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SYNTHESIS OF BOUNDARY CONDITIONS CONTROL AND STATE MEASUREMENT POINTS MOTION FOR FAST DAMPING OF STRING VIBRATIONS

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Abstract. The problem of synthesis of boundary controls for damping transverse vibrations of a string is investigated. The problem has two important features. Firstly, feedback is achieved by measuring the state (vertical movement) of the string points by remote sensors of the state of the string points moving outside the string. Secondly, to form the modes of operation of boundary controls, the values of state measurements at measurement points are used both at the current and past moments of time. The paper presents a formulation of the problem of synthesis of boundary controls, in which the parameters of the feedback involved in the linear dependencies between the measurements of the state of the string at the measurement points and the modes of boundary controls and the velocities of movement of the measuring devices are optimized. To determine the values of the optimized parameters involved in these dependencies, formulas are obtained that allow the use of effective numerical methods of first-order optimization. The results of numerical experiments on test problems are presented.

1. Introduction

The paper considers the problem of controlling the process of stabilizing transverse vibrations of a string by controlling the state of the string at the ends (boundary conditions) [10, 14, 15, 16]. Controls occurs with feedback. Measurements of the state (vertical movement) of the string points are made by remote sensors moving above the string. The values of the control actions in the boundary conditions are assigned depending on the values of the measurements of the state of the string points both at the current and past moments of time. The velocities of movement of the measuring devices are controlled and are assigned depending on the measured state values.

The formulation of the problem under study arises in many applied problems [5, 10, 11, 14, 15, 22, 16]. A special case of formulation of this problem arises,

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for example, in the controlling of transient processes of pipeline transportation of carbon raw materials [5, 11, 13].

It should be noted that the problem of damping string vibrations using program control has been studied by many authors. In the works of the authors [10, 14, 15] the problem of optimal damping to stabilization of string vibrations due to boundary conditions was studied. The dependence of the minimum time for damping the string vibration on its length was established. In the work [6] the problem of boundary conditions controlling under constraints on control actions was investigated. In the works [6, 8, 17] the problem of damping vibrations at the expense of stabilizers installed on the string itself was investigated. In works [4, 20] the problem of optimizing the placement of vibration stabilizers for a thin membrane string was solved. The above studies considered the problem of constructing optimal program control.

In contrast to previously studied formulations of problems of control of the vibration stabilization process, this paper considers, firstly, the problem of boundary control with feedback. Secondly, information from measuring devices moving along the string, both current and past moments in time is used for feedback. Thirdly, the velocity of movement of the sensors is also controlled and depends on the values of the current state measurements obtained by the sensors themselves.

Formulas are proposed for the dependence of the values of the boundary control and the velocities of movement of the sensors on the measured values of the state at the current and previous moments of time. The constant coefficients involved in these dependencies are the optimized feedback parameters. Formulas for the gradient components of the objective functional of the source problem based on the optimized parameters are obtained. The formulas allow the use of effective first-order optimization methods for the numerical determination of optimal parameter values.

The paper presents the results of numerical experiments and an analysis of the behavior of the stabilization process in the presence of noise (error) in the measurements of the current state carried out by the sensors.

2. Statement of the problem

The problem of damping transverse vibrations of a thin string by stabilizers installed at its ends is considered. The process of vibration at $t > 0$ is described by the following initial–boundary-value problem [10, 14, 26]:

$$u_{tt}(x,t) = a^2 u_{xx}(x,t) - \lambda u_t(x,t), \quad x \in (0,l), \tag{2.1}$$

$$u(x,t) = 0, \quad u_t(x,0) = \sum_{s=1}^{N_0} q_s \mu_\sigma(x;\theta_s), \quad t \le 0, \quad x \in [0,l], \tag{2.2}$$

$$u(0,t) = \vartheta_1(t), \tag{2.3}$$

$$u(l,t) = \vartheta_2(t). \tag{2.4}$$

Here: $u(x,t)$ is the function defining the amplitude of transverse vibrations of the string at point x at time t; $a^2 > 0$, $\lambda \ge 0$ are given constants determined by the physical properties of the string and the medium in which it is located; q_s is the magnitude of the initial state splash at N_0 points of the string θ_s, $s = 1, 2, \ldots, N_0$;

$\vartheta_1(t)$ and $\vartheta_2(t)$ are functions defining the control actions of the stabilizers at the left and right ends of the string, respectively, $x = 0$ and $x = l$.

The given piecewise continuous function in $x \in [0, l]$ $\mu_\sigma(x; \theta_s) \geq 0$ determines the intensity distribution of the splash of the initial state of the string in the σ neighborhood of the point $\breve{\theta} \in [\sigma, l - \sigma]$. It has the following properties:

$$\mu_\sigma\left(x; \breve{\theta}\right) \begin{cases} \geq 0 \text{ for } x \in \left[\breve{\theta} - \sigma, \breve{\theta} + \sigma\right], \\ = 0 \text{ for } x \notin \left[\breve{\theta} - \sigma, \breve{\theta} + \sigma\right], \end{cases} \qquad \int\limits_{\breve{\theta}-\sigma}^{\breve{\theta}+\sigma} \delta_\sigma\left(x; \breve{\theta}\right) dx = 1. \tag{2.5}$$

It is assumed that the values of the magnitudes $q = (q_1, q_2, \ldots, q_{N_0})$ of the initial state splash at the concentration points $\theta = (\theta_1, \theta_2, \ldots, \theta_{N_0})$ are not known exactly. The sets of their possible values q_s are given:

$$Q_s = \left\{q_s \in \mathbb{R} : \ \underline{q_s} \leq q_s \leq \overline{q_s}\right\}, \quad s = 1, 2, \ldots, N_0, \tag{2.6}$$

$$Q = Q_1 \times Q_2 \times \cdots \times Q_{N_0},$$

and the distribution density functions of their values $\rho_{Q_s}(q_s) \geq 0$ such that

$$\int\limits_{Q_s} \rho_{Q_s}(q_s) dq_s = 1, \quad s = 1, 2, \ldots, N_0.$$

The locations θ_s of possible initial state splashes are determined by the sets Θ_s, $s = 1, 2, \ldots, N_0$:

$$\theta_s \in \Theta_s \subset [\sigma, l - \sigma], \quad s = 1, 2, \ldots, N_0, \quad \Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_{N_0}, \tag{2.7}$$

with given distribution density functions $\rho_{\Theta s}(\theta_s) \geq 0$ such that

$$\int\limits_{\Theta_s} \rho_{\Theta_s}(\theta_s) d\theta_s = 1, \quad s = 1, 2, \ldots, N_0.$$

The functions $\vartheta_1(t)$ and $\vartheta_2(t)$ are optimized controls for the vibration damping process under consideration and satisfy the technological constraints:

$$\underline{\vartheta_1} \leq \vartheta_1(t) \leq \overline{\vartheta_1}, \quad \underline{\vartheta_2} \leq \vartheta_2(t) \leq \overline{\vartheta_2}, \quad t \geq 0. \tag{2.8}$$

It is known that the classical solution $u(x, t)$ of the initial–boundary-value problem (2.1)–(2.4) for the given control actions $\vartheta_1(t)$ and $\vartheta_2(t)$ exists and is unique [26].

The problem under consideration of the velocity of control of the process of damping the vibrations of a string consists in determining the control values of the vibration dampers $\vartheta = (\vartheta_1(t), \vartheta_2(t))$ that minimize the time T_f, after which the ε-settling condition is satisfied:

$$T_f \to \min, \tag{2.9}$$

$$J(T, \vartheta) \leq \varepsilon \quad \text{at} \quad T \geq T_f. \tag{2.10}$$

Here,

$$J(T_f, \vartheta) = \int\limits_{Q} \int\limits_{\Theta} I(T_f, \vartheta; q, \theta) \rho_Q(q) \rho_\Theta(\theta) d\theta dq, \tag{2.11}$$

$$I\left(T_f,\vartheta;q,\theta\right)=\int\limits_{T_f}^{T_1}\int\limits_{0}^{l}\mu_1\left(x\right)\left[u\left(x,t\right)\right]^2dxdt \tag{2.12}$$

$$+\varepsilon_1\|\vartheta_1\left(t\right)-\hat{\vartheta}_1\|^2_{L_2[0,T_1]}+\varepsilon_2\|\vartheta_2\left(t\right)-\hat{\vartheta}_2\|^2_{L_2[0,T_1]}.$$

The condition (2.10) will be called the ε-settling condition for a string.

Here the function $u\left(x,t\right)=u\left(x,t;\vartheta,q,\theta\right)$ is a solution of the initial–boundary-value problem (2.1)–(2.4) for the given concentration points θ_s of the magnitude of splashes of the state q_s at the initial time, $s=1,2,\ldots,N_0$, and damper modes $\vartheta=\left(\vartheta_1\left(t\right),\vartheta_2\left(t\right)\right)$; $\mu_1\left(x\right)\geq 0$ is a weight function that determines the significance of damping of vibrations at the point of the string $x\in[0,l]$. The second and third terms in (2.12) serve for regularization the functional, $\varepsilon_1\geq 0$, $\varepsilon_2\geq 0$, $\hat{\vartheta}_1\in\mathbb{R}$ and $\hat{\vartheta}_2\in\mathbb{R}$ are regularization parameters. $\varepsilon\geq 0$ is a given number characterizing the degree of "settling" of vibrations, specified from practical considerations.

Let us note the specificity of the functional (2.11), (2.12). Here the given value $\Delta T>0$ determines the duration of the time closed interval $[T_f,T_1]$, $T_1=T_f+\Delta T$, during which the degree of stabilization of the string vibrations are assessed. The functional (2.11), (2.12) estimates the quality of the control parameters $\vartheta=\left(\vartheta_1\left(t\right),\vartheta_2\left(t\right)\right)$ for the state of the string over the time interval $t\in[T_f,T_1]$ when controlling the process of damping vibrations over the time closed interval $t\in[0,T_1]$ on average over all values of the parameters of external splashes θ, q, satisfying the conditions (2.6), (2.7).

The meaning of the functional (2.11), (2.12) is that it evaluates the quality of the boundary condition control $\vartheta\left(t\right)$ of the string damping process on the closed interval $[0,T_f]$ on average over all sets of possible values of the splash magnitudes and the locations of their concentration at the initial moment of time.

To synthesize the current control values $\vartheta\left(t\right)$, $t\in[0,T_1]$, we will use information about the state of the string points at the current and previous moments of time, measured continuously in time by N remote sensors moving above it. The movements of the sensors are defined by the functions $\xi_i\left(t\right)\in[0,l]$, $i=1,2,\ldots,N$. The sensors continuously measure the values of the string transverse displacement (amplitude) at the current locations of the measuring devices:

$$\breve{u}_i\left(t\right)=u\left(\xi_i\left(t\right),t\right),\quad t\in\left[0,T_1\right],\quad i=1,2,\ldots,N. \tag{2.13}$$

The movements of the sensors are controllable and are described by the following Cauchy problems

$$\dot{\xi}_i\left(t\right)=a_i\left(t\right)\xi_i\left(t\right)+w_i\left(t\right),\quad t\in\left(0,T_1\right],\quad i=1,2,\ldots,N, \tag{2.14}$$

$$\xi_i\left(0\right)=\xi_i^0,\quad i=1,2,\ldots,N. \tag{2.15}$$

Here: $\xi_i\left(t\right)$ is location of i-th sensor at time t, $t\in[0,T_1]$; $w_i\left(t\right)$ is optimized control action on the movement of sensor; ξ_i^0, $i=1,2,\ldots,N$, are specified initial locations of sensors on the string.

