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SOLARITY OF BOUNDEDLY Δ-COMPACT MONOTONE PATH-CONNECTED SETS

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Abstract. In 1980, F. Deutsch introduced a general type of convergence of nets and sequences. In general, such convergences are nontopological. Convergences of this type proved useful in establishing existence theorems for many classical and abstract objects appearing in approximation theory. In the present paper, we introduce a special type of Deutsch convergence, which we call Δ-convergence. We study boundedly Δ-compact sets in normed linear spaces — by definition, any net in the intersection of such a set with any closed ball contains a subnet which Δ-converges to a point of this intersection. We show that any boundedly Δ-compact monotone path-connected set is a sun.

1. Introduction

We study boundedly Δ-compact sets — these are the sets such that the intersection of such a set with any closed ball is compact relative to Δ-convergence (see Definition 2.2), which is a particular case of fairly general Deutsch convergence (see Definition 2.1). We study the problem of solarity of boundedly Δ-compact monotone path-connected sets in normed linear spaces.

The main results are Theorems 2.1 and 2.2.

Given a nonempty subset M of a normed linear space X over $\mathbb{R}$, let $\varrho(y, M) := \inf_{z\in M} \|z - y\|$ be the distance from a point $y \in M$ to the set M. Let $P_M x = P_M(x) := \{y \in M \mid \|y - x\| = \varrho(x, M)\}$ be the set of all nearest points from M for $x \in X$. The mapping $x \mapsto P_M x$ is the metric projection onto the set M. Next,

$$B(x, r) = \{y \in X \mid \|y - x\| \leqslant r\} \quad \text{and} \quad S(x, r) = \{y \in X \mid \|y - x\| = r\}$$

are, respectively, the closed ball and sphere with center x and radius $r \geqslant 0$; $B = B(0, 1)$ and $S = S(0, 1)$ are, respectively, the unit ball and the unit sphere. Further, $\mathring{B}(x, r) := \{y \in X \mid \|y - x\| < r\}$ is the open ball with center x and radius r. As usual, X^* is the dual space of X, $S^* = S^*(X^*)$ is the dual unit sphere.

Definition 1.1. Let $\varnothing \neq M \subset X$. A point $x \in X \setminus M$ is a *solar point* for M if there exists a point $y \in P_M x \neq \varnothing$ (called a *luminosity point*) such that $y \in$

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$P_M((1-\lambda)y+\lambda x)$ for each $\lambda \geqslant 0$ (geometrically, this means that there is a (solar) ray emanating from y and passing through x such that y is a nearest point from M for any point from this ray.

Definition 1.2. A closed set $M \subset X$ is called a *sun* if every point $x \in X \setminus M$ is a solar point for M.

Suns are the most natural objects for which the generalized Kolmogorov criterion for an element of best approximation is fulfilled. They have certain properties of separability: a ball can be separated from such set using a larger ball or a support cone (see, e.g., [4]).

As already noted, our aim here is to obtain fairly general conditions on a subset of a normed linear space which guarantee that this set is a sun. We will show that any boundedly Δ-compact (in the sense of Definition 2.2) monotone path-connected subset of a normed linear space is a sun (Theorems 2.1 and 2.2). It is well known that any sun in X is convex if and only if X is smooth (see [4]).

2. Solarity of monotone path-connectedness of boundedly Δ-compact sets

Following Deutsch (see, [10], and also [4, § 4.4]), we recall the definition of a general type of convergence of nets (sequences) in a normed linear space.

Definition 2.1. Let X be a normed linear space. A Deutsch convergence (or τ-convergence) $x_\delta \overset{\tau}{\to} x$ of nets (sequences) in X satisfies the following axioms:

(i) τ is translation invariant, i.e., $x_\delta \overset{\tau}{\to} x$ implies that $x_\delta + y \overset{\tau}{\to} x + y$ for each $y \in X$;

(ii) τ is majorized by norm convergence, i.e., $x_\delta \overset{\tau}{\to} x$ implies that $\|x\| \le \limsup \|x_\delta\|$;

(iii) τ is homogeneous, i.e., $x_\delta \overset{\tau}{\to} x$ implies that $\alpha x_\delta \overset{\tau}{\to} \alpha x$ for each $\alpha \in \mathbb{R}$.

