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FRACTIONAL SPACES GENERATED BY THE SECOND ORDER DIFFERENTIAL OPERATOR WITH PERIODIC CONDITIONS

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Abstract. In this study, we consider the second-order differential operator A^x defined by

$$A^x u = -\left(a(x)u_x(x)\right)_x + \sigma u(x), \sigma \geqslant 0, x \in \mathbb{R},$$

with domain

$$D(A^x) = \left\{ u : u, u'' \in C\left(\mathbb{R}\right), u(x) = u(x+2\pi), x \in \mathbb{R}, \int_0^{2\pi} u(x)dx = 0 \right\}.$$

Estimates for the Green's function are obtained. It is proved that for any $\alpha \in (0, \frac{1}{2})$, the norms in the spaces $E_\alpha = E_\alpha(\mathring{C}\left(\mathbb{R}\right), A^x)$ and $\mathring{C}^{2\alpha}\left(\mathbb{R}\right)$ are equivalent. The positivity of the operator A^x in Hölder spaces $\mathring{C}^{2\alpha}\left(\mathbb{R}\right), \alpha \in (0, \frac{1}{2})$, is proved. As an application, theorems on well-posedness of local and nonlocal boundary value problems for elliptic equations in Hölder spaces are established.

1. Introduction

The role played by positivity of differential and difference operators in a Banach space in the study of various properties of boundary value problems for partial differential equations, of stability of difference schemes for partial differential equations and of summation Fourier series is well-known (see [1, 8, 11, 20, 22, 23, 24, 27, 28, 31] and the references therein). The positivity of a wider class of differential and difference operators with local boundary conditions in Banach spaces has been studied by many researchers (see [2, 3, 4, 5, 6, 7, 9, 10, 13, 14, 15, 16, 17, 18, 19, 21, 25, 26, 29, 30] and the references therein).

Let E be a Banach space and $A : D(A) \subset E \to E$ be a linear unbounded operator densely defined in E. A is called a positive operator in the Banach space if the operator $(\lambda I + A)$ has a bounded inverse in E and for any $\lambda \geqslant 0$, the following estimate holds [11]:

$$\left\|(\lambda I + A)^{-1}\right\|_{E \to E} \leqslant \frac{M}{\lambda + 1}.$$

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Throughout the present paper, M denotes positive constants, which may differ in time and thus is not a subject of precision. However, we will use $M(\alpha,\beta,...)$ to stress the fact that the constant depends only on $\alpha,\beta,...$.

For a positive operator A in the Banach space E, let us introduce the fractional spaces $E_\alpha = E_\alpha(E,A), (0<\alpha<1)$ consisting of those $v\in E$ for which the norm

$$\|v\|_{E_\alpha} = \sup_{\lambda>0} \lambda^\alpha \left\|A(\lambda I+A)^{-1}v\right\|_E + \|v\|_E$$

is finite.

Let us introduce the Banach space $C^\beta(\mathbb{R})$, $\beta\in(0,\frac{1}{2})$ of all continuous 2π periodic functions $\varphi(x)$ defined on $\mathbb{R}$ and satisfying a Hölder condition for which the following norm is finite

$$\|\varphi\|_{C^\beta(\mathbb{R})} = \|\varphi\|_{\mathring{C}(\mathbb{R})} + \sup_{x,x+\tau\in[0,2\pi],\tau\neq 0} \frac{|\varphi(x+\tau)-\varphi(x)|}{|\tau|^\beta},$$

where $\mathring{C}(\mathbb{R})$ is the Banach space of all continuous 2π periodic functions $\varphi(x)$ defined on $\mathbb{R}$ with the norm

$$\|\varphi\|_{\mathring{C}(\mathbb{R})} = \max_{x\in[0,2\pi]} |\varphi(x)|.$$

In paper [28], a new method of summations of Fourier series converging in

$$\mathring{C}(\mathbb{R}) = \left\{\varphi(x)\in C(\mathbb{R}) : \varphi(x)=\varphi(x+2\pi), x\in\mathbb{R}, \int_0^{2\pi}\varphi(x)dx=0\right\}$$

is presented. It is based on the following result on the positivity of the differential operator A^x defined by the formula

$$A^x u = -u_{xx}(x) + \sigma u(x), \sigma\geqslant 0 \tag{1.1}$$

with domain

$$D(A^x) = \left\{u : u(x), u''(x)\in C(\mathbb{R}), u(x)=u(x+2\pi), x\in\mathbb{R}, \int_0^{2\pi} u(x)dx = 0\right\}.$$

Theorem 1.1. [28] *The operator $(A^x+\lambda)$ has a bounded in $\mathring{C}(\mathbb{R})$ inverse for $\sigma=0$, $\lambda\geqslant 0$ and the following estimate holds:*

$$\left\|(A^x+\lambda I)^{-1}\right\|_{\mathring{C}(\mathbb{R})\to\mathring{C}(\mathbb{R})} \leqslant \frac{1+16\pi^2}{1+\lambda}. \tag{1.2}$$

The positivity of differential and difference operators with nonlocal boundary conditions in Banach spaces and its applications have not been studied well.

In the present paper, we consider the differential operator A^x defined by the formula

$$A^x u(x) = -\left(a(x)u_x(x)\right)_x + \sigma u(x), \sigma>0, x\in\mathbb{R} \tag{1.3}$$

with domain

$$D(A^x) = \left\{u : u, u''\in C(\mathbb{R}), u(x)=u(x+2\pi), x\in\mathbb{R}, \int_0^{2\pi} u(x)dx = 0\right\}.$$

Assume that $a(x) = a(x + 2\pi), x \in \mathbb{R}$ and $a(x) \geqslant a > 0$ is a continuously differentiable function defined on $\mathbb{R}$. We will be interested in obtaining the resolvent of the operator A^x, i.e., in solving the resolvent equation

$$A^x u(x) + \lambda u(x) = \varphi(x), x \in \mathbb{R}, \tag{1.4}$$

where $\int_0^{2\pi} \varphi(x)dx = 0$.