The control actions are the velocities of movement of the measuring sensors, which must satisfy the technological limitations:

$$\underline{w_i}\leq w_i\left(t\right)\leq\overline{w_i},\quad t\in\left[0,T_1\right],\quad i=1,2,\ldots,N. \tag{2.16}$$

During the movement the sensors themselves should not approach the boundary of the string and should not go beyond the boundaries of individual non-intersecting segments specified for each sensor:

$$\xi_i(t) \in \left[\underline{\xi_i}, \overline{\xi_i}\right] \subset [0,l], \quad t \in [0,T_1], \quad i = 1,2,\ldots,N, \tag{2.17}$$
$$\left[\underline{\xi_i}, \overline{\xi_i}\right] \cap \left[\underline{\xi_j}, \overline{\xi_j}\right] = \emptyset, \quad i \neq j, \quad i,j = 1,2,\ldots,N.$$

To form the current value of the damping mode $\vartheta(t)$, we use the following dependence, which determines the continuous linear feedback of control actions from the state of the string at the measurement points (2.13) at the current and past moments of time:

$$\vartheta_1(t;\mathbf{k}) = \sum_{i=1}^{N}\left[k_i^1 \check{u}_i(t) + k_i^2 \check{u}_i(t-\tau)\right] \tag{2.18}$$
$$= \sum_{i=1}^{N}\left[k_i^1 u(\xi_i(t),t) + k_i^2 u(\xi_i(t-\tau),t-\tau)\right], \quad t \in [0,T_1],$$

$$\vartheta_2(t;\mathbf{k}) = \sum_{i=1}^{N}\left[k_i^3 \check{u}_i(t) + k_i^4 \check{u}_i(t-\tau)\right] \tag{2.19}$$
$$= \sum_{i=1}^{N}\left[k_i^3 u(\xi_i(t),t) + k_i^4 u(\xi_i(t-\tau),t-\tau)\right], \quad t \in [0,T_1].$$

Here: $k_1 = \left(k_1^1, k_2^1, \ldots, k_N^1\right)$, $k^2 = \left(k_1^2, k_2^2, \ldots, k_N^2\right)$, $k^3 = \left(k_1^3, k_2^3, \ldots, k_N^3\right)$, $k^4 = \left(k_1^4, k_2^4, \ldots, k_N^4\right)$ are the vectors of synthesized values of the gain coefficients.

Similarly, for the dependence of the control actions $w_i(t)$ on the movement of the sensors on the measured current values of the string displacement at the measurement points, we use the formula

$$w_i(t;\mathbf{k}) = k_i^5 \check{u}_i(t) + k_i^6 \check{u}_i(t-\tau) \tag{2.20}$$
$$= k_i^5 u(\xi_i(t),t) + k_i^6 u(\xi_i(t-\tau),t-\tau), \quad t \in [0,T_1], \quad i = 1,2,\ldots,N,$$

where $k^5 = \left(k_1^5, k_2^5, \ldots, k_N^5\right)$ and $k^6 = \left(k_1^6, k_2^6, \ldots, k_N^6\right)$ are vectors of feedback parameters.

Let denote $\mathbf{k} = \left(k_1, k^2, k^3, k^4, k^5, k^6\right) \in \mathbb{R}^{6N}$ the optimized vector of feedback parameters.

The value $\tau > 0$ is given and is determined depending on the intensity of the vibration process, namely, the more intense the damping, the smaller its value.

Substituting (2.18) and (2.19) into the boundary conditions (2.3), (2.4)

$$u(0,t) = \sum_{i=1}^{N}\left[k_i^1 u(\xi_i(t),t) + k_i^2 u(\xi_i(t-\tau),t-\tau)\right], \quad t \in (0,T_1], \tag{2.21}$$

$$u(l,t) = \sum_{i=1}^{N}\left[k_i^3 u(\xi_i(t),t) + k_i^4 u(\xi_i(t-\tau),t-\tau)\right], \quad t \in (0,T_1], \tag{2.22}$$

we obtain non-local (loaded) boundary conditions non-separated from the values of the state at the interior points of the string. Substituting the expressions for the velocities $w_i(t)$, $i = 1,2,\ldots,N$ from formula (2.20) into the differential equations

(2.14), we obtain the following differential equations, in which the solution of the initial–boundary-value problem (2.1), (2.2), (2.21), (2.22) participates:

$$\dot{\xi}_i(t) = a_i(t)\,\xi_i(t) + k_i^5 u(\xi_i(t), t) + k_i^6 u(\xi_i(t-\tau), t-\tau), \tag{2.23}$$

$$t \in (0, T_1], \quad i = 1, 2, \ldots, N.$$

Note that the initial–boundary-value problem (2.1), (2.2), (2.21), (2.22) and the Cauchy problems (2.23), (2.15) must be solved simultaneously. The issues of existence and uniqueness of the solution of linearly loaded initial–boundary-value problems and numerical methods for their solution were studied in such works as [1, 3, 9, 12, 19, 21].

It is clear that the solutions of the initial–boundary-value problem (2.1), (2.2), (2.21), (2.22) and the Cauchy problems (2.23), (2.15) depend on the synthesized constant parameters $\mathbf{k}$.

In this case, the functionality (2.11), (2.12) will also depend on the synthesized feedback parameters

$$J(T_f, \mathbf{k}) = \int\limits_Q \int\limits_\Theta I(T_f, \mathbf{k}; q, \theta)\, \rho_Q(q) \rho_\Theta(\theta)\, dq d\theta, \tag{2.24}$$

$$I(T_f, \mathbf{k}; q, \theta) = \int\limits_{T_f}^{T_1} \int\limits_0^l \mu_1(x)\, [u(x, t; \mathbf{k})]^2 dx dt + \varepsilon_3 \|\mathbf{k} - \hat{\mathbf{k}}\|^2_{\mathbb{R}^{6N}}. \tag{2.25}$$

Here: $\hat{\mathbf{k}} \in \mathbb{R}^{6N}$ are the regularization parameters of the functional (2.24), (2.25), the values of which are assigned using known regularization methods [27].

From the dependencies (2.18) and (2.19) it is clear that the controls $\vartheta_1(t; \mathbf{k})$, $\vartheta_2(t; \mathbf{k})$ and the velocities $w_i(t; \mathbf{k})$, $i = 1, 2, \ldots, N$ depend nonlinearly on the synthesized parameters. Consequently, the state $u(x, t; \mathbf{k})$ and the locations of the sensor points $\xi_i(t; \mathbf{k})$ are essentially nonlinear in terms of the synthesized parameters. The functional (2.25) represents a complex nonlinear dependence on the optimized parameters and it is not possible to conduct any studies of its convexity. This implies the possibility of multi-extremality of the functional in the space of synthesized feedback parameters $\mathbf{k} \in \mathbb{R}^{6N}$.

Let us substitute formulas (2.18) and (2.19) into constraints (2.8), and formulas (2.20) into constraints (2.16). We obtain the following constraints on the feedback parameters:

$$\underline{\vartheta_1} \le \sum_{i=1}^N \left[k_i^1 u(\xi_i(t), t) + k_i^2 u(\xi_i(t-\tau), t-\tau) \right] \le \overline{\vartheta_1}, \quad t \in [0, T_1], \tag{2.26}$$

$$\underline{\vartheta_2} \le \sum_{i=1}^N \left[k_i^3 u(\xi_i(t), t) + k_i^4 u(\xi_i(t-\tau), t-\tau) \right] \le \overline{\vartheta_2}, \quad t \in [0, T_1], \tag{2.27}$$

$$\underline{w_i} \le k_i^5 u(\xi_i(t), t) + k_i^6 u(\xi_i(t-\tau), t-\tau) \le \overline{w_i}, \quad t \in [0, T_1], \tag{2.28}$$

$$i = 1, 2, \ldots, N.$$

Let the range of the value of transverse vibrations of the string during the process be known for all admissible controls, the magnitudes of the initial splashes $q \in Q$ and the points of their concentrations $\theta \in \Theta$,

$$\underline{u} \le u\left(x,t\right) \le \overline{u}, \quad x \in \left[0,l\right], \quad t \in \left[0,T_1\right]. \tag{2.29}$$

Then, taking into account the conditions (2.29) and the dependencies (2.26)–(2.28), we obtain the following linear constraints on the synthesized feedback parameters:

$$\begin{cases} \underline{\vartheta_1} \le \sum\limits_{i=1}^{N} k_i^1 \underline{u} + k_i^2 \underline{u} \le \overline{\vartheta_1}, & \underline{\vartheta_1} \le \sum\limits_{i=1}^{N} k_i^1 \underline{u} + k_i^2 \overline{u} \le \overline{\vartheta_1}, \\ \underline{\vartheta_1} \le \sum\limits_{i=1}^{N} k_i^1 \overline{u} + k_i^2 \underline{u} \le \overline{\vartheta_1}, & \underline{\vartheta_1} \le \sum\limits_{i=1}^{N} k_i^1 \overline{u} + k_i^2 \underline{u} \le \overline{\vartheta_1}, \\ \underline{\vartheta_2} \le \sum\limits_{i=1}^{N} k_i^3 \underline{u} + k_i^4 \underline{u} \le \overline{\vartheta_2}, & \underline{\vartheta_2} \le \sum\limits_{i=1}^{N} k_i^3 \underline{u} + k_i^4 \overline{u} \le \overline{\vartheta_2}, \\ \underline{\vartheta_2} \le \sum\limits_{i=1}^{N} k_i^3 \overline{u} + k_i^4 \underline{u} \le \overline{\vartheta_2}, & \underline{\vartheta_2} \le \sum\limits_{i=1}^{N} k_i^3 \overline{u} + k_i^4 \underline{u} \le \overline{\vartheta_2}, \\ \underline{w_i} \le k_i^5 \underline{u} + k_i^6 \underline{u} \le \overline{w_i}, & \underline{w_i} \le k_i^5 \underline{u} + k_i^6 \overline{u} \le \overline{w_i}, \quad i = 1,2,\ldots,N, \\ \underline{w_i} \le k_i^5 \overline{u} + k_i^6 \underline{u} \le \overline{w_i}, & \underline{w_i} \le k_i^5 \overline{u} + k_i^6 \overline{u} \le \overline{w_i}, \quad i = 1,2,\ldots,N. \end{cases} \tag{2.30}$$

We will reduce the constraints (2.17) to the following equivalent constraints:

$$g_i\left(t;\xi_i\left(t\right)\right) = \left|g_i^0\left(t;\xi_i\left(t\right)\right)\right| - \frac{\overline{\xi_i} - \underline{\xi_i}}{2} \le 0, \quad i = 1,2,\ldots,N, \quad t \in \left[0,T_1\right], \tag{2.31}$$

$$g_i^0\left(t;\xi_i\left(t\right)\right) = \xi_i\left(t\right) - \frac{\overline{\xi_i} + \underline{\xi_i}}{2}.$$

To take into account the constraints (2.31) we use the following external penalty functional relative to the functional (2.24), (2.25):

$$\tilde{J}\left(T_f,\mathbf{k}\right) = \int\limits_{Q}\int\limits_{\Theta} \tilde{I}\left(T_f,\mathbf{k};q,\theta\right)\rho_Q(q)\rho_\Theta\left(\theta\right)dqd\theta, \tag{2.32}$$

$$\tilde{I}\left(T_f,\mathbf{k};q,\theta\right) = \int\limits_{T_f}^{T_1}\int\limits_{0}^{l} \mu_1\left(x\right)\left[u\left(x,t\right)\right]^2 dxdt + \varepsilon_3\|\mathbf{k} - \hat{\mathbf{k}}\|^2_{\mathbb{R}^{6N}} + \mathcal{R}G, \tag{2.33}$$

$$G = \sum_{i=1}^{N}\int\limits_{0}^{T_1}\left[g_i^+\left(t;\xi_i\left(t\right)\right)\right]^2 dt,$$

where $\mathcal{R} \ge 0$ is a positive penalty coefficient tending to $+\infty$. The notation $g_i^+\left(t;\xi_i\left(t\right)\right)$ means that $g_i^+\left(t;\xi_i\left(t\right)\right) = g_i\left(t;\xi_i\left(t\right)\right)$ if $g_i\left(t;\xi_i\left(t\right)\right) > 0$ and $g_i^+\left(t;\xi_i\left(t\right)\right) = 0$ if $g_i\left(t;\xi_i\left(t\right)\right) \le 0$, $i = 1,2,\ldots,N$.