Example 2.1. Let us give some particular cases of Deutsch convergences.

a) *norm convergence*: $x_\delta \overset{\mathrm{n}}{\to} x$ if $\|x_\delta - x\| \to 0$;

b) *weak convergence*: $x_\delta \overset{\mathrm{w}}{\to} x$ if $x^*(x_\delta) \to x^*(x)$ for each $x^* \in X^*$;

c) $*$-*weak convergence* in the dual space $X = Y^*$: $y^*_\delta \overset{\mathrm{w}^*}{\to} y^*$ if $y^*_\delta(y) \to y^*(y)$ for each $y \in Y$;

d) *pointwise convergence in* $C(Q)$ *on a* (*net-specific*) *subset of the compact set* Q: $x_\delta \overset{\Delta}{\to} x$ if there exists a dense subset Q_0 of Q on which $x_\delta(t) \to x(t)$ for each $t \in Q_0$;

e) *pointwise convergence in* $C[a,b]$ *at all points of* $[a,b]$ *except finitely many points*: $x_\delta \overset{\varphi}{\to} x$ if there exists $Q_0 \subset Q$ such that $[a,b]\backslash Q_0$ is finite and $x_\delta(t) \to x(t)$ for each $t \in Q_0$;

f) *convergence of sequences almost everywhere in* $L^p(\mu)$: $x_n \overset{\mathrm{a.e.}}{\to} x$ if $\mu(\{t \mid x_n(t) \not\to x(t)\}) = 0$;

g) *almost weak convergence*: $x_\delta \overset{\mathrm{aw}}{\to} x$ if there exists a w^*-dense subset Λ of the set of extreme points of the dual unit ball of X^* such that $x^*(x_\delta) \to x^*(x)$ for each $x^* \in \Lambda$. In $C(Q)$, the aw-convergence coincides with the Δ-convergence from item d).

Let us define another natural case of Deutsch convergence. Here and in what follows, $\operatorname{ext} S^*$ is the set of all extreme functionals of the dual unit sphere S^*.

Definition 2.2. A net $\{x_\gamma\}_\gamma$ is said to Δ-*converge* to x if there exists an at most countable set $q \subset \operatorname{ext} S^*$ which is w^*-dense in $\operatorname{ext} S^*$ and such that $\{x^*(x_\gamma)\}_\gamma$ converges to x on any point $x^* \in q$. (This convergence is a particular case of aw-convergence from item g)).

Let τ be a Deutsch convergence. A set $M \subset X$ is *boundedly Deutsch compact* (*boundedly Deutsch sequentially compact*) if any bounded net (sequence) from M contains a subnet (a subsequence) Deutsch convergent to a point from M. (If τ is the norm convergence, we get the classical definition of boundedly compact sets.)

It is well known that any boundedly Deutsch compact (boundedly Deutsch sequentially compact) set is an existence set (see, e.g., [4, Theorem 4.4]).

Definition 2.3. A set $M \subset X$ is *boundedly* Δ-*compact* if any bounded net from M has a Δ-convergent subnet to a point from M.

Particular cases of Deutsch convergence in $C(Q)$ were used, in particular, for derivation of theorems on existence of best approximants by rational functions, exponential sums, and splines with free knots (see, e.g., [8], [10], [12], [13]).

Deutsch convergence (sequential convergence) τ is topological if there exists a topology on X relative to which the convergence of nets (sequences) is equivalent to τ-convergence. In a)–c) of Example 2.1, the convergence is topological. In d)–g), the Deutsch convergences are not topological (see [10]).

We next consider a directed set $\mathcal{A}$. Let X and Y_α ($\alpha \in \mathcal{A}$) be normed linear spaces with norms $\|\cdot\|$ and $\|\cdot\|_\alpha$, and $\{A_\alpha\}_{\alpha\in\mathcal{A}} : X \overset{\text{onto}}{\longrightarrow} Y_\alpha$ be a family of continuous linear operators, $\|A_\alpha\| \leqslant 1$. Given a set $M \subset X$. We set $M_\alpha := A_\alpha(M)$ ($\alpha \in \mathcal{A}$). Note that $\varrho_\alpha(A_\alpha(x), M_\alpha) \leqslant \varrho(x, M)$ for all $x \in X$ and $\alpha \in \mathcal{A}_\alpha$, because $\|A_\alpha\| \leqslant 1$ for each $\alpha \in \mathcal{A}_\alpha$.

Below, we will obtain an analogue of the theorem and the remark after it from [14] but under weaker conditions. The following result holds.