Note that in general A^x is not self-adjoint. However, under the condition $\int_0^{2\pi} \varphi(x)dx = 0$ it follows A^x is self-adjoint. Actually, under this condition and the definition of $D(A^x)$, we have $\int_0^{2\pi} A^x u(x)dx = 0$. That means conditions $\int_0^{2\pi} u(x)dx = 0$ and $\int_0^{2\pi} A^x u(x)dx = 0$ are equivalent.

Note that equation (1.4) can be written in the following boundary value problem

$$\begin{cases} -\left(a(x)u_x(x)\right)_x + (\sigma + \lambda)\, u(x) = \varphi(x), x \in \mathbb{R}, \\ u(x) = u(x + 2\pi), \ \int_0^{2\pi} u(s)ds = 0, x \in \mathbb{R} \end{cases} \tag{1.5}$$

for the second order differential equation with periodic conditions.

The Green function of A^x is constructed. The estimates for the Green function are obtained. It is proved that for any $\alpha \in (0, \frac{1}{2})$, the norms in the spaces $E_\alpha = E_\alpha(\mathring{C}\left(\mathbb{R}\right), A^x)$ and $\mathring{C}^{2\alpha}\left(\mathbb{R}\right)$ are equivalent. Here, $\mathring{C}^{2\alpha}\left(\mathbb{R}\right)$ is the subspace of $C^{2\alpha}\left(\mathbb{R}\right)$ such that $\int_0^{2\pi} \varphi(x)dx = 0$. The positivity of the operator A^x in the Hölder spaces $\mathring{C}^{2\alpha}\left(\mathbb{R}\right), \alpha \in (0, \frac{1}{2})$ is proved. In the applications, theorems well-posedness of local and nonlocal boundary value problems for elliptic equations in Hölder spaces are obtained.

2. The Green's function of the second order differential operator with periodic conditions

Assume that $G^x(x, y; \lambda)$ is the fundamental solution of (1.4). Then,

$$u(x) = \int_0^{2\pi} G^x(x, y; \lambda)\varphi(y)dy, x \in \mathbb{R}. \tag{2.1}$$

It is well-known that $G^x(x, y; \lambda)$ is defined as the solution of (1.4) for the special right-hand case

$$\delta(x, y) = \begin{cases} \infty \ , x = y, \\ 0 \ , x \neq y. \end{cases} \tag{2.2}$$

Therefore,

$$G^x(x, y; \lambda) = \int_0^{2\pi} G^x(x, z; \lambda)\delta(z, y)dz. \tag{2.3}$$

Lemma 2.1. *For any $x_0 \in \mathbb{R}$, the following formula for $G^x(x,y;\lambda)$ holds*

$$G^x(x,y;\lambda) = G^{x_0}(x,y;\lambda) + \int_0^{2\pi} G^x(x,z;\lambda)\left(A^{x_0} - A^z\right) G^{x_0}(z,y;\lambda)dz, x, y \in \mathbb{R}, \tag{2.4}$$

where $G^{x_0}(x,y;\lambda)$ is the fundamental solution of resolvent equation with constant coefficient $a(x_0)$

$$A^{x_0}u(x) + \lambda u(x) = \varphi(x), x \in \mathbb{R}. \tag{2.5}$$

Proof. It is easy to see that (1.4) is equivalent to the following equation

$$A^{x_0}u(x) + \lambda u(x) = \psi(x),$$

where

$$\psi(x) = \left(A^{x_0} - A^x\right) u(x) + \varphi(x). \tag{2.6}$$

Thus, using the definition of Green's function and formula (2.1), we can write

$$u(x) = \int_0^{2\pi} G^x(x,y;\lambda)\psi(y)dy$$

$$= \int_0^{2\pi} G^x(x,y;\lambda)\varphi(y)dy + \int_0^{2\pi} G^x(x,y;\lambda)\left(A^y - A^{x_0}\right)\int_0^{2\pi} G^{x_0}(y,z;\lambda)\varphi(z)dzdy$$

$$= \int_0^{2\pi}\left[G^x(x,y;\lambda) + \int_0^{2\pi} G^x(x,z;\lambda)\left(A^z - A^{x_0}\right) G^{x_0}(z,y;\lambda)dz\right]\varphi(y)dy.$$

Using formula (2.1), we obtain

$$\int_0^{2\pi} G^{x_0}(x,y;\lambda)\varphi(y)dy = \int_0^{2\pi}\left[G^x(x,y;\lambda)\right.$$

$$\left.+ \int_0^{2\pi} G^x(x,z;\lambda)\left(A^z - A^{x_0}\right) G^{x_0}(z,y;\lambda)dz\right]\varphi(y)dy.$$

This equation holds for any function $\varphi(x) \in C(\mathbb{R})$. Hence,

$$G^{x_0}(x,y;\lambda) = G^x(x,y;\lambda) + \int_0^{2\pi} G^x(x,z;\lambda)\left(A^z - A^{x_0}\right) G^{x_0}(z,y;\lambda)dz, x, y \in \mathbb{R}$$

or

$$G^x(x,y;\lambda) = G^{x_0}(x,y;\lambda) + \int_0^{2\pi} G^x(x,z;\lambda)\left(A^{x_0} - A^z\right) G^{x_0}(z,y;\lambda)dz, x, y \in \mathbb{R}.$$

Lemma 2.1 is proved. □

We note that

$$\left(A^{x_0}-A^z\right)G^{x_0}(z,y;\lambda)=\frac{a(x_0)-a(z)}{a(x_0)}\left(A^{x_0}-\sigma I\right)G^{x_0}(z,y;\lambda). \tag{2.7}$$