The obtained problem (2.1), (2.2), (2.21), (2.22), (2.23), (2.15), (2.24), (2.25) can be classified as a class of parametric problems of optimal control of an object with distributed parameters [7, 24, 27]. The main features of the task are: 1) nonlinear participation in the problem of optimized parameters, which indicates the non-convexity of the control synthesis problem as a whole, and, consequently, the possible multi-extremality of the functional; 2) the presence of an non-separated boundary condition, in which the states of the process at the internal points of the string participate, which causes additional difficulties in solving the corresponding boundary value problems for given values of the control parameters [1, 3, 9, 12].

The statement of the problem (2.1)–(2.12) can be classified as class of problems of parametric optimal performance of objects with distributed parameters.

Note that, the problem under consideration is nonlinear and, in general, nonconvex in the parameters being optimized; therefore, it can be expected that it is multi-extremal. In this case, the solution to the problem is understood as finding some local optimum, in particular, the closest to some given initial approximation $\mathbf{k}^0$. In this work we will use the "multi-start" method, one of the most common methods of global optimization.

Thus, in the problem under consideration (2.1)–(2.12) it is required to determine the following parameters and functions: T_f is the minimum time of the transient process with initial conditions (2.2) to a state satisfying condition (2.10); admissible control actions of the stabilizers at the ends of the string $\vartheta_1(t)$ and $\vartheta_2(t)$, determined by formulas (2.18), (2.19); the controls $w_i(t)$, $i = 1, 2, \ldots, N$ of the sensor movement determined by formulas (2.20) and the corresponding trajectories $\xi_i(t)$ from formulas (2.23), (2.15).

Let us note an important specificity of the process described by the equation (2.1). In the case of $\lambda > 0$, which corresponds to the influence of elastic forces (resistance) of the string or the medium on the vibration process, the vibration process under consideration, even without the functioning of the dampers (i.e. $\vartheta_1(t) \equiv 0$ and $\vartheta_2(t) \equiv 0$), is self-soothing, i.e. at $t \to \infty$ the state of the string as a whole will tend to zero. Therefore, there will be a finite time T_f at which the amplitude of the string vibration will be small, and its state will satisfy the condition (2.10).

3. Approach and derivation of formulas for numerical solution of the problem

A two-level approach to solving the problem under consideration is proposed, which consists of separate optimization of the transient process time T_f and the optimized feedback parameters $\mathbf{k}$. At the top level, the transition time T_f is optimized using some one-dimensional optimization method to find the minimum possible value at which the condition (2.10) is satisfied. For each fixed value of T_f, the selected one-dimensional optimization algorithm solves the auxiliary problem of minimizing the functional to determine the corresponding feedback parameters $\mathbf{k}^*_{T_f}$:

$$\tilde{J}_{T_f}(\mathbf{k}) = \tilde{J}(T_f, \mathbf{k}), \tag{3.1}$$

$$\tilde{I}_{T_f}(\mathbf{k}; q, \theta) = \tilde{I}(T_f, \mathbf{k}; q, \theta),$$

under constraints (2.30) and (2.31).

As an algorithm for one-dimensional optimization for T_f, one can use, for example, the bisection method with a preliminary search for an uncertainty interval

$\left[\underline{T_f}, \overline{T_f}\right]$ such that

$$\tilde{J}\left(\underline{T_f}, \mathbf{k}^*_{\underline{T_f}}\right) = \min_{\mathbf{k}} \tilde{J}\left(\underline{T_f}, \mathbf{k}\right) \geq \varepsilon, \qquad (3.2)$$
$$\tilde{J}\left(\overline{T_f}, \mathbf{k}^*_{\overline{T_f}}\right) = \min_{\mathbf{k}} \tilde{J}\left(\overline{T_f}, \mathbf{k}\right) \leq \varepsilon.$$

It is clear that the main complexity of the proposed approach lies in carrying out the second level of optimization of determining the optimal values of the feedback parameters $\mathbf{k}$ for a given time T_f of the end of the control process, i.e. solving the auxiliary optimal control problem (2.1), (2.2), (2.21), (2.22), (2.23), (2.15), (2.32), (2.33), (3.1) with a fixed time T_f.

To minimize the penalty functional (2.32), (2.33) for finding the interval $\left[\underline{T_f}, \overline{T_f}\right]$ at each given T_f and penalty coefficient $\mathcal{R}$ taking into account the linearity of the constraints (2.30), it is proposed to use the iterative gradient projection method,

$$\mathbf{k}^{n+1} = \mathcal{P}_{(2.30)}\left[\mathbf{k}^n - \alpha^n \; \mathbf{grad}_{\mathbf{k}} \tilde{J}_{T_f}\left(\mathbf{k}^n\right)\right], \quad n = 0, 1, \ldots . \qquad (3.3)$$

Here $\mathcal{P}_{(2.30)}\left[\,\cdot\,\right]$ is the operator of projection of the optimized feedback parameters $\mathbf{k}$ onto the acceptable domain defined by the constraints (2.30).

In (3.3) the following notation is used: $6N$-dimensional gradient vector of the objective functional (2.32)

$$\mathbf{grad}_{\mathbf{k}} \tilde{J}_{T_f}\left(\mathbf{k}\right) = \left(\frac{\partial \tilde{J}_{T_f}\left(\mathbf{k}\right)}{\partial k_1}, \frac{\partial \tilde{J}_{T_f}\left(\mathbf{k}\right)}{\partial k^2}, \frac{\partial \tilde{J}_{T_f}\left(\mathbf{k}\right)}{\partial k^3}, \frac{\partial \tilde{J}_{T_f}\left(\mathbf{k}\right)}{\partial k^4}, \frac{\partial \tilde{J}_{T_f}\left(\mathbf{k}\right)}{\partial k^5}, \frac{\partial \tilde{J}_{T_f}\left(\mathbf{k}\right)}{\partial k^6}\right), \qquad (3.4)$$

$\alpha^n \geq 0$ is a step in the direction of the antigradient, determined by some known method that ensures the condition of monotonicity of the iterative process [27]

$$\tilde{J}_{T_f}\left(\mathbf{k}^{n+1}\right) \leq \tilde{J}_{T_f}\left(\mathbf{k}^n\right), \quad n = 0, 1, \ldots$$

In particular, one-dimensional minimization methods can be used to find α^n,

$$\alpha^n = \arg\min_{\alpha \geq 0} \tilde{J}_{T_f}\left(\mathcal{P}_{(2.30)}\left[\mathbf{k}^n - \alpha \; \mathbf{grad}_{\mathbf{k}} \tilde{J}_{T_f}\left(\mathbf{k}^n\right)\right]\right), \quad n = 0, 1, \ldots .$$

The following theorem presents formulas for the components of the gradient (3.4) of the functional $\tilde{J}_{T_f}\left(\mathbf{k}\right) = \tilde{J}\left(T_f, \mathbf{k}\right)$ with respect to $\mathbf{k}$ for a given completion time of the control process T_f.

Theorem 3.1. *If the above conditions are satisfied on the functions and parameters involved in the initial–boundary-value problem* (2.1), (2.2), (2.21), (2.22) *and in the Cauchy problems* (2.23), (2.15), *the functional* (3.1) *is differentiable with respect to the feedback parameters* $\mathbf{k}$, *and the components of its gradient are determined by the formulas:*

$$\frac{\partial \tilde{J}_{T_f}\left(\mathbf{k}\right)}{\partial k_i^1} = \int_Q \int_\Theta \left\{ - \int_0^{T_1} a^2 \psi_x\left(0, t\right) u\left(\xi_i\left(t\right), t\right) dt + 2\varepsilon\left(k_i^1 - \hat{k}_i^1\right)\right\} \qquad (3.5)$$
$$\times \rho_Q(q) \rho_\Theta\left(\theta\right) d\theta dq,$$

$$\frac{\partial \tilde{J}_{T_f}(\mathbf{k})}{\partial k_i^2} = \int_Q \int_\Theta \left\{ -\int_0^{T_1-\tau} a^2 \psi_x(0, t+\tau)\, u(\xi_i(t), t)\, dt + 2\varepsilon_3 \left(k_i^2 - \hat{k}_i^2 \right) \right\} \times \rho_Q(q) \rho_\Theta(\theta)\, d\theta dq, \tag{3.6}$$

$$\frac{\partial \tilde{J}_{T_f}(\mathbf{k})}{\partial k_i^3} = \int_Q \int_\Theta \left\{ -\int_0^{T_1} a^2 \psi_x(l, t)\, u(\xi_i(t), t)\, dt + 2\varepsilon_3 \left(k_i^3 - \hat{k}_i^3 \right) \right\} \times \rho_Q(q) \rho_\Theta(\theta)\, d\theta dq, \tag{3.7}$$

$$\frac{\partial \tilde{J}_{T_f}(\mathbf{k})}{\partial k_i^4} = \int_Q \int_\Theta \left\{ -\int_0^{T_1-\tau} a^2 \psi_x(l, t+\tau)\, u(\xi_i(t), t)\, dt + 2\varepsilon_3 \left(k_i^4 - \hat{k}_i^4 \right) \right\} \times \rho_Q(q) \rho_\Theta(\theta)\, d\theta dq, \tag{3.8}$$

$$\frac{\partial \tilde{J}_{T_f}(\mathbf{k})}{\partial k_i^5} = \int_Q \int_\Theta \left\{ -\int_0^{T_1} \varphi_i(t)\, u(\xi_i(t), t)\, dt + 2\varepsilon_3 \left(k_i^5 - \hat{k}_i^5 \right) \right\} \times \rho_Q(q) \rho_\Theta(\theta)\, d\theta dq, \tag{3.9}$$

$$\frac{\partial \tilde{J}_{T_f}(\mathbf{k})}{\partial k_i^6} = \int_Q \int_\Theta \left\{ -\int_0^{T_1-\tau} \varphi_i(t+\tau)\, u(\xi_i(t), t)\, dt + 2\varepsilon_3 \left(k_i^6 - \hat{k}_i^6 \right) \right\} \times \rho_Q(q) \rho_\Theta(\theta)\, d\theta dq, \tag{3.10}$$