Theorem 2.1. *Let $M \subset X$ be a boundedly Deutsch compact set and M_α be a sun in Y_α ($\alpha \in \mathcal{A}$), Assume that the following conditions are satisfied*:

1) *for any net $(x_\gamma)_{\gamma\in\mathcal{A}} \subset M$ with limit point $x \in M$ and for each $\varepsilon > 0$, there exists $\beta_0 = \beta_0(x, \varepsilon)$ such that $\|A_{\alpha_0}(x)\| \leq \limsup_\gamma \|A_{\alpha_0}(x_\gamma)\|_{\alpha_0} + \varepsilon$ for each $\alpha_0 \geqslant \beta_0$;*
2) *$\|A_\alpha(x)\|_\alpha \leqslant \|A_\beta(x)\|_\beta$ for all $\alpha, \beta \in \mathcal{A}$, $\alpha \leqslant \beta$, and each $x \in X$;*
3) *$\|A_\alpha(x)\|_\alpha \to \|x\|$ for each $x \in M$.*

Then M is a sun in X.

Proof. Let $x_0 \in X \setminus M$ be an arbitrary point. We claim that there exists a point $y_0 \in P_M(x_0)$ such that $y_0 \in P_M(y_0 + \lambda(x_0 - y_0))$ for each $\lambda > 0$. We can assume without loss of generality that $x_0 = 0$, $\varrho(x_0, M) = 1$. Hence $A_\alpha(x_0) = 0$ for each α. Since M_α is a sun in X_α, there exists a point $y_\alpha \in M_\alpha$ such that $y_\alpha \in P_{M_\alpha}(y_\alpha + \lambda(0 - y_\alpha))$ for all $\lambda > 0$ and $\alpha \in \mathcal{A}$. Therefore,

$$\varrho_\alpha((1-\lambda)y_\alpha, M_\alpha) = \|\lambda y_\alpha\|_\alpha = \lambda \|y_\alpha\|_\alpha \quad \text{for all } \lambda > 0 \text{ and } \alpha \in \mathcal{A}.$$

Consider an arbitrary point $x_\alpha \in M$ such that $A_\alpha(x_\alpha) = y_\alpha$ $(\alpha \in \mathcal{A})$. Let $x \in X$ be any (Deutsch) limit point. By condition 2) of the theorem,

$$\|A_{\alpha_0}(x_\alpha)\|_{\alpha_0} \leqslant \|A_\alpha(x_\alpha)\|_\alpha = \|y_\alpha\|_\alpha = \varrho_\alpha(0, M_\alpha) \leqslant 1$$

for all $\alpha_0, \alpha \in \mathcal{A}$ with $\alpha_0 \leqslant \alpha$, and hence, by condition 1) of the theorem, for each $\varepsilon > 0$, there exists $\beta_0 = \beta_0(x, \varepsilon)$ such that $\|A_{\alpha_0}(x)\| \leq 1 + \varepsilon$ for each $\alpha_0 \geqslant \beta_0$. Hence $\|x\| \leqslant 1$ by condition 3) of the theorem. Since $1 = \varrho(0, M) \leqslant \|x\| \leqslant 1$, it follows that $\|x\| = 1$ and $P_M(0)$.

We now claim that $\varrho(x + \lambda(0 - x), M) = \lambda\|x\|$. Assume on the contrary that, for some $\lambda > 0$, there exists a point $z \in M \cap \mathring{B}((1-\lambda)x, \lambda\|x\|)$, i.e., $\|z - (1-\lambda)x\| \leqslant \lambda\|x\| - \varepsilon$ for some $\varepsilon > 0$. Let $z_\alpha := A_\alpha(z)$ $(z_\alpha \in M_\alpha)$. Since M_α is a sun in Y_α, we have $\|z_\alpha - (1-\lambda)x_\alpha\|_\alpha \geqslant \lambda\|x_\alpha\|_\alpha$ for each α. Passing to the limit in the inequality, using condition 3), employing the inequality $\|z_\alpha - (1-\lambda)x_\alpha\|_\alpha = \|A_\alpha(z - (1-\lambda)x)\|_\alpha \to \|z - (1-\lambda)x\|$ and the convergence $\lambda\|x_\alpha\|_\alpha \to \lambda\|x\|$, we have $\|z - (1-\lambda)x\| \geqslant \lambda\|x\|$, a contradiction. So,

$$M \cap \mathring{B}((1-\lambda)x, \lambda\|x\|) = \varnothing \quad \text{and} \quad \varrho((1-\lambda)x, M) \geqslant \lambda\|x\|.$$