Applying (2.2), we get

$$\lambda G^{x_0}(z,y;\lambda)+A^{x_0}G^{x_0}(z,y;\lambda)=\delta\left(z,y\right).$$

Then

$$\left(A^{x_0}-A^z\right)G^{x_0}(z,y;\lambda)=\frac{a(x_0)-a(z)}{a(x_0)}\left\{-(\sigma+\lambda)G^{x_0}(z,y;\lambda)+\delta\left(z,y\right)\right\}. \tag{2.8}$$

Applying (2.4) and (2.8), we get

$$\begin{aligned} G^x(x,y;\lambda) &= G^{x_0}(x,y;\lambda) \\ &+ \int_0^{2\pi} G^x(x,z;\lambda)\frac{a(x_0)-a(z)}{a(x_0)}\{-(\sigma+\lambda)G^{x_0}(z,y;\lambda)+\delta(z,y)\}dz. \end{aligned} \tag{2.9}$$

We note that (2.9) holds for any $x, y, x_0 \in \mathbb{R}$ and this identity can be considered as an equation for the unknown function $G(x,y;\lambda)$.

It can be written as the operator equation

$$G^x(x,y;\lambda)=G^{x_0}(x,y;\lambda)+K(x_0,\lambda)G^x(x,y;\lambda). \tag{2.10}$$

Here, $K(x_0,\lambda)$ is a linear operator defined by the formula

$$\begin{aligned} &K(x_0,\lambda)G^x(x,y;\lambda) \\ &= \int_0^{2\pi} G^x(x,z;\lambda)\frac{a(x_0)-a(z)}{a(x_0)}\left\{-(\sigma+\lambda)G^{x_0}(z,y;\lambda)+\delta\left(z,y\right)\right\}dz. \end{aligned}$$

We will study operator $K(x_0,\lambda)$ in some space where it is contraction. Then, there exists a unique solution of this operator equation and norm of $G^x(x,y;\lambda)$ is estimated by norm of $G^{x_0}(x,y;\lambda)$ in this space (Theorem 2.1, Lemma 2.2).

Putting $y=x_0$ in (2.9), we get

$$\begin{aligned} G^x(x,x_0;\lambda) &= G^{x_0}(x,x_0;\lambda) \\ &+ \int_0^{2\pi} G^x(x,z;\lambda)\frac{a(x_0)-a(z)}{a(x_0)}\left\{-(\sigma+\lambda)G^{x_0}(z,x_0;\lambda)+\delta\left(z,x_0\right)\right\}dz. \end{aligned}$$

or

$$\begin{aligned} G^x(x,x_0;\lambda) &= G^{x_0}(x,x_0;\lambda) \\ &+ \int_0^{2\pi} G^x(x,z;\lambda)\frac{a(x_0)-a(z)}{a(x_0)}\left\{-(\sigma+\lambda)G^{x_0}(z,x_0;\lambda)\right\}dz. \end{aligned} \tag{2.11}$$

Putting $x_0=y$ in (2.11), we get

$$G^x(x,y;\lambda)=G^y(x,y;\lambda)-(\sigma+\lambda)\int_0^{2\pi} G^x(x,z;\lambda)\frac{a(y)-a(z)}{a(y)}G^y(z,y;\lambda)dz. \tag{2.12}$$

We will prove that equation (2.12) has a unique solution in a Banach space E_1 with norm

$$|G(\cdot,\cdot;\lambda)|_1 = \sqrt{\sigma+\lambda} \max_{x,y\in[0,2\pi]} |G(x,y;\lambda)| \, e^{\frac{a}{2}\sqrt{\sigma+\lambda}\,\gamma(x,y)}, \tag{2.13}$$

where here and throughout the article

$$\gamma(x,y) = \min\{2\pi - |x-y|, |x-y|\}.$$

Note that we have

$$\gamma(x,y) = \begin{cases} \alpha(x,y) & , y \leqslant x, \\ \beta(x,y) & , x \leqslant y, \end{cases}$$

where

$$\alpha(x,y) = \min\{2\pi - x + y, x - y\}, \beta(x,y) = \min\{2\pi + x - y, y - x\}.$$

It is easy to see that

$$\alpha(x,y) = \begin{cases} 2\pi - x + y & , 0 \leqslant y \leqslant x - \pi, \\ x - y & , x - \pi \leqslant y \leqslant x, \end{cases} \tag{2.14}$$

$$\beta(x,y) = \begin{cases} y - x & , x \leqslant y \leqslant x + \pi, \\ 2\pi + x - y & , x + \pi \leqslant y \leqslant 2\pi. \end{cases} \tag{2.15}$$

Note that we have the following pointwise estimates for $G^y(x,y;\lambda)$ and its derivative with respect to x [8]:

$$|G^y(x,y;\lambda)| \leqslant \frac{M(\sigma)}{\sqrt{\sigma+\lambda}} \, e^{-a\sqrt{\sigma+\lambda}\,\gamma(x,y)}, \tag{2.16}$$

$$|G^y_x(x,y;\lambda)| \leqslant M(\sigma) \, e^{-a\sqrt{\sigma+\lambda}\,\gamma(x,y)}. \tag{2.17}$$

Now, we consider operator equation (2.10) in E_1.

Theorem 2.1. *The operator $K(y,\lambda)$ defined by formula*

$$K(y,\lambda)G^x(x,y;\lambda) = -(\sigma+\lambda) \int_0^{2\pi} G^x(x,z;\lambda) \frac{a(y)-a(z)}{a(y)} G^y(z,y;\lambda) dz \tag{2.18}$$

is the contractive operator in E_1.