$i = 1, 2, \ldots, N$. *For arbitrarily given admissible feedback parameters* $\mathbf{k}$*, magnitudes* q *and locations of* θ *splashes in the initial conditions* (2.2) *the functions* $\psi(x,t) = \psi(x,t;\mathbf{k},q,\theta)$ *and* $\varphi_i(t) = \varphi_i(t;\mathbf{k},q,\theta)$*,* $i = 1, 2, \ldots, N$ *are solutions of the following conjugate boundary-value problem and Cauchy problems:*

$$\psi_{tt}(x,t) = a^2 \psi_{xx}(x,t) + \lambda \psi_t(x,t) - 2\mu(x)\, u(x,t)\, \chi_{[T_f, T_1]}(t), \tag{3.11}$$

$$x \in \left(\xi_i\left(t^+\right), \xi_{i+1}\left(t^-\right) \right), \quad i = 0, 1, \ldots, N, \quad t \in [0, T_1),$$

$$\psi_x\left(\xi_i\left(t^+\right), t\right) = \psi_x\left(\xi_i\left(t^-\right), t\right) - k_i^1 \psi_x(0,t) - k_i^2 \psi_x(0, t+\tau) + k_i^3 \psi_x(l,t) + k_i^4 \psi_x(l, t+\tau) - \frac{k_i^5}{a^2} \varphi_i(t) - \frac{k_i^6}{a^2} \varphi_i(t+\tau), \tag{3.12}$$

$$t \in [0, T_1 - \tau), \quad i = 1, 2, \ldots, N,$$

$$\psi_x\left(\xi_i\left(t^+\right), t\right) = \psi_x\left(\xi_i\left(t^-\right), t\right) - k_i^1 \psi_x(0,t) + k_i^3 \psi_x(l,t) - \frac{k_i^5}{a^2} \varphi_i(t), \tag{3.13}$$

$$t \in [T_1 - \tau, T_1), \quad i = 1, 2, \ldots, N,$$

$$\psi\left(\xi_i\left(t^+\right), t\right) = \psi\left(\xi_i\left(t^-\right), t\right), \tag{3.14}$$

$$t \in [0, T_1), \quad i = 1, 2, \ldots, N,$$

$$\psi(x, T_1) = 0, \quad \psi_t(x, T_1) = \lambda \psi(x, T_1), \quad x \in [0, l], \tag{3.15}$$

$$\psi(0,t) = 0, \quad t \in [0, T_1), \tag{3.16}$$

$$\psi(l,t) = 0, \quad t \in [0, T_1), \tag{3.17}$$

$$\dot{\varphi}_i(t) = -a_i(t)\, \varphi_i(t) + 2\mathcal{R} sgn\left(g_i^0(t; \xi_i(t)) \right) g_i^+(t; \xi_i(t)) - \left[a^2 k_i^1 \psi_x(0,t) - a^2 k_i^3 \psi_x(l,t) + k_i^5 \varphi_i(t) \right. \tag{3.18}$$

$$+a^2k_i^2\psi_x(0,t+\tau)-a^2k_i^4\psi_x(l,t+\tau)+k_i^6\varphi_i(t+\tau)\big]u_x(\xi_i(t),t),$$
$$t\in[0,T_1-\tau),\quad i=1,2,\ldots,N,$$
$$\dot\varphi_i(t)=-a_i(t)\varphi_i(t)+2\mathcal{R}sgn\left(g_i^0(t;\xi_i(t))\right)g_i^+(t;\xi_i(t)) \tag{3.19}$$
$$-\left[a^2k_i^1\psi_x(0,t)-a^2k_i^3\psi_x(l,t)+k_i^5\varphi_i(t)\right]u_x(\xi_i(t),t),$$
$$t\in[T_1-\tau,T_1),\quad i=1,2,\ldots,N,$$
$$\varphi_i(T_1)=0,\quad i=1,2,\ldots,N. \tag{3.20}$$

Proof. To prove the differentiability of the functional $\tilde J_{T_f}(\mathbf{k})$ with respect to $\mathbf{k}$, we use the increment method [18, 27].

From the mutual independence of the sets of powers (2.4) and the locations of the concentration of the splashes in the initial conditions (2.6) the following relation holds:

$$\mathbf{grad_k}\tilde J_{T_f}(\mathbf{k})=\mathbf{grad_k}\int\limits_Q\int\limits_\Theta \tilde I_{T_f}(\mathbf{k};q,\theta)\rho_Q(q)\rho_\Theta(\theta)d\theta dq= \tag{3.21}$$
$$=\int\limits_Q\int\limits_\Theta \mathbf{grad_k}\tilde I_{T_f}(\mathbf{k};q,\theta)\rho_Q(q)\rho_\Theta(\theta)d\theta dq.$$

Using (3.21), we will work on obtaining the gradient components of the functional (2.33) for arbitrary but specifically specified values of the splash magnitudes q and the locations of their concentration centers θ.

Let, for given admissible control parameters $\mathbf{k}=\left(k_1,k^2,k^3,k^4,k^5,k^6\right)\in\mathbb{R}^{6N}$, splash magnitudes q and their concentration locations θ in the initial conditions (2.1), (2.2), (2.18), (2.19) the function $u(x,t)=u(x,t;\mathbf{k},q,\theta)$ be the solution of the initial–boundary-value problem, where the function $\tilde u(x,t;\tilde{\mathbf{k}},q,\theta)=u(x,t;\mathbf{k},q,\theta)+\Delta u(x,t)$ is the solution of the problem for admissible control parameters $\tilde{\mathbf{k}}=\mathbf{k}+\Delta\mathbf{k}$, which have received a sufficiently small increment $\Delta\mathbf{k}=\left(\Delta k_1,\Delta k^2,\Delta k^3,\Delta k^4,\Delta k^5,\Delta k^6\right)$.

Similarly, for admissible control parameters $\mathbf{k}$, the functions $\xi_i(t)=\xi_i(t;\mathbf{k})$, $i=1,2,\ldots,N$ will be solutions of the Cauchy problems (2.23), (2.15), where the functions $\tilde\xi_i(t;\tilde{\mathbf{k}})=\xi_i(t;\mathbf{k})+\Delta\xi_i(t)$, $i=1,2,\ldots,N$ are solutions of the problems for admissible control parameters $\tilde{\mathbf{k}}$.

Then it is easy to verify that $\Delta u(x,t)$ and $\Delta\xi_i(t)$, $i=1,2,\ldots,N$, up to terms of the second order of smallness, are solutions to the following initial-boundary value problem:

$$\Delta u_{tt}(x,t)=a^2\Delta u_{xx}(x,t)-\lambda\Delta u_t(x,t),\quad x\in[0,l],\quad t\in(0,T_1], \tag{3.22}$$
$$\Delta u(x,t)=0,\quad \Delta u_t(x,0)=0,\quad t\le 0,\quad x\in[0,l], \tag{3.23}$$
$$\Delta u(0,t)=\sum_{i=1}^N\left[\Delta k_i^1u(\xi_i(t),t)+k_i^1\Delta u(\xi_i(t),t)+k_i^1u_x(\xi_i(t),t)\Delta\xi_i(t)\right] \tag{3.24}$$
$$+\sum_{i=1}^N\Big[\Delta k_i^2u(\xi_i(t-\tau),t-\tau)+k_i^2\Delta u(\xi_i(t-\tau),t-\tau)$$
$$+k_i^2u_x(\xi_i(t-\tau),t-\tau)\Delta\xi_i(t-\tau)\Big]+\sum_{i=1}^N\Big[\Delta k_i^1\Delta u(\xi_i(t),t)$$

$$+\Delta k_i^2 \Delta u\left(\xi_i\left(t-\tau\right), t-\tau\right)\Big] + \mathcal{O}\left(\|\Delta\xi\left(t\right)\|_{\mathbb{R}^N}^2\right) + \mathcal{O}\left(\|\Delta\xi\left(t-\tau\right)\|_{\mathbb{R}^N}^2\right),$$

$$\Delta u\left(l,t\right) = \sum_{i=1}^{N}\Big[\Delta k_i^3 u\left(\xi_i\left(t\right),t\right) + k_i^3\Delta u\left(\xi_i\left(t\right),t\right) + k_i^3 u_x\left(\xi_i\left(t\right),t\right)\Delta\xi_i\left(t\right)\Big] \quad (3.25)$$

$$+\sum_{i=1}^{N}\Big[\Delta k_i^4 u\left(\xi_i\left(t-\tau\right), t-\tau\right) + k_i^4\Delta u\left(\xi_i\left(t-\tau\right), t-\tau\right)$$

$$+k_i^4 u_x\left(\xi_i\left(t-\tau\right), t-\tau\right)\Delta\xi_i\left(t-\tau\right)\Big] + \sum_{i=1}^{N}\Big[\Delta k_i^3\Delta u\left(\xi_i\left(t\right),t\right)$$

$$+\Delta k_i^4 \Delta u\left(\xi_i\left(t-\tau\right), t-\tau\right)\Big] + \mathcal{O}\left(\|\Delta\xi\left(t\right)\|_{\mathbb{R}^N}^2\right) + \mathcal{O}\left(\|\Delta\xi\left(t-\tau\right)\|_{\mathbb{R}^N}^2\right),$$

and Cauchy problems:

$$\Delta\dot{\xi}_i\left(t\right) = a_i\left(t\right)\Delta\xi_i\left(t\right) + \big[\Delta k_i^5 u\left(\xi_i\left(t\right),t\right) + k_i^5\Delta u\left(\xi_i\left(t\right),t\right) \quad (3.26)$$

$$+k_i^5 u_x\left(\xi_i\left(t\right),t\right)\Delta\xi_i\left(t\right) + \Delta k_i^6 u\left(\xi_i\left(t-\tau\right), t-\tau\right) + k_i^6\Delta u\left(\xi_i\left(t-\tau\right), t-\tau\right)$$

$$+k_i^6 u_x\left(\xi_i\left(t-\tau\right), t-\tau\right)\Delta\xi_i\left(t-\tau\right)\big] + \Delta k_i^5\Delta u\left(\xi_i\left(t\right),t\right)$$

$$+\Delta k_i^6\Delta u\left(\xi_i\left(t-\tau\right), t-\tau\right) + \mathcal{O}\left(\|\Delta\xi\left(t\right)\|_{\mathbb{R}^N}^2\right) + \mathcal{O}\left(\|\Delta\xi\left(t-\tau\right)\|_{\mathbb{R}^N}^2\right),$$

$$t \in (0, T_1], \quad i = 1,2,\ldots,N,$$

$$\Delta\xi_i\left(0\right) = 0, \quad i = 1,2,\ldots,N. \quad (3.27)$$

It is known that solutions of initial–boundary-value problems with respect to differential equations of hyperbolic type, in particular, of the form (3.22)–(3.25) [18], and ordinary differential equations of the form (3.26), (3.27), under the assumptions made on the functions involved in the formulation of the problem under study, continuously depend on the parameters and functions involved in them. Therefore, with respect to the increment of solutions to problems (2.1), (2.2), (2.21), (2.22) and (2.23), (2.15), obtained due to sufficiently small increments of the parameters $\Delta\mathbf{k}$, there are estimates that we write in a fairly general form:

$$\|\Delta u\left(x,t\right)\|_{L_2([0,l]\times[0,T_1])}^2 \le c_1\left(\|\Delta\mathbf{k}\|_{\mathbb{R}^{6N}}^2\right),$$

$$\|\Delta\xi\left(t\right)\|_{L_2^N[0,T_1]}^2 \le c_2\left(\|\Delta\mathbf{k}\|_{\mathbb{R}^{6N}}^2\right),$$

$c_1 > 0$, $c_2 > 0$ are constants that do not depend on $\Delta\mathbf{k}$.