On the other hand, $\|(1-\lambda)x - x\| = \lambda\|x\|$ and $x \in P_M(0)$, i.e., x is a luminosity point from M for $x_0 = 0$ in X. Hence M is a sun. This proves the theorem. □

Remark 2.1. Let $M \subset X$ be a boundedly Deutsch compact set satisfying conditions 1)–3) of Theorem 2.1, and let $\delta > 0$, $R \geqslant \varrho(x_0, M) + \delta$, $M_R := M \cap B(x, 3R)$. Assume that, for each $\varepsilon > 0$, there exist $\alpha_0 \in \mathcal{A}$ and a point $y^\varepsilon_\alpha \in (M_R)_\alpha$ such that

$$\|z - (y^\varepsilon_\alpha + \lambda((x_0)_\alpha - y^\varepsilon_\alpha))\|_\alpha \geqslant \|y^\varepsilon_\alpha - (y^\varepsilon_\alpha + \lambda((x_0)_\alpha - y_\alpha)^\varepsilon)\|_\alpha - \varepsilon = \lambda\|(x_0)_\alpha - y^\varepsilon_\alpha\|_\alpha - \varepsilon$$

for all $\alpha \geqslant \alpha_0$, $y^\varepsilon_\alpha + \lambda((x_0)_\alpha - y^\varepsilon_\alpha) \in B_\alpha((x_0)_\alpha, \delta)$ $(\lambda \geqslant 0)$, and $z \in (M_R)_\alpha$. Then there is a point $x \in M$ which is a nearest point in M for each point $x_0 + \lambda(x_0 - x) \in B(x_0, \delta)$ $(\lambda \geqslant 0)$ in X.

Definition 2.4. A set $M \subset X$ is *monotone path-connected* if any two points from M can be connected by a continuous monotone curve (arc) $k(\,\cdot\,) \subset M$; a continuous curve $k(\tau)$, $0 \leqslant \tau \leqslant 1$, in a normed linear space X is *monotone* if $f(k(\tau))$ is a monotone function of τ for each $f \in \operatorname{ext} S^*$. For further details, see [4, § 7.7.3], [5], [6], and [7].

Given a bounded set M in X, we let $\mathrm{m}(M)$ denote the intersection of all closed balls which contain M; in particular, $\mathrm{m}(x, y)$ is the intersection of all closed balls containing the point x, y (the *Banach–Mazur hull* of points x and y, or the bull hull of x and y).

Definition 2.5. A set $M \subset X$ is *Menger connected* if $\mathrm{m}(\{x, y\}) \cap M \neq \{x, y\}$ for all different points $x, y \in M$ (see [9] and [4]).

Remark 2.2. In any separable Banach space, each boundedly weakly compact Menger connected set is monotone path-connected (see, e.g., Theorem 3.24 in [5]).

The class[1] (BHS) of normed spaces X is defined by the property

$$\mathrm{m}(x, y) = [\![x, y]\!] \quad \text{for each } x, y \in X$$

[1](BHS) comes from the phrase "the ball hulls are precisely the segments".

(see, e.g., [4, § 7.7.2]), where $[\![x,y]\!]$ is the segment $[\![x,y]\!]$ is defined by

$$[\![x,y]\!] := \{z \in X \mid \min\{\varphi(x),\varphi(y)\} \leqslant \varphi(z) \leqslant \max\{\varphi(x),\varphi(y)\} \ \ \forall\, \varphi \in \operatorname{ext} S^*\}. \tag{2.1}$$

(The inclusion $\mathrm{m}(x,y) \supset [\![x,y]\!]$ always holds; see, e.g., Remark 24 in [6].)

K. Franchetti and S. Roversi (see, e.g., [4]) introduced the class of normed spaces (which we denote by (FR)) defined by the property

$$\operatorname{ext} S^* \text{ is } w^*\text{-separable.}$$

Here, in the definition of the class (FR) we always assume that the set $F = (f_i)_{i\in I} \subset \operatorname{ext} S^*$ is w^*-dense in $\operatorname{ext} S^*$, $\operatorname{card} I \leqslant \aleph_0$. Below, we assume without loss of generality that $I = \mathbb{N}$. Any separable space lies in the class (FR) ∩ (BHS).