Proof. Let $0 \leqslant y \leqslant x, x \in [0,2\pi]$. Then, using (2.12) and triangle inequality, we get

$$\sqrt{\sigma+\lambda} \, |G^x(x,y;\lambda)| \, e^{\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)} \leqslant \sqrt{\sigma+\lambda} \, |G^y(x,y;\lambda)| \, e^{\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)}$$

$$+(\sigma+\lambda)^{\frac{3}{2}} \int_0^{2\pi} e^{\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)} \, |G^x(x,z;\lambda)| \, \frac{|a(y)-a(z)|}{a(y)} \, |G^y(z,y;\lambda)| \, dz \,. \tag{2.19}$$

Applying the definition of norm $|\cdot|_1$, estimate (2.16) and inequality (2.19), we get

$$\sqrt{\sigma+\lambda} \, |G^x(x,y;\lambda)| \, e^{\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)} \leqslant M e^{-a\sqrt{\sigma+\lambda}\alpha(x,y)} e^{\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)}$$

$$+M_1(\sigma+\lambda)^{\frac{3}{2}} \int_0^{y} e^{\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)} \, |G^x(x,z;\lambda)| \, (y-z) \, |G^y(z,y;\lambda)| \, dz$$

$$+M_1(\sigma+\lambda)^{\frac{3}{2}}\int_y^x e^{\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)}\left|G^x(x,z;\lambda)\right|(z-y)\left|G^y(z,y;\lambda)\right|dz$$

$$+M_1(\sigma+\lambda)^{\frac{3}{2}}\int_x^{2\pi} e^{\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)}\left|G^x(x,z;\lambda)\right|(z-y)\left|G^y(z,y;\lambda)\right|dz$$

$$\leqslant Me^{-\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)}+M_1(\sigma)B(x,y,\lambda)\left|G(\cdot,\cdot;\lambda)\right|_1,$$

where

$$B(x,y,\lambda)=(\sigma+\lambda)^{\frac{1}{2}}\int_0^y e^{\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)}e^{-\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,z)}(y-z)e^{-a\sqrt{\sigma+\lambda}\beta(z,y)}dz$$

$$+(\sigma+\lambda)^{\frac{1}{2}}\int_y^x e^{\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)}e^{-\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,z)}(z-y)e^{-a\sqrt{\sigma+\lambda}\alpha(z,y)}dz$$

$$+(\sigma+\lambda)^{\frac{1}{2}}\int_x^{2\pi} e^{\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,y)}e^{-\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,z)}(z-y)e^{-a\sqrt{\sigma+\lambda}\alpha(z,y)}dz$$

$$=B_1(x,y,\lambda)+B_2(x,y,\lambda)+B_3(x,y,\lambda).$$

We will prove that

$$B_k(x,y,\lambda)\leqslant M(a,\sigma,\lambda), k=1,2,3, \tag{2.20}$$

where

$$M(a,\sigma,\lambda)\to 0 \text{ when } \sigma\to\infty. \tag{2.21}$$

First, we estimate $B_1(x,y,\lambda)$. From identities (2.14), (2.15) and the fact that $y\leqslant x$ it follows for $0\leqslant z\leqslant y$ that

$$2\beta(z,y)+\alpha(x,z)-\alpha(x,y)=2(y-z)+(x-z)-(x-y)=3(y-z) \tag{2.22}$$

Using (2.22), we can write

$$B_1(x,y,\lambda)=(\sigma+\lambda)^{\frac{1}{2}}\int_0^y e^{-\frac{a}{2}\sqrt{(\sigma+\lambda)}\,3(y-z)}(y-z)dz.$$

The substitution $p=\frac{a}{2}\sqrt{\sigma+\lambda}\,3(y-z)$ yields

$$B_1(x,y,\lambda)\leqslant\frac{9}{a^2\sqrt{\sigma+\lambda}}. \tag{2.23}$$

Applying (2.23), we get

$$B_1(x,y,\lambda)\leqslant\frac{M_1(a)}{\sqrt{\sigma+\lambda}}. \tag{2.24}$$

Thus, estimate (2.20) for $k=1$ follows from estimate (2.24).

Next, we estimate $B_2(x,y,\lambda)$. By identity (2.14) and the fact that $y\leqslant x$, we have for $y\leqslant z\leqslant x$ that

$$2\alpha(z,y)+\alpha(x,z)-\alpha(x,y)=2(z-y)+(x-z)-(x-y)=z-y. \tag{2.25}$$

Then, using (2.25), we can write

$$B_2(x,y,\lambda) = (\sigma+\lambda)^{\frac{1}{2}} \int\limits_y^x e^{-\frac{a}{2}\sqrt{\sigma+\lambda}(z-y)}(z-y)dz.$$

By the substitution $p = \frac{a}{2}\sqrt{\sigma+\lambda}\ (z-y)$, we have

$$B_2(x,y,\lambda) \leqslant \frac{4}{a^2\sqrt{\sigma+\lambda}}.$$

Applying this estimate, we get

$$B_2(x,y,\lambda) \leqslant \frac{M_1(a)}{\sqrt{\sigma+\lambda}}. \tag{2.26}$$

Hence, estimate (2.20) for $k=2$ follows from estimate (2.26).
Finally, let us estimate $B_3(x,y,\lambda)$. The fact $y \leqslant x$ and identities (2.14), (2.15) yield

$$2\alpha(z,y) + \beta(x,z) - \alpha(x,y) = 2\,(z-y) + (z-x) - (x-y) = 3z - y - 2x.$$

From this it follows

$$\begin{aligned} B_3(x,y,\lambda) &= (\sigma+\lambda)^{\frac{1}{2}} \int\limits_x^{2\pi} e^{-\frac{a}{2}\sqrt{\sigma+\lambda}(3z-y-2x)}(z-y)dz \\ &\leqslant (\sigma+\lambda)^{\frac{1}{2}} \int\limits_x^{2\pi} e^{-\frac{a}{2}\sqrt{\sigma+\lambda}(z-y)}(z-y)dz, \end{aligned}$$

where the last estimate uses the fact $x \leqslant z \leqslant 2\pi$.