In this case, the functional $\tilde{I}_{T_f}\left(\mathbf{k};q,\theta\right)$ will receive an increment that, with an accuracy of terms of the first order of smallness relative to the increment of the parameters $\Delta\mathbf{k}$ and, accordingly, the increment of the solution of the initial–boundary-value problem, we write in the form:

$$\Delta\tilde{I}_{T_f}\left(\mathbf{k};q,\theta\right) = \tilde{I}_{T_f}\left(\mathbf{k}+\Delta\mathbf{k};q,\theta\right) - \tilde{I}_{T_f}\left(\mathbf{k};q,\theta\right) = \Delta I_{T_f}\left(\mathbf{k};q,\theta\right) \quad (3.28)$$

$$+\mathcal{R}\Delta G = I_{T_f}\left(\mathbf{k}+\Delta\mathbf{k};q,\theta\right) - I_{T_f}\left(\mathbf{k};q,\theta\right) + \mathcal{R}\left(G\left(\xi\left(t\right)+\Delta\xi\left(t\right)\right) - G\right)$$

$$= \int_{T_f}^{T_1}\int_0^l 2\mu\left(x\right)u\left(x,t\right)\Delta u\left(x,t\right)dxdt + 2\varepsilon_3\langle\mathbf{k}-\hat{\mathbf{k}},\Delta\mathbf{k}\rangle$$

$$+\mathcal{R}\sum_{i=1}^{N}\int_{0}^{T_1} 2\mathrm{sgn}\left(g_i^0\left(t;\xi_i\left(t\right)\right)\right) g_i^+\left(t;\xi_i\left(t\right)\right)\Delta\xi_i\left(t\right)dt+\mathcal{O}\left(\|\Delta\mathbf{k}\|^2_{\mathbb{R}^{6N}}\right).$$

Here $\langle\cdot,\cdot\rangle$ denotes the scalar product of vectors.

Having moved all the terms of the equation (3.22) to the left, we multiply both parts of the resulting equality by an arbitrary function $\psi\left(x,t\right)=\psi\left(x,t;\mathbf{k}\right)=\psi\left(x,t;\mathbf{k},q,\theta\right)$ from the class of functions continuously differentiable with respect to $t\in[0,T_1]$ and twice continuously differentiable with respect to x for $x\in[0,l]$. Similarly, after moving all the terms of the equations (3.26) to the left, we multiply both parts of the obtained equalities by the still arbitrary functions $\varphi_i\left(t\right)=\varphi_i\left(t;\mathbf{k}\right)$, $i=1,2,\ldots,N$. Integrating the left-hand sides of the obtained equalities, equal to zero, respectively, over $x\in[0,l]$, $t\in[0,T_1]$ and $t\in[0,T_1]$, and adding to (3.28), we will have:

$$\Delta\tilde{I}_{T_f}\left(\mathbf{k};q,\theta\right)=\int_{T_f}^{T_1}\int_{0}^{l} 2\mu\left(x\right)u\left(x,t\right)\Delta u\left(x,t\right)dxdt \tag{3.29}$$

$$+2\varepsilon_3\langle\mathbf{k}-\hat{\mathbf{k}},\Delta\mathbf{k}\rangle+\mathcal{R}\sum_{i=1}^{N}\int_{0}^{T_1} 2\mathrm{sgn}\left(g_i^0\left(t;\xi_i\left(t\right)\right)\right) g_i^+\left(t;\xi_i\left(t\right)\right)\Delta\xi_i\left(t\right)dt$$

$$+\int_{0}^{T_1}\int_{0}^{l}\psi\left(x,t\right)\left[\Delta u_{tt}\left(x,t\right)-a^2\Delta u_{xx}\left(x,t\right)+\lambda\Delta u_t\left(x,t\right)\right]dxdt$$

$$+\sum_{i=1}^{N}\int_{0}^{T_1}\varphi_i\left(t\right)\Big[\Delta\dot{\xi}_i\left(t\right)-a_i\left(t\right)\Delta\xi_i\left(t\right)-\Delta k_i^5 u\left(\xi_i\left(t\right),t\right)-k_i^5\Delta u\left(\xi_i\left(t\right),t\right)$$

$$-k_i^5 u_x\left(\xi_i\left(t\right),t\right)-\Delta k_i^6 u\left(\xi_i\left(t-\tau\right),t-\tau\right)$$

$$-k_i^6\Delta u\left(\xi_i\left(t-\tau\right),t-\tau\right)-k_i^6 u_x\left(\xi_i\left(t-\tau\right),t-\tau\right)\Delta\xi_i\left(t-\tau\right)\Big]dt+\mathcal{O}\left(\|\Delta\mathbf{k}\|^2_{\mathbb{R}^{6N}}\right).$$

In the fourth term of formula (3.29) we split the integration over $x\in(0,l)$ into integration over intervals $\left(\xi_i\left(t^+\right),\xi_{i+1}\left(t^-\right)\right)$, $i=0,1,\ldots,N$, $\xi_0\left(t\right)=0$, $\xi_{N+1}\left(t\right)=l$, then after simple calculations, integrating by parts over x and t, taking into account estimates (3.23), (3.24), (3.27), we obtain:

$$\Delta\tilde{I}_{T_f}\left(\mathbf{k};q,\theta\right)=\Delta\tilde{I}_1+\Delta\tilde{I}_2+\Delta\tilde{I}_3, \tag{3.30}$$

$$\Delta\tilde{I}_1=\sum_{i=1}^{N}\Big\{-\int_{0}^{T_1}a^2\psi_x\left(0,t\right)u\left(\xi_i\left(t\right),t\right)\Delta k_i^1 dt+2\varepsilon_3\left(k_i^1-\hat{k}_i^1\right)\Delta k_i^1\Big\}$$

$$+\sum_{i=1}^{N}\Big\{-\int_{0}^{T_1-\tau}a^2\psi_x\left(0,t+\tau\right)u\left(\xi_i\left(t\right),t\right)\Delta k_i^2 dt+2\varepsilon_3\left(k_i^2-\hat{k}_i^2\right)\Delta k_i^2\Big\}$$

$$+\sum_{i=1}^{N}\Big\{-\int_{0}^{T_1}a^2\psi_x\left(l,t\right)u\left(\xi_i\left(t\right),t\right)\Delta k_i^3 dt+2\varepsilon_3\left(k_i^3-\hat{k}_i^3\right)\Delta k_i^3\Big\}$$

$$+\sum_{i=1}^{N}\left\{-\int_{0}^{T_1-\tau}a^2\psi_x\left(l,t+\tau\right)u\left(\xi_i\left(t\right),t\right)\Delta k_i^4dt+2\varepsilon_3\left(k_i^4-\hat{k}_i^4\right)\Delta k_i^4\right\}$$

$$+\sum_{i=1}^{N}\left\{-\int_{0}^{T_1}\varphi_i\left(t\right)u\left(\xi_i\left(t\right),t\right)\Delta k_i^5dt+2\varepsilon_3\left(k_i^5-\hat{k}_i^5\right)\Delta k_i^5\right\}$$

$$+\sum_{i=1}^{N}\left\{-\int_{0}^{T_1-\tau}\varphi_i\left(t+\tau\right)u\left(\xi_i\left(t\right),t\right)\Delta k_i^6dt+2\varepsilon\left(k_i^6-\hat{k}_i^6\right)\Delta k_i^6\right\},$$

$$\Delta\tilde{I}_2=\int_{0}^{l}\psi\left(x,T_1\right)\Delta u_t\left(x,T_1\right)dx+\int_{0}^{l}\left[-\psi_t\left(x,T_1\right)+\lambda\psi\left(x,T_1\right)\right]\Delta u\left(x,T_1\right)dx$$

$$+\sum_{i=0}^{N}\int_{0}^{T_1}\int_{\xi_i(t^+)}^{\xi_{i+1}\left(t^-\right)}\left[\psi_{tt}\left(x,t\right)-a^2\psi_{xx}\left(x,t\right)-\lambda\psi_t\left(x,t\right)\right]\Delta u\left(x,t\right)dxdt$$

$$+\int_{T_f}^{T_1}\int_{0}^{l}2\mu\left(x\right)u\left(x,t\right)\Delta u\left(x,t\right)dxdt$$

$$+a^2\sum_{i=1}^{N}\int_{0}^{T_1}\left[\psi_x\left(\xi_i\left(t^-\right),t\right)-\psi_x\left(\xi_i\left(t^+\right),t\right)\right]\Delta u\left(\xi_i\left(t\right),t\right)dt$$

$$+a^2\sum_{i=1}^{N}\int_{0}^{T_1}\left[k_i^3\psi_x\left(l,t\right)-k_i^1\psi_x\left(0,t\right)-\frac{k_i^5}{a^2}\varphi_i\left(t\right)\right]\Delta u\left(\xi_i\left(t\right),t\right)dt$$

$$+a^2\sum_{i=1}^{N}\int_{0}^{T_1-\tau}\left[k_i^4\psi_x\left(l,t+\tau\right)-k_i^2\psi_x\left(0,t+\tau\right)-\frac{k_i^6}{a^2}\varphi_i\left(t+\tau\right)\right]\Delta u\left(\xi_i\left(t\right),t\right)dt$$

$$-a^2\sum_{i=1}^{N}\int_{0}^{T_1}\left[\psi\left(\xi_i\left(t^-\right),t\right)-\psi\left(\xi_i\left(t^+\right),t\right)\right]\Delta u_x\left(\xi_i\left(t\right),t\right)dt$$

$$+a^2\int_{0}^{T_1}\left[\psi\left(0,t\right)\Delta u_x\left(0,t\right)-\psi\left(l,t\right)\Delta u_x\left(l,t\right)\right]dt,$$

$$\Delta\tilde{I}_3=\sum_{i=1}^{N}\varphi_i\left(T_1\right)\Delta\xi_i\left(T_1\right)-\sum_{i=1}^{N}\int_{0}^{T_1}\left[\dot{\varphi}_i\left(t\right)+a_i\left(t\right)\varphi_i\left(t\right)\right]\Delta\xi_i\left(t\right)dt$$

$$+\sum_{i=1}^{N}\int_{0}^{T_1}\left[a^2k_i^3\psi_x\left(l,t\right)-a^2k_i^1\psi_x\left(0,t\right)-k_i^5\varphi_i\left(t\right)\right]u_x\left(\xi_i\left(t\right),t\right)\Delta\xi_i\left(t\right)dt$$

$$+\sum_{i=1}^{N}\int_{0}^{T_1-\tau}\left[a^2k_i^4\psi_x\left(l,t+\tau\right)-a^2k_i^2\psi_x\left(0,t+\tau\right)-k_i^6\varphi_i\left(t+\tau\right)\right]u_x\left(\xi_i\left(t\right),t\right)\Delta\xi_i\left(t\right)dt$$

$$+2\mathcal{R}\sum_{i=1}^{N}\int_{0}^{T_1}\operatorname{sgn}\left(g_i^0\left(t;\xi_i\left(t\right)\right)\right)g_i^+\left(t;\xi_i\left(t\right)\right)\Delta\xi_i\left(t\right)dt+\mathcal{O}\left(\|\Delta\mathbf{k}\|^2_{\mathbb{R}^{6N}}\right).$$

Since no requirements have been imposed on the functions $\psi\left(x,t\right)$ and $\varphi_i\left(t\right)$, $i=1,2,\ldots,N$, we will require that the conditions $\Delta\tilde{I}_2=\Delta\tilde{I}_3=0$ be satisfied, i.e. that they be solutions of the initial-boundary value problem (3.11)–(3.17) and the Cauchy problems (3.18)–(3.20), respectively.