Remark 2.3. The class (FR)∩(BHS) includes all $C(Q)$-spaces (even non-separable), Q is a compact set.

Monotone path-connectedness and Menger connectedness are frequently useful in deriving solar-like properties of sets. In particular, in $C(Q)$, a boundedly compact set is a sun if and only if it is monotone path-connected (see, e.g., [5] and [6]). For some recent advances here, see [1], [2], [3], [14], [15].

Example 2.2. The set

$$\mathbf{Ri} := \operatorname{cl}\left\{ a e^{(\boldsymbol{\lambda},\mathbf{x})} + c \;\middle|\; \boldsymbol{\lambda} \in \mathbb{R}^d,\ a, c \in \mathbb{R} \right\} \tag{2.2}$$

is monotone path-connected in $C(D)$, where D is a convex compact body in $\mathbb{R}^d$ (here $\operatorname{cl} A$ is the closure of a set A). The set **Ri** is boundedly compact, and hence **Ri** is a sun in $C(D)$. The univariate case of (2.2) was examined by Rice [11].

Theorem 2.2. *Let $X \in$ (FR). Then any boundedly Δ-compact monotone path-connected set in X is a sun in X.*

Remark 2.4. The conclusion of Theorem 2.2 also holds in any normed linear space for boundedly aw-compact monotone path-connected sets (the definition of aw-convergence is given in Example 2.1, g)).

We introduce some notation. We assume that $\mathcal{A}$ is a subset of $\operatorname{ext} S^*$ such that $\mathcal{A} \sqcup -\mathcal{A} = \operatorname{ext} S^*$ and $\mathcal{A}$ contains no pair of antipodal functionals. Let $\alpha = \{x_i\}_{i=1}^n \subset \mathcal{A}$ be a finite tuple of linearly independent functionals. On the set of all such tuples, we introduce the partial order defined by

$$\alpha_1 \geqslant \alpha_2 \quad \Leftrightarrow \quad \alpha_2 \subset \alpha_1.$$

For each such α, consider the subspace

$$\mathcal{L}_\alpha := \{x \in X \mid x_k^*(x) = 0,\ k = \overline{1,n}\}.$$

By Y_α we denote the space $\ell_\alpha := (x_1^*, \dots, x_n^*)(X)$ and equip it with the norm

$$\|\cdot\|_\alpha := \max_{k=\overline{1,n}} |x_k^*(x)|.$$

Note that Y_α is a finite-dimensional space. Let $\mathcal{P}_\alpha : X \to Y_\alpha$ be the linear operator $\mathcal{P}_\alpha : X \to Y_\alpha$ defined by $\mathcal{P}_\alpha(x) := x_\alpha := (x_1^*, \dots, x_n^*)(x)$. Note that $\mathcal{P}_y^{-1}(x_\alpha) := x + \mathcal{L}_\alpha$ for each $x \in X$ (we assume that $\mathcal{P}_\alpha^{-1}(x_\alpha)$ is embedded in X as a subspace). The norm of the operator $\mathcal{P}_\alpha$ is 1. Further, let $B_\alpha(x,r)$ and

$\mathring{B}_\alpha(x,r)$ denote, respectively, the closed and open balls in Y_α, and B_α and $\mathring{B}_\alpha$ be, respectively, the closed and open unit balls in Y_α. We have $\mathcal{P}_\alpha(B) = B_\alpha$. Given an arbitrary set $E \subset X$, we define $E_\alpha := \mathcal{P}_\alpha(E)$.

Proof. The finite-dimensional case being trivial, we assume that $\dim X = \infty$. By assumption, there exists a countable family $E = \{x_i\}_{i\in\mathbb{N}} \subset \operatorname{ext} S^*$ which is w^*-dense in $\operatorname{ext} S^*$. For any finite family $\alpha = (y_1^*, \dots, y_n^*)$, consider the set $M_\alpha := \mathcal{P}_\alpha(M)$ and points $x_\alpha := \mathcal{P}_\alpha(x)$ $(x \in X)$. It is easily seen that

$$[\![x_\alpha, y_\alpha]\!] = \mathcal{P}_\alpha([\![x,y]\!]) \qquad x, y \in X, \quad x_\alpha(t) := \mathcal{P}_\alpha(x(t)),$$

is a monotone path in Y_α if $x(t) \in C([0,1], X)$ is a monotone path in X. In addition, if the monotone path $x(\,\cdot\,)$ connects points $x, y \in X$ (i.e., $x(0) = x$ and $x(1) = y$), then $x(t) \in [\![x,y]\!]$ for each $t \in [0,1]$, and the monotone path $x_\alpha(t) \in [\![x_\alpha, y_\alpha]\!]$ connects the points $x_\alpha, y_\alpha \in Y_\alpha$. The set M is monotone path-connected in X, and hence M_α is monotone path-connected in the finite-dimensional space Y_α. Hence M_α is a sun in Y_α (see, e.g., Theorem 7 in [6]).

