mu\right)\left(x\left(l\right)\right)-\left(T^N\mu\right)\left(x\left(l\right)\right)\right|\le$$

$$\le M\Bigg(\left\|\mu\right\|_\infty N^{-\frac{\alpha}{2(1+\alpha)}}+\int\limits_0^{N^{-\frac{1}{2(1+\alpha)}}}\frac{\omega\left(\mu,t\right)}{t}dt+N^{-\frac{1}{2(1+\alpha)}}\int\limits_{N^{-\frac{1}{2(1+\alpha)}}}^{diam\Omega}\frac{\omega\left(\mu,t\right)}{t^2}dt\Bigg).$$

Acknowledgements. The authors are grateful to the reviewer for careful reading of the paper and valuable comments.

References

[1] A.Yu. Anfinogenov, I.K. Lifanov and P.I. Lifanov, On certain one-and two-dimensional hypersingular integral equations, *Sb. Math.*, **192**, (2001), no 8, 1089–1131.

[2] D.L. Colton and R. Kress, *Integral equation methods in scattering theory*, Mir, Moscow, 1983 (in Russian).

[3] N.M. Gunter, *Potential theory and its applications to basic problems of mathematical physics*, Gostekhizdat, Moscow, 1953 (in Russian).

[4] A.I. Guseinov and Kh.Sh. Mukhtarov, *Introduction to the theory of nonlinear singular integral equations*, Nauka, Moscow, 1980 (in Russian).

[5] E.H. Khalilov, Cubic formula for the potential of simple layer and its normal derivative, *Proc. IMM NAS Azerb.*, **37**, (2012), 77-84.

[6] E.H. Khalilov, Some properties of the operators generated by a derivative of the acoustic double layer potential, *Siberian Math. Journal*, **55**, (2014), no 3, 564-573.

[7] E.H. Khalilov, Cubic formula for the normal derivative of a double layer acoustic potential, *Transac. NAS Azerb., ser. of phys.-tech. and math. Sciences*, **34**, (2014), no1, 73-82.

[8] E.H. Khalilov, Justification of the collocation method for a class of surface integral equations, *Math. Notes*, **107**, (2020), no 4, 663–678.

[9] V.S. Vladimirov, *The equations of mathematical physics*, Nauka, Moscow, 1976 (in Russian).

Elnur H. Khalilov
Azerbaijan State Oil and Industry University, Baku, Azerbaijan,
Western Caspian University, Baku, Azerbaijan
E-mail address: elnurkhalil@mail.ru

Vafa O. Safarova
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
E-mail address: vefa-seferova-91@bk.ru

Received: March 29, 2025; Revised: July 10, 2025; Accepted: July 30, 2025