For the components (3.4) of the gradient vector of the functional (2.33), taking into account that they are determined by the linear parts of the increment of the functional with the corresponding increments of the parameters, from $\Delta\tilde{I}_1$ we obtain the following formulas:

$$\frac{\partial\tilde{I}_{T_f}\left(\mathbf{k};q,\theta\right)}{\partial k_i^1}=-\int_{0}^{T_1}a^2\psi_x\left(0,t\right)u\left(\xi_i\left(t\right),t\right)dt+2\varepsilon_3\left(k_i^1-\hat{k}_i^1\right),\tag{3.31}$$

$$\frac{\partial\tilde{I}_{T_f}\left(\mathbf{k};q,\theta\right)}{\partial k_i^2}=-\int_{0}^{T_1-\tau}a^2\psi_x\left(0,t+\tau\right)u\left(\xi_i\left(t\right),t\right)dt+2\varepsilon_3\left(k_i^2-\hat{k}_i^2\right),\tag{3.32}$$

$$\frac{\partial\tilde{I}_{T_f}\left(\mathbf{k};q,\theta\right)}{\partial k_i^3}=-\int_{0}^{T_1}a^2\psi_x\left(l,t\right)u\left(\xi_i\left(t\right),t\right)dt+2\varepsilon_3\left(k_i^3-\hat{k}_i^3\right),\tag{3.33}$$

$$\frac{\partial\tilde{I}_{T_f}\left(\mathbf{k};q,\theta\right)}{\partial k_i^4}=-\int_{0}^{T_1-\tau}a^2\psi_x\left(l,t+\tau\right)u\left(\xi_i\left(t\right),t\right)dt+2\varepsilon_3\left(k_i^4-\hat{k}_i^4\right),\tag{3.34}$$

$$\frac{\partial\tilde{I}_{T_f}\left(\mathbf{k};q,\theta\right)}{\partial k_i^5}=-\int_{0}^{T_1}\varphi_i\left(t\right)u\left(\xi_i\left(t\right),t\right)dt+2\varepsilon_3\left(k_i^5-\hat{k}_i^5\right),\tag{3.35}$$

$$\frac{\partial\tilde{I}_{T_f}\left(\mathbf{k};q,\theta\right)}{\partial k_i^6}=-\int_{0}^{T_1-\tau}\varphi_i\left(t+\tau\right)u\left(\xi_i\left(t\right),t\right)dt+2\varepsilon_3\left(k_i^6-\hat{k}_i^6\right).\tag{3.36}$$

Taking into account (3.21), from formulas (3.31)–(3.36) we obtain the desired formulas (3.5)–(3.10). □

Remark 3.1. The conjugate equation (3.11), if including the conditions (3.12)–(3.14) for the jumps of the conjugate function at the measurement points using the Dirac δ-function, can be written as

$$\psi_{tt}\left(x,t\right)=a^2\psi_{xx}\left(x,t\right)+\lambda\psi_t\left(x,t\right)-2\mu\left(x\right)u\left(x,t\right)\chi_{\left[T_f,T_1\right]}\left(t\right)$$

$$+a^2\sum_{i=1}^{N}\delta\left(x-\xi_i\left(t\right)\right)\left[k_i^1\psi_x\left(0,t\right)+k_i^2\psi_x\left(0,t+\tau\right)-k_i^3\psi_x\left(l,t\right)-k_i^4\psi_x\left(l,t+\tau\right)\right],$$

$$+\sum_{i=1}^{N}\delta\left(x-\xi_i\left(t\right)\right)\left[k_i^5\varphi_i\left(t\right)+k_i^6\varphi_i\left(t+\tau\right)\right]$$
$$x\in\left(0,l\right),\quad t\in\left[0,T_1-\tau\right),$$
$$\psi_{tt}\left(x,t\right)=a^2\psi_{xx}\left(x,t\right)+\lambda\psi_t\left(x,t\right)-2\mu\left(x\right)u\left(x,t\right)\chi_{\left[T_f,T_1\right]}\left(t\right)$$
$$+\sum_{i=1}^{N}\delta\left(x-\xi_i\left(t\right)\right)\left[a^2k_i^1\psi_x\left(0,t\right)-a^2k_i^3\psi_x\left(l,t\right)+k_i^5\varphi_i\left(t\right)\right],$$
$$x\in\left(0,l\right),\quad t\in\left[T_1-\tau,T_1\right).$$

Thus, the approach to the numerical solution of the problem (2.1), (2.2), (2.21), (2.22), (2.23), (2.15), (2.32), (2.33) is as follows. As was said at the beginning of this section, to find the minimum time T_f required for the damping process, i.e. to satisfy the condition (2.10), the "incremental search method" is used to determine the uncertainty interval for the minimum calming time $T_f\in\left[\underline{T_f},\overline{T_f}\right]$ with the subsequent application of some one-dimensional search method (golden section search, bisection method, etc.) to refine the optimal value of T_f.

At each given time T_f, to determine the corresponding optimal values of the feedback parameters $\mathbf{k}$, a minimizing sequence is constructed using, for example, the procedure (3.3).

4. Results of numerical experiments

Let us present the results of numerical experiments obtained in solving the problem under consideration (2.1)–(2.25) with the following values of the data involved in the formulation:

$$a^2=1,\quad\lambda=0,\quad\Delta T=0.5,\quad\tau=0.02,\quad N_0=3,\quad N=4,$$
$$\mu_1\left(x\right)\equiv1,\quad\mu_2\left(x\right)\equiv1,\quad x\in\left[0,1\right],\quad\varepsilon_3=1.0,\quad\mathcal{R}=1,\quad\varepsilon=0.0002,$$
$$q_1\in Q_1=\left[1.32,\ 1.42\right],\ \rho_{Q_1}\left(q_1\right)=10,\quad\theta_1\in\Theta_1=\left[0.20,\ 0.24\right],\ \rho_{\Theta_1}\left(\theta_1\right)=25,$$
$$q_2\in Q_2=\left[1.29,\ 1.39\right],\ \rho_{Q_2}\left(q_2\right)=10,\quad\theta_2\in\Theta_2=\left[0.54,\ 0.58\right],\ \rho_{\Theta_2}\left(\theta_2\right)=25,$$
$$q_3\in Q_3=\left[1.28,\ 1.38\right],\ \rho_{Q_3}\left(q_3\right)=10,\quad\theta_3\in\Theta_3=\left[0.78,\ 0.82\right],\ \rho_{\Theta_3}\left(\theta_3\right)=25,$$
$$a_1\left(t\right)\equiv0,\quad a_2\left(t\right)\equiv0,\quad a_3\left(t\right)\equiv0,\quad a_4\left(t\right)\equiv0,$$
$$\xi_1^0=0.11,\quad\left[\underline{\xi_1},\overline{\xi_1}\right]=\left[0.03,\ 0.22\right],\quad\xi_2^0=0.37,\quad\left[\underline{\xi_2},\overline{\xi_2}\right]=\left[0.27,\ 0.48\right],$$
$$\xi_3^0=0.62,\quad\left[\underline{\xi_3},\overline{\xi_3}\right]=\left[0.52,\ 0.73\right],\quad\xi_4^0=0.86,\quad\left[\underline{\xi_4},\overline{\xi_4}\right]=\left[0.77,\ 0.97\right].$$

The splash magnitude values have uniform distributions in Q_1, Q_2, and Q_3, and their possible impact points are uniformly distributed in the given admissible regions Θ_1, Θ_2, and Θ_3.

The dimension of the optimized vector of problem parameters is 24.

For one-dimensional minimization of the completion time of the process T_f, under conditions (3.2), we use the method of bisection method with preliminary determining the initial bracketing interval $\left[\underline{T_f},\overline{T_f}\right]$.

Let us describe the general scheme of implementation of the iterative procedure (3.3) for a given value of T_f with the aim of minimizing the functional (2.32), (2.33) using the methods of penalty function under constraints (2.31) and gradient projection under constraints (2.30).

For each value of the penalty coefficients, the functional was regularized using known schemes [25, 27]. In this case, the regularization parameters were changed three times, namely, at the initial value it was reduced by 10 times after the completion of the iterations, and the optimal value $\mathbf{k}$ obtained at the previous step was assigned as $\hat{\mathbf{k}}$ for the regularization of the functional (2.32), (2.33). The initial value of the penalty coefficient $\mathcal{R}$ was set to 1, which was increased by a factor of 10 at each subsequent stage. These stages were carried out until the value of the main functional of the problem (2.32), (2.33) differed by more than 0.0001 at two consecutive stages.

To solve the direct and conjugate initial–boundary-value problems (2.1), (2.2), (2.21), (2.22) and (3.11)–(3.17), a modified implicit three-layer grid method [23] was used. To solve loaded initial–boundary-value problems, an implicit finite-difference approximation scheme was used, which was investigated in the works [3, 6]. To solve finite-difference approximating initial–boundary-value problems, numerical methods proposed in the work [2] were used.

To solve the direct (2.23), (2.15) and the conjugate (3.18)–(3.20) Cauchy problems, a modified scheme of the Euler method with a time step $h_t = 0.01$ was used.

In numerical calculations under the initial conditions (2.2) the value of σ for the neighborhood of points θ_s was equal to 0.03.

In our computer experiments, we used the following everywhere continuously differentiable function as the function $\mu_\sigma\left(x;\check{\theta}\right)$:

$$\mu_\sigma\left(x;\check{\theta}\right) = \begin{cases} \dfrac{1}{2\sigma}\left[1+\cos\left(\dfrac{x-\check{\theta}}{\sigma}\pi\right)\right], & x \in \left[\check{\theta}-\sigma, \check{\theta}+\sigma\right], \\ 0, & x \notin \left[\check{\theta}-\sigma, \check{\theta}+\sigma\right], \end{cases}$$

which, as can be easily verified, satisfies the property (2.5).

Table 1 shows the results obtained for different given times T_f by minimizing the functional $\tilde{J}_{T_f}(\mathbf{k})$ from (3.1), in which different values of the feedback parameter vector $\mathbf{k}^0$ were used as the initial approximation for the iterative process (3.3). As can be seen from the Table 1, the condition ε-settling of vibrations (2.10) in the experiments carried out is fulfilled at $T_f \geq 1.65$.

Table 2 presents the results of solving the auxiliary problem (3.1), obtained using the three given initial approximations for the given time $T_f = 1.5$. As was indicated above about the possible multi-extremality of the objective functional, the optimization results obtained from different starting points differ in arguments, although the difference in the values of the functional is not significant. Here it is also necessary to take into account (as other specially conducted numerical experiments have shown) that the functional of the problem has a ravine structure.

Computer experiments were conducted to control the process of vibration damping using the obtained optimal values of the synthesized feedback parameters under the assumption that the state measurements are carried out with noises (error), namely:

$$\check{u}_i^{\chi}(t) = u\left(\xi_i(t), t\right)\left[1+\chi_i(t)\right], \quad t \in [0, T_1], \quad i = 1, 2, \ldots, N.$$

TABLE 1. Results of minimization of the functional (3.1) for different T_f and initial approximations $\mathbf{k}^0$.