As linear operators A_α (see Theorem 2.1) we consider the operators $\mathcal{P}_\alpha$, and as a directed index set $\mathcal{A}$ we consider the inclusion ordered set of all finite tuples $\alpha = (y_1^*, \dots, y_n^*)$ from E,

$$\alpha \leqslant \beta \quad \Leftrightarrow \quad \alpha \subset \beta.$$

Let $\{x_\gamma\}_{\gamma\in\mathcal{A}}$ be an arbitrary net in X with some limit point $x_0 \in X$. By assumption, there exists a functional $x_0^* \in \operatorname{ext} S^*$ such that $x_0^*(x_0) = \|x_0\|$. Hence, for each $\varepsilon > 0$, there exists a functional $y_0^* \in E$ such that $y_0^*(x_0) \geqslant \|x_0\| - \varepsilon$. Then $\|(x_0)_\alpha\|_\alpha \leqslant \|x_0\| \leqslant y_0^*(x_0) + \varepsilon$ for each tuple α. Hence, for each $\alpha_0 \geqslant \{y_0^*\}$. we have

$$\|A_{\alpha_0}(x_0)\| \leqslant \limsup_{\gamma} \|A_{\alpha_0}(x_\gamma)\|_{\alpha_0} + \varepsilon,$$

i.e., condition 1) of Theorem 2.1 is met. It is easily seen that conditions 2) and 3) of Theorem 2.1 are also satisfied. Now Theorem 2.1 implies that M is a sun in X. $\square$

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References

[1] A. R. Alimov, Hereditary properties of lower semicontinuous metric projection and solar properties of sets, *J. Analysis* **33** (2025), no. 2, 743–753.

[2] A. R. Alimov, Balayage theorems for connectedness problems in uniformly convex spaces, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **50** (2024), no. 1, 53–61.

[3] A. R. Alimov, N. A. Ilyasov, Approximative compactness and acting points, *Russ. J. Math. Phys.* **32** (2025), no. 2, 219–227.

[4] A.R. Alimov, I. G. Tsar'kov, *Geometric approximation theory*, Springer, Cham (2021).

[5] A. R. Alimov, I. G. Tsar'kov, *Modern Problems of Geometric Approximation Theory*, Moscow, Moscow University Press (2025) [in Russian].

[6] A. R. Alimov, I. G. Tsar'kov, Monotone path-connected sets in geometric approximation theory and their applications, *Matem. Teork. Komp. Nauki* **2** (2024), no. 2, 30–46,

[7] A. R. Alimov, I. G. Tsar'kov, Stability of approximation in classical problems of geometric approximation theory, *Izv. Math.* **89** (2025), no. 6, 1087–1107.

[8] D. Braess, Über die Vorzeichenstruktur der Exponentialsummen, *J. Approx. Theory* **3** (1970), 101–113.

[9] A. L. Brown, Suns in normed linear spaces which are finite dimensional, *Math. Ann.* **279** (1987), no. 1, 87–101.

[10] F. Deutsch, Existence of best approximations, *J. Approx. Theory* **28** (1980), no. 2, 132–154.

[11] J. R. Rice, Chebyshev approximation by $ab^x + c$, *SIAM J. Appl. Math.* **8** (1960), no. 4, 691–702.

[12] E. Schmidt, Stetigkeitsaussagen bei der Tschebyscheff-Approximation mit Exponentialsummen, *Math. Z.* **113** (1970), no. 2, 159–170,

[13] E. Schmidt, Zur Kompaktheit bei Exponcntialsummen, *J. Approx. Theory* **3** (1970), 445–454.

[14] I. G. Tsar'kov, Solarity of Deutsch boundedly compact sets, *Russian J. Math. Phys.* **33** (2026), no. 1.

[15] I. G. Tsar'kov, Local and global suns, *Russian J. Math. Phys.* **31** (2024), no. 4, 765–773.

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