Using the substitution $p = \frac{a}{2}\sqrt{\sigma+\lambda}\ (z-y)$, we obtain

$$B_3(x,y,\lambda) \leqslant \frac{M_1(a)}{\sqrt{\sigma+\lambda}}. \tag{2.27}$$

Therefore, estimate (2.20) for $k=3$ follows from estimate (2.27).

Now, let us assume that $x \leqslant y \leqslant 2\pi, x \in [0,2\pi]$. Then, using (2.12) and triangle inequality, we get

$$\begin{aligned} &\sqrt{\sigma+\lambda}\,|G^x(x,y;\lambda)|\, e^{\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)} \leqslant \sqrt{\sigma+\lambda}\,|G^y(x,y;\lambda)|\, e^{\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)} \\ &+(\sigma+\lambda)^{\frac{3}{2}} \int\limits_0^{2\pi} e^{\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)}\,|G^x(x,z;\lambda)|\, \frac{|a(y)-a(z)|}{a(y)}\,|G^y(z,y;\lambda)|\,dz\ . \end{aligned} \tag{2.28}$$

Applying the definition of norm $|\cdot|_1$, estimate (2.16) and inequality (2.28), we get

$$\begin{aligned} &\sqrt{\sigma+\lambda}\,|G^x(x,y;\lambda)|\, e^{\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)} \leqslant M e^{-a\sqrt{\sigma+\lambda}\beta(x,y)} e^{\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)} \\ &+M_1(\sigma+\lambda)^{\frac{3}{2}} \int\limits_0^{x} e^{\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)}\,|G^x(x,z;\lambda)|\,(y-z)\,|G^y(z,y;\lambda)|\,dz \end{aligned}$$

$$+M_1(\sigma+\lambda)^{\frac{3}{2}}\int\limits_x^y e^{\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)}\left|G^x(x,z;\lambda)\right|(z-y)\left|G^y(z,y;\lambda)\right|dz$$

$$+M_1(\sigma+\lambda)^{\frac{3}{2}}\int\limits_y^{2\pi} e^{\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)}\left|G^x(x,z;\lambda)\right|(z-y)\left|G^y(z,y;\lambda)\right|dz$$

$$\leqslant Me^{-\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)}+M_1(\sigma)D(x,y,\lambda)\left|G(\cdot,\cdot;\lambda)\right|_1,$$

where

$$D(x,y,\lambda)=(\sigma+\lambda)^{\frac{1}{2}}\int\limits_0^x e^{\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)}e^{-\frac{a}{2}\sqrt{\sigma+\lambda}\alpha(x,z)}(y-z)e^{-a\sqrt{\sigma+\lambda}\beta(z,y)}dz$$

$$+(\sigma+\lambda)^{\frac{1}{2}}\int\limits_x^y e^{\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)}e^{-\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,z)}(y-z)e^{-a\sqrt{\sigma+\lambda}\beta(z,y)}dz$$

$$+(\sigma+\lambda)^{\frac{1}{2}}\int\limits_y^{2\pi} e^{\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,y)}e^{-\frac{a}{2}\sqrt{\sigma+\lambda}\beta(x,z)}(z-y)e^{-a\sqrt{\sigma+\lambda}\alpha(z,y)}dz$$

$$=D_1(x,y,\lambda)+D_2(x,y,\lambda)+D_3(x,y,\lambda).$$

We will prove that

$$D_k(x,y,\lambda)\leqslant M(a,\sigma,\lambda),k=1,2,3, \tag{2.29}$$

where

$$M(a,\sigma,\lambda)\to 0 \text{ when } \sigma\to\infty. \tag{2.30}$$

We first estimate $D_1(x,y,\lambda)$. Using identities (2.14), (2.15) and the fact that $x\leqslant y$, we have

$$2\beta(z,y)+\alpha(x,z)-\beta(x,y)=2\,(y-z)+(x-z)-(y-x)=2x+y-3z.$$

This results

$$\begin{aligned} D_1(x,y,\lambda) &= (\sigma+\lambda)^{\frac{1}{2}}\int\limits_0^x e^{-\frac{a}{2}\sqrt{\sigma+\lambda}\,(2x+y-3z)}(y-z)dz \\ &\leqslant (\sigma+\lambda)^{\frac{1}{2}}\int\limits_0^x e^{-\frac{a}{2}\sqrt{\sigma+\lambda}\,(y-z)}(y-z)dz, \end{aligned}$$

where the last estimate follows from fact that $0\leqslant z\leqslant x$.

Using the substition $p=\frac{a}{2}\sqrt{\sigma+\lambda}\,(y-z)$, we get

$$D_1(x,y,\lambda)\leqslant\frac{M\,(a)}{\sqrt{\sigma+\lambda}}. \tag{2.31}$$

Estimate (2.29) for $k=1$ follows from estimate (2.31).
Next, we estimate $D_2(x,y,\lambda)$. Identity (2.15) and the fact that $x\leqslant y$ yield

$$2\beta(z,y)+\beta(x,z)-\beta(x,y)=2\,(y-z)+(z-x)-(y-x)=y-z.$$

Thus, we can write

$$D_2(x,y,\lambda) = (\sigma+\lambda)^{\frac{1}{2}} \int\limits_x^y e^{-\frac{a}{2}\sqrt{\sigma+\lambda}\,(y-z)}(y-z)dz.$$