T_f	$\mathbf{k}^0$				$\tilde{J}_{T_f}(\mathbf{k}^0)$	$\mathbf{k}^*$				$\tilde{J}_{T_f}(\mathbf{k}^*)$
0.50	0.1758	0.8195	-0.9487	-0.1336	0.05712	-0.5686	0.2927	-1.5831	-0.2576	0.00188
	0.1389	0.8103	-0.6741	-0.3989		0.8313	0.6252	0.3298	-0.2426	
	0.2944	0.4316	0.3106	0.4219		0.3440	0.4425	-1.7175	-0.6691	
	0.0817	0.7539	0.2294	0.5541		0.0819	1.0531	0.0468	0.3898	
	0.0203	-0.1043	0.4114	-0.1574		-0.1117	-0.1550	0.3043	0.2918	
	-0.0231	-0.0661	0.4020	-0.1632		-0.1135	-0.1154	0.1867	0.2630	
0.75	-0.1177	-0.1465	-0.2245	-0.1747	0.05435	-0.0099	0.2451	-0.0651	-0.0267	0.00145
	-0.1036	-0.1247	-0.1964	-0.1575		0.0113	0.0346	-0.0297	-0.0064	
	-0.1179	-0.1346	-0.2193	-0.1865		-0.0113	0.0301	-0.0305	-0.0344	
	-0.1048	-0.1087	-0.1846	-0.1659		0.0037	0.0598	0.0213	-0.0113	
	0.0051	-0.0041	0.0053	0.0004		0.0706	-0.0152	-0.0837	0.0390	
	0.0075	-0.0038	0.0058	0.0025		0.0698	-0.0155	-0.0910	0.0384	
1.00	-0.6416	0.3797	-0.4275	-0.2041	0.05247	0.0121	0.0537	0.0270	0.0018	0.00114
	0.4165	0.1263	0.6844	0.0989		0.0295	0.0838	0.0558	0.0184	
	0.0373	0.0299	-0.6083	-0.9425		0.0176	0.0837	0.0567	0.0052	
	0.6841	0.8113	0.5803	0.1782		0.0295	0.1077	0.0820	0.0217	
	-0.2138	0.1255	0.3835	-0.2482		0.0858	0.0416	-0.0141	-0.0124	
	-0.2671	0.1601	0.3161	-0.3036		0.0946	0.0359	-0.0418	-0.0414	
1.25	-0.1622	-0.0770	-0.2142	-0.2480	0.04882	0.0619	-0.0578	-0.3777	-0.1444	0.00083
	-0.1369	-0.0313	-0.1564	-0.2166		0.1430	0.1681	-0.1763	-0.0653	
	-0.1431	-0.0422	-0.2035	-0.2883		0.1226	-0.0804	-0.2664	-0.0516	
	-0.1320	0.0162	-0.1207	-0.2466		0.2020	0.0483	-0.1270	0.0428	
	0.0014	-0.0136	0.0231	-0.0009		0.0641	0.0492	-0.0557	-0.0825	
	0.0011	-0.0128	0.0220	-0.0052		0.0562	0.0505	-0.0575	-0.0753	
1.50	-0.0425	-0.0362	-0.1243	-0.0855	0.04768	0.2547	-0.0291	-0.7072	-0.4322	0.00055
	-0.0246	0.0043	-0.0774	-0.0693		0.6009	0.7103	-0.1448	-0.2776	
	-0.0374	-0.0923	-0.1067	-0.1023		0.2036	0.1012	-0.5550	-0.3712	
	-0.0271	0.0335	-0.0544	-0.0806		0.30064	0.3622	-0.1795	0.0661	
	-0.0024	-0.0272	0.0121	0.0326		-0.0206	-0.1043	0.1712	-0.0265	
	-0.0043	-0.0206	0.0218	0.0127		-0.0169	-0.1088	0.1675	-0.0299	
1.65	-0.0888	-0.0104	-0.0668	-0.0256	0.01594	-0.1263	-0.1904	-0.4470	-0.3183	0.00018
	0.0724	0.0165	-0.0378	-0.0107		0.2266	0.4969	0.1615	-0.0321	
	-0.0950	0.0187	-0.0352	-0.0318		-0.2512	-0.0901	-0.3884	-0.2830	
	0.0176	0.0431	-0.0768	-0.0138		-0.0449	0.3687	0.1087	0.0541	
	-0.0073	0.0071	-0.0239	0.0004		0.0370	-0.0589	0.0198	-0.0541	
	-0.0691	0.0065	-0.0251	0.0415		0.0378	-0.0651	0.0195	-0.0524	
2.00	0.0121	0.0537	0.0270	0.0018	0.02459	-0.0333	0.0057	-0.0358	-0.0499	0.00042
	0.0295	0.0838	0.0558	0.0184		0.0414	0.1237	0.0821	0.0180	
	0.0176	0.0837	0.0567	0.0052		-0.0187	0.0298	-0.0418	-0.0691	
	0.0295	0.1077	0.0820	0.0217		0.0363	0.1468	0.0885	0.0166	
	0.0085	0.0141	-0.0114	-0.0014		-0.0158	-0.0630	0.0025	0.0194	
	0.0084	0.0141	-0.0213	-0.0154		-0.0166	-0.0120	0.0026	0.0183	

Here $\chi_i(t)$ for each t is a random variable uniformly distributed on the closed interval $[-\zeta, \zeta]$, ζ determines the maximum noise level. In the experiments conducted, the values of ζ were chosen equal to 0.01, 0.03, 0.05, which corresponds to a measurement noise of 1%, 3% and 5% from the exact (calculated) values of the measured quantities.

In tables 1 and 2 the values of the feedback parameters $\mathbf{k}$ are given row by row in the following order: $\left(k_1^1, k_2^1, \ldots, k_4^1\right)$, $\left(k_1^2, k_2^2, \ldots, k_4^2\right)$, $\left(k_1^3, k_2^3, \ldots, k_4^3\right)$, $\left(k_1^4, k_2^4, \ldots, k_4^4\right)$, $\left(k_1^5, k_2^5, \ldots, k_4^5\right)$, $\left(k_1^6, k_2^6, \ldots, k_4^6\right)$.

Fig. 1 shows the graphs of the function $\tilde{J}(T_f, \mathbf{k}^*)$ defining the vibration process on the interval $[T_f, T_f + \Delta T]$, obtained with optimal feedback parameters for $T_f \in [0; 2]$ (solid line). The dashed line shows the graph of the function $\tilde{J}(T_f, \mathbf{k}^*)$ obtained by solving the original problem under the assumption that the measurement points do not move, but their locations are optimized. In this case, the feedback parameters form a vector $\mathbf{k} = (k_1, k^2, k^3, k^4)$. It is evident from these graphs that the quality of control of the string vibration damping

TABLE 2. Results of minimization of the functional (3.1) and different initial approximations $\mathbf{k}^0$ for $T_f = 1.5$.

	$\mathbf{k}^0$				$\tilde{J}_{T_f}(\mathbf{k}^0)$	$\mathbf{k}^*$				$\tilde{J}_{T_f}(\mathbf{k}^*)$
1	-0.0981	-0.0437	-0.1301	-0.1556	0.03752	0.2347	-0.4161	-0.8229	-0.2424	0.00043
	-0.0343	0.0785	-0.0213	-0.1044		0.5242	0.4761	0.3562	0.4296	
	-0.1058	-0.0361	-0.1118	-0.1560		-0.0871	-0.1411	-0.7610	-0.4026	
	-0.1058	-0.0361	-0.1118	-0.1560		0.5674	0.3676	0.1836	0.4674	
	-0.1058	-0.0361	-0.1118	-0.1560		-0.1285	-0.0214	-0.2573	-0.1286	
	-0.1058	-0.0361	-0.1118	-0.1560		-0.1255	-0.0133	-0.2489	-0.1282	
2	-0.0166	-0.1743	-0.2436	-0.0190	0.04250	0.1875	-0.0953	-0.8034	-0.4290	0.00045
	0.1048	0.0596	-0.0130	0.0909		0.4944	0.7165	-0.2233	-0.3276	
	-0.0144	-0.1490	-0.2453	-0.0519		0.0612	0.2653	-0.8156	-0.5590	
	0.0836	0.0517	-0.0227	0.0796		0.2754	0.5308	-0.4748	0.0577	
	-0.0898	0.0137	-0.0157	0.0447		-0.1225	-0.1985	0.3272	-0.1289	
	-0.0721	0.0132	-0.0251	0.0420		-0.1226	-0.2207	0.2482	-0.1189	
3	-0.0241	-0.0968	-0.2097	-0.0955	0.03892	0.3277	-0.0780	-0.5070	-0.1548	0.00041
	0.0254	-0.0018	-0.1225	-0.0487		0.6291	0.4457	-0.0347	0.1181	
	-0.0001	-0.1082	-0.2008	-0.0601		0.0379	0.0936	-0.1688	-0.1422	
	0.0468	-0.0399	-0.1365	-0.0082		0.1713	0.3202	0.0951	0.0449	
	0.0264	0.1421	-0.1452	-0.0194		0.1645	0.0744	0.1280	0.1517	
	0.0141	0.1283	-0.1042	-0.1891		0.0758	0.0764	0.1878	0.1487	

process when the measuring points move is significantly better compared to the case when the measuring device locations are stationary. As can be seen from the figures, ε-settling on the string is not achieved when the measuring points are stationary.

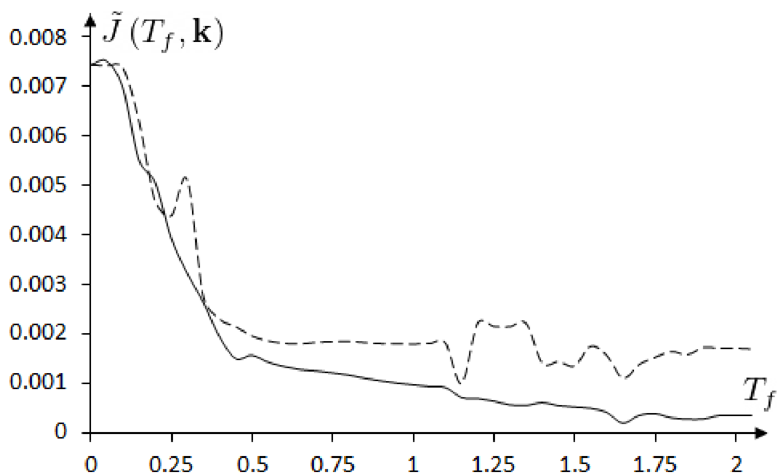

FIGURE 1. Graphs of the function $\tilde{J}(T_f, \mathbf{k}^*)$ obtained with moving measurement points (——), and with their optimal stationary locations (– – –).

An important indicator of the quality of control of the damping process when using, in particular, with optimal feedback parameters, is the function

$$\hat{J}(t, \mathbf{k}^*) = \int_Q \int_\Theta \left\{ \int_t^{t+\Delta T} \int_0^l \mu(x) [u(x,\tau)]^2 dx d\tau \right\} \rho_Q(q) \rho_\Theta(\theta)\, dq d\theta. \qquad (4.1)$$

This function characterizes the result of process control on average for all possible values of the initial conditions (2.2).

Fig. 2 solid line shows the graphs of the function (4.1) over time $t \in [0;\ 2]$ for the synthesized optimal vector of feedback parameters $\mathbf{k}^*$. The dashed line shows the graph of the function (4.1) in the case where the measurement points did not move, but their stationary resolution locations and feedback parameters $\mathbf{k} = (k_1, k^2, k^3, k^4)$ were optimized.

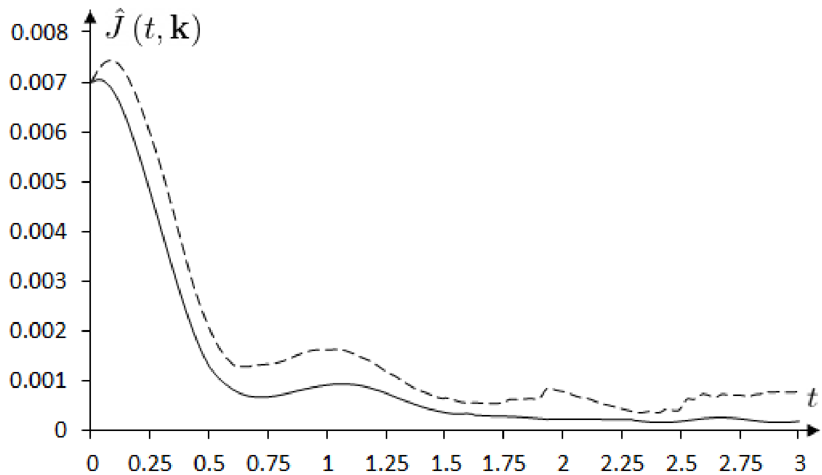

Figure 2. Graphs of the function $\hat{J}\,(t, \mathbf{k}^*)$ for moving measurement points (——) and for stationary measurement points (– – –).

Fig. 3 shows the graph of the function

$$F\,(t; \mathbf{k}^*) = \int\limits_Q \int\limits_\Theta \left\{ \int\limits_0^l \mu\,(x)\,[u\,(x,t)]^2 dx \right\} \rho_Q(q) \rho_\Theta\,(\theta)\,dq d\theta,$$

for the synthesized optimal vector of feedback parameters $\mathbf{k}^*$ for $t \in [0;\ 2]$.

Fig. 4 shows the graphs of the trajectories of the measurement sensors $\xi_i\,(t)$ and the corresponding controls $w_i\,(t)$ for the initial value of the feedback parameter vector $\mathbf{k}^0$ (dashed lines) and for the synthesized optimal vector $\mathbf{k}^*$ (solid lines).

Fig. 5 shows two graphs of the boundary control functions $\vartheta\,(t) = (\vartheta_1(t), \vartheta_2(t))$ for the initial values of the parameter vector $\mathbf{k}^0$ (dashed lines) and for the synthesized optimal feedback parameter vector $\mathbf{k}^*$ (solid lines).

Fig. 6 shows the graphs of the final state of the string $u\,(x, T_f)$, $x \in [0; 1]$, $T_f = 1.65$ for the synthesized optimal parameter vector $\mathbf{k}^*$ (solid line) without noise and with noise of 3% and 5% (dashed lines).

As can be seen from the results of computer experiments given in table 1, when using measurements at past moments in time in the feedback, the values of the corresponding coefficients k^2, k^4, k^6, according to the optimization results, differ from zero. Consequently, their presence reduces the value of the functionality, which proves the effectiveness of using measurements at past moments in time for feedback.

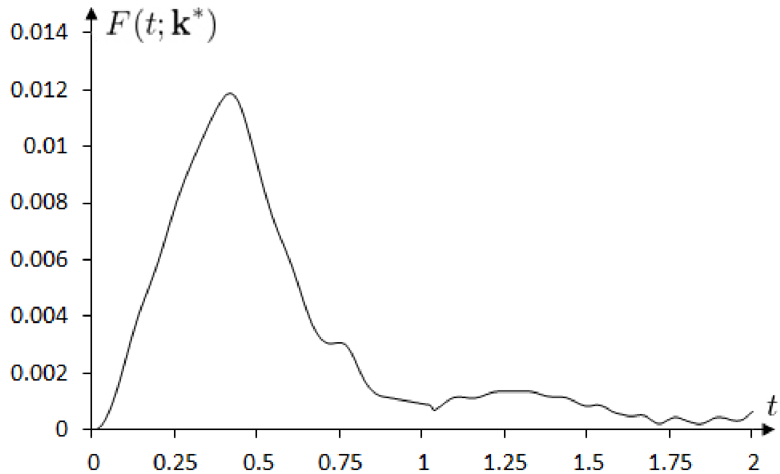

FIGURE 3. Graphs of the function $F(t;\mathbf{k}^*)$ for moving measurement points for the synthesized optimal vector of feedback parameters $\mathbf{k}^*$.

Conclusion

The paper studies the problem of synthesis of boundary control for the process of damping the vibration of a string. The main feature of the problem statement under consideration is that feedback is carried out using measurements taken by state sensors moving along the string. The current values of the boundary controls and the velocity of movement of the sensors are assigned depending on the measured values of the state at the points where the sensors are located, both at the current and past moments in time. Formulas for dependencies implementing feedback are proposed. With respect to the constant coefficients (feedback parameters) involved in these dependencies, formulas for the derivatives of the objective functional were obtained. The formulas are used for the numerical solution of the problem of determining the optimal values of feedback parameters using first-order optimization methods.

The formulation of the problem presented in the work, the methodology for its study, including the numerical solution of the control synthesis problem can be used to solve problems of controlling the stabilization of membrane and plate vibrations, as well as for feedback control of other technological processes, objects with distributed parameters described by other types of initial–boundary-value problems with other partial differential equations.

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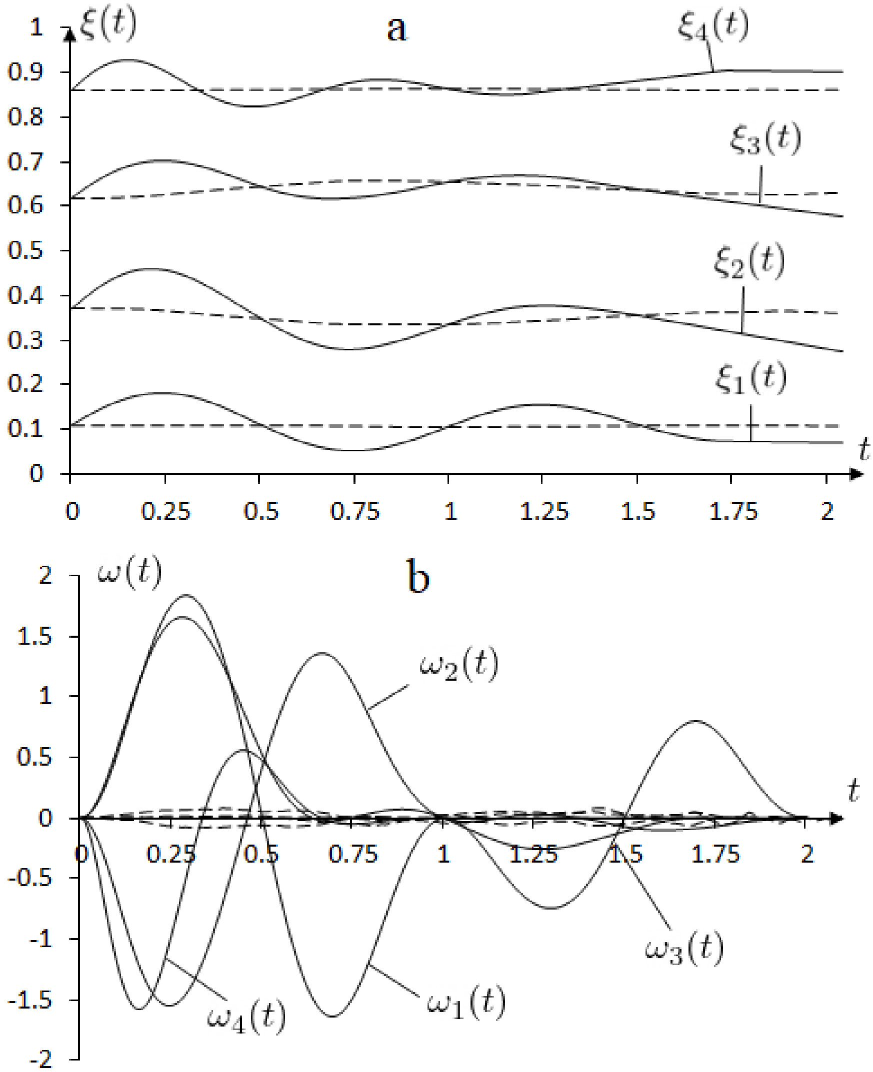

FIGURE 4. Graphs of the trajectories of the measurement sensors (a) and the corresponding controls (b) for the initial value of the feedback parameter vector $\mathbf{k}^0$ (– – –) and for the synthesized optimal vector $\mathbf{k}^*$ (——).

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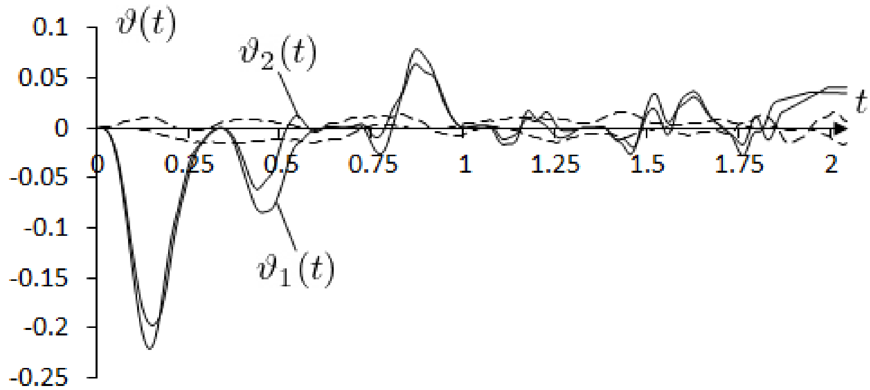

FIGURE 5. Graphs of boundary controls for the initial parameter vector $\mathbf{k}^0$ (– – –) and for the synthesized optimal parameter vector (——).

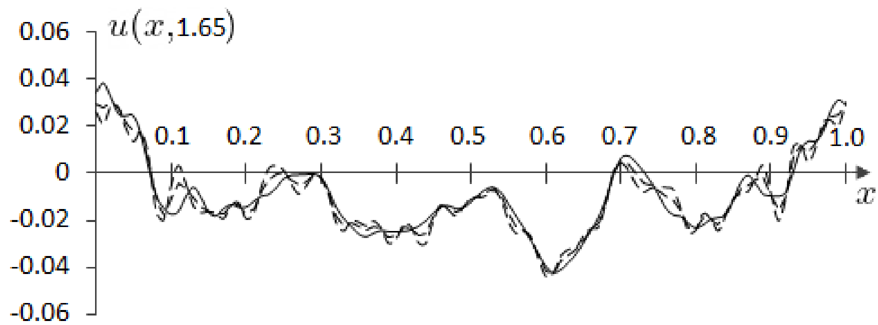

FIGURE 6. Graphs of the final state of the string $u\,(x, T_f)$, $T_f =$ 1.65 for the synthesized optimal parameter vector $\mathbf{k}^*$ (solid line) without noise and with noise of 3% and 5% (dashed lines: – – – , - - - - - respectively).

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