The substition $p = \frac{a}{2}\sqrt{\sigma+\lambda}\ (y-z)$ yields

$$D_2(x,y,\lambda) \leqslant \frac{M(a)}{\sqrt{\sigma+\lambda}}. \tag{2.32}$$

Estimate (2.29) for $k=2$ follows from estimate (2.32).
Finally, let us estimate $D_3(x,y,\lambda)$. From identities (2.14), (2.15) and the fact that $x \leqslant y$ it follows that

$$2\alpha(z,y) + \beta(x,z) - \beta(x,y) = 2\,(z-y) + (z-x) - (y-x) = 3\,(z-y)\,.$$

Using this, we have

$$\begin{aligned} D_3(x,y,\lambda) &= (\sigma+\lambda)^{\frac{1}{2}} \int\limits_y^{2\pi} e^{-\frac{a}{2}\sqrt{\sigma+\lambda}\,3(z-y)}(z-y)dz \\ &\leqslant (\sigma+\lambda)^{\frac{1}{2}} \int\limits_y^{2\pi} e^{-\frac{a}{2}\sqrt{\sigma+\lambda}\,(z-y)}(z-y)dz. \end{aligned}$$

By using the substition $p = \frac{a}{2}\sqrt{\sigma+\lambda}\ (y-z)$, we obtain

$$D_3(x,y,\lambda) \leqslant \frac{M(a)}{\sqrt{\sigma+\lambda}}. \tag{2.33}$$

Estimate (2.29) for $k=3$ follows from estimate (2.33).
Applying the triangle inequality and estimates (2.20) and (2.29), we can write

$$|K(\cdot,\lambda)G(\cdot,\cdot;\lambda)|_1 \leq \frac{M_1(a)}{\sqrt{\sigma+\lambda}}\,|G(\cdot,\cdot;\lambda)|_1\,, \tag{2.34}$$

$$|G(\cdot,\cdot;\lambda)|_1 \leq M_2(a) + \frac{M_1(a)}{\sqrt{\sigma+\lambda}}\,|G(\cdot,\cdot;\lambda)|_1\,. \tag{2.35}$$

From (2.34) it follows that for sufficiently large σ the operator $K(y,\lambda)$ defined by formula (2.18) is the contractive operator in E_1. Moreover, using estimate (2.35), we can get

$$|G(\cdot,\cdot;\lambda)|_1 \leqslant M_3(a,\sigma), \tag{2.36}$$

where $M_3(a,\sigma) = \frac{M_2(a)}{1-\frac{M_1(a)}{\sqrt{\sigma}}}$. □

By estimate (2.36), we have the following pointwise estimate for $G^x(x,y;\lambda)$.

Lemma 2.2 (Pointwise estimate for the Green function)**.** *For sufficiently large σ and $\lambda \geqslant 0$, the following estimate holds:*

$$|G^x(x,y;\lambda)| \leqslant \frac{M(a,\sigma)}{\sqrt{\sigma+\lambda}}\, e^{-a\sqrt{\sigma+\lambda}\,\gamma(x,y)} \tag{2.37}$$

□

In a similar manner, one can obtain the following pointwise estimate for derivative of $G^x(x,y;\lambda)$ with respect to x.

Lemma 2.3 (Pointwise estimate for derivative of the Green function)**.** *For sufficiently large σ and $\lambda \geqslant 0$, the following estimate*

$$|G_x^x(x,y;\lambda)| \leqslant M(a,\sigma)\, e^{-a\sqrt{\sigma+\lambda}\,\gamma(x,y)}. \tag{2.38}$$

is valid. □

3. Structure of fractional spaces generated by the second order differential operator with periodic conditions

We will study the positivity of A^x in $\overset{\circ}{C}(\mathbb{R})$.

Theorem 3.1. *For sufficiently large σ and $\lambda \geq 0$, the operator $(A^x+\lambda)$ has a bounded inverse in $\overset{\circ}{C}(\mathbb{R})$ and the following estimate holds:*

$$\left\|(A^x+\lambda I)^{-1}\right\|_{\overset{\circ}{C}(\mathbb{R})\to\overset{\circ}{C}(\mathbb{R})} \leqslant \frac{M_1(a,\sigma)}{\sigma+\lambda}. \tag{3.1}$$

Proof. Applying formula (2.1), the triangle inequality and estimate (2.37), we can write

$$|u(x)| \leqslant \frac{M(a,\sigma)}{\sqrt{\sigma+\lambda}}\left[\int_0^{x-\pi} e^{-a\sqrt{\sigma+\lambda}(2\pi-x+y)}dy + \int_{x-\pi}^{x+\pi} e^{-a\sqrt{\sigma+\lambda}|x-y|}dy\right.$$

$$\left.+\int_{x+\pi}^{2\pi} e^{-a\sqrt{\sigma+\lambda}(2\pi+x-y)}dy\right]\max_{0\leqslant y\leqslant 2\pi}|\varphi(y)| \leqslant \frac{M_1(a,\sigma)}{\sigma+\lambda}\,\|\varphi\|_{\overset{\circ}{C}(\mathbb{R}^1)}$$

for any $x\in\mathbb{R}^1$. Therefore,

$$\|u\|_{\overset{\circ}{C}(\mathbb{R}^1)} \leqslant \frac{M_1(a,\sigma)}{\sigma+\lambda}\,\|\varphi\|_{\overset{\circ}{C}(\mathbb{R}^1)}\,.$$

From that it follows estimate (3.1). Theorem 3.1 is proved. □

Clearly, the operator A^x and its resolvent $(A^x+\lambda I)^{-1}$commute. By the definition of the norm in the fractional space $E_\alpha = E_\alpha(\overset{\circ}{C}(\mathbb{R}), A^x)$, we get

$$\left\|(A^x+\lambda I)^{-1}\right\|_{E_\alpha\to E_\alpha} \leqslant \left\|(A^x+\lambda I)^{-1}\right\|_{\overset{\circ}{C}(\mathbb{R})\to\overset{\circ}{C}(\mathbb{R})}\,.$$

Thus, from Theorem 3.1 it follows that A^x is a positive operator in the fractional spaces $E_\alpha(\overset{\circ}{C}(\mathbb{R}), A^x)$. Moreover, we have the following result.

Theorem 3.2. *For $\alpha\in(0,\frac{1}{2})$, the norms of the spaces $E_\alpha(\overset{\circ}{C}(\mathbb{R}), A^x)$ and the Hölder space $\overset{\circ}{C}{}^{2\alpha}(\mathbb{R})$ are equivalent.*

Proof. For any $\lambda \geqslant 0$, we have the obvious equality

$$A^x(A^x+\lambda I)^{-1}\varphi(x) = \varphi(x) - \lambda(A^x+\lambda I)^{-1}\varphi(x).$$

By formulas (1.4) and (2.1), we can write

$$A^x(A^x+\lambda I)^{-1}\varphi(x) = \frac{\sigma}{\sigma+\lambda}\varphi(x)+\lambda\int\limits_0^{2\pi} G^x(x,s;\lambda)(\varphi(x)-\varphi(s))\,ds$$

$$+ \lambda\int\limits_0^{2\pi}\left[G^s(x,s;\lambda)-G^x(x,s;\lambda)\right]ds\,\varphi(x). \tag{3.2}$$

Then,

$$\lambda^\alpha A^x(A^x+\lambda I)^{-1}\varphi(x)$$

$$=\frac{\sigma\lambda^\alpha}{\sigma+\lambda}\varphi(x)+\lambda^{\alpha+1}\int\limits_0^{2\pi} G^x(x,s;\lambda)(\varphi(x)-\varphi(s))\,ds$$

$$+\lambda^{\alpha+1}\int\limits_0^{2\pi}\left[G^s(x,s;\lambda)-G^x(x,s;\lambda)\right]ds\,\varphi(x)=P_1(x)+P_2(x)+P_3(x),$$

where

$$P_1(x)=\frac{\sigma\lambda^\alpha}{\sigma+\lambda}\varphi(x),\ P_2(x)=\lambda^{\alpha+1}\int\limits_0^{2\pi} G^x(x,s;\lambda)(\varphi(x)-\varphi(s))\,ds,$$

$$P_3(x)=\lambda^{\alpha+1}\int\limits_0^{2\pi}\left[G^s(x,s;\lambda)-G^x(x,s;\lambda)\right]ds\,\varphi(x).$$

Let us estimate $P_k(x), k=1,2,3$ in $\mathring{C}(\mathbb{R})$ norm. First, we will estimate $P_1(x)$ in $\mathring{C}(\mathbb{R})$ norm. Using the definition of the norm of space $\mathring{C}^{2\alpha}(\mathbb{R})$ and $\frac{\lambda^\alpha\sigma^{1-\alpha}}{\sigma+\lambda}\leqslant 1$, we can write

$$|P_1(x)|\leqslant\sigma^\alpha\|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})}$$

for any $x\in[0,2\pi]$. Then,

$$\|P_1\|_{\mathring{C}(\mathbb{R})}\leqslant\sigma^\alpha\|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})}. \tag{3.3}$$

Next, we will estimate $P_2(x)$ in $\mathring{C}(\mathbb{R})$ norm. Using estimate (2.37), we get

$$|P_2(x)|\leqslant\frac{M(a,\sigma)\lambda^{\alpha+1}}{\sqrt{\sigma+\lambda}}\left(\int\limits_0^{x-\pi} e^{-a\sqrt{\sigma+\lambda}(2\pi-x+s)}|\varphi(x)-\varphi(s)|\,ds\right.$$

$$+\int\limits_{x-\pi}^{x} e^{-a\sqrt{\sigma+\lambda}(x-s)}|\varphi(x)-\varphi(s)|\,ds+\int\limits_{x}^{x+\pi} e^{-a\sqrt{\sigma+\lambda}(s-x)}|\varphi(x)-\varphi(s)|\,ds$$

$$\left.+\int\limits_{x+\pi}^{2\pi} e^{-a\sqrt{\sigma+\lambda}(2\pi+x-s)}|\varphi(x)-\varphi(s)|\,ds\right)=P_{21}(x)+P_{22}(x)+P_{23}(x)+P_{24}(x),$$

where

$$P_{21}(x) = \frac{M(a,\sigma)\lambda^{\alpha+1}}{\sqrt{\sigma+\lambda}} \int_0^{x-\pi} e^{-a\sqrt{\sigma+\lambda}(2\pi-x+s)} \left|\varphi(x)-\varphi(s)\right| ds,$$

$$P_{22}(x) = \frac{M(a,\sigma)\lambda^{\alpha+1}}{\sqrt{\sigma+\lambda}} \int_{x-\pi}^{x} e^{-a\sqrt{\sigma+\lambda}(x-s)} \left|\varphi(x)-\varphi(s)\right| ds,$$

$$P_{23}(x) = \frac{M(a,\sigma)\lambda^{\alpha+1}}{\sqrt{\sigma+\lambda}} \int_{x}^{x+\pi} e^{-a\sqrt{\sigma+\lambda}(s-x)} \left|\varphi(x)-\varphi(s)\right| ds,$$

$$P_{24}(x) = \frac{M(a,\sigma)\lambda^{\alpha+1}}{\sqrt{\sigma+\lambda}} \int_{x+\pi}^{2\pi} e^{-a\sqrt{\sigma+\lambda}(2\pi+x-s)} \left|\varphi(x)-\varphi(s)\right| ds.$$

Using the condition $\varphi(s) = \varphi(s+2\pi)$, the definition of the norm of space $\mathring{C}^{2\alpha}(\mathbb{R})$ and the definition of Gamma function, we have

$$P_{21}(x) \leqslant \|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})} \frac{M_1(a,\sigma)\lambda^{\alpha+1}}{(\sigma+\lambda)^{\alpha+1}} \Gamma(2\alpha+1)$$

for any $x \in [0,2\pi]$. Then, we have

$$\max_{x\in[0,2\pi]} P_{21}(x) \leqslant \|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})} M_1(a,\sigma)\Gamma(2\alpha+1). \quad (3.4)$$

Let us estimate $P_{22}(x)$. From the definition of the norm of space $\mathring{C}^{2\alpha}(\mathbb{R})$ and the definition of Gamma function it follows that for each $x \in [0,2\pi]$

$$P_{22}(x) \leqslant \|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})} \frac{M_1(a,\sigma)\lambda^{\alpha+1}}{(\sigma+\lambda)^{\alpha+1}} \Gamma(2\alpha+1).$$

This yields

$$\max_{x\in[0,2\pi]} P_{22}(x) \leqslant \|\varphi\|_{C^{2\alpha}(\mathbb{R})} M_1(a,\sigma)\Gamma(2\alpha+1). \quad (3.5)$$

Let us estimate $P_{23}(x)$. By the definition of the norm of space $\mathring{C}^{2\alpha}(\mathbb{R})$ and the definition of Gamma function, we get

$$P_{23}(x) \leqslant \|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})} \frac{M_1(a,\sigma)\lambda^{\alpha+1}}{(\sigma+\lambda)^{\alpha+1}} \Gamma(2\alpha+1)$$

for any $x \in [0,2\pi]$. From this it follows that

$$\max_{x\in[0,2\pi]} P_{23}(x) \leqslant \|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R}^1)} M_1(a,\sigma)\Gamma(2\alpha+1). \quad (3.6)$$

Using the condition $\varphi(x) = \varphi(x+2\pi)$, the definition of the norm of space $\mathring{C}^{2\alpha}(\mathbb{R})$ and the definition of Gamma function, we obtain that for any $x \in [0,2\pi]$

$$P_{24}(x) \leqslant \|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})} \frac{M_1(a,\sigma)\lambda^{\alpha+1}}{(\sigma+\lambda)^{\alpha+1}} \Gamma(2\alpha+1).$$

Thus, we have

$$\max_{x\in[0,2\pi]} P_{24}(x) \leqslant \|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})} M_1(a,\sigma)\Gamma(2\alpha+1). \quad (3.7)$$

Combining estimates (3.4)-(3.7), we obtain

$$\max_{x\in[0,2\pi]} |P_2(x)| \leqslant \|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})} M_2(a,\sigma)\Gamma(2\alpha+1). \tag{3.8}$$

Finally, we will estimate $P_3(x)$ in $\mathring{C}(\mathbb{R})$ norm. Applying formula (2.12), we get

$$P_3(x) = \lambda^{\alpha+1}(\sigma+\lambda)\int_0^{2\pi}\int_0^{2\pi} G^x(x,z;\lambda)\frac{a(s)-a(z)}{a(s)}G^s(z,s;\lambda)dzds\,\varphi(x). \tag{3.9}$$

We have

$$\max_{x\in[0,2\pi]} |P_3(x)| \leqslant B \max_{x\in[0,2\pi]} |\varphi(x)|, \tag{3.10}$$

where $B = \max_{x\in[0,2\pi]} B(x)$ and

$$B(x) = \lambda^{\alpha+1}(\sigma+\lambda)\left|\int_0^{2\pi}\int_0^{2\pi} G^x(x,z;\lambda)\frac{a(s)-a(z)}{a(s)}G^s(z,s;\lambda)dzds\right|.$$

We will prove that

$$B(x) \leqslant M(a,\sigma) \tag{3.11}$$

for any $x \in [0,2\pi]$ and $\alpha \in [0,\frac{1}{2})$. Using the triangle inequality, estimates (2.16)

and (2.37), we get

$$\begin{aligned}
B(x) &\leqslant \lambda^{\alpha+1}(\sigma+\lambda)\int_0^{2\pi}\int_0^{2\pi} \left|G^x(x,z;\lambda)\frac{a(s)-a(z)}{a(s)}G^s(z,s;\lambda)\right| dzds \\
&\leqslant \lambda^{\alpha+1}(\sigma+\lambda)\int_0^{x}\int_0^{z} |G^x(x,z;\lambda)| \frac{|a(s)-a(z)|}{a(s)} |G^s(z,s;\lambda)| dsdz \\
&+ \lambda^{\alpha+1}(\sigma+\lambda)\int_0^{x}\int_z^{2\pi} |G^x(x,z;\lambda)| \frac{|a(s)-a(z)|}{a(s)} |G^s(z,s;\lambda)| dsdz \\
&+ \lambda^{\alpha+1}(\sigma+\lambda)\int_x^{2\pi}\int_0^{z} |G^x(x,z;\lambda)| \frac{|a(s)-a(z)|}{a(s)} |G^s(z,s;\lambda)| dsdz \\
&+ \lambda^{\alpha+1}(\sigma+\lambda)\int_x^{2\pi}\int_z^{2\pi} |G^x(x,z;\lambda)| \frac{|a(s)-a(z)|}{a(s)} |G^s(z,s;\lambda)| dsdz
\end{aligned}$$

$$
\begin{aligned}
&\leqslant M(a,\sigma)\lambda^{\alpha+1}\int_0^x\int_0^z e^{-a\sqrt{\sigma+\lambda}\,\alpha(x,z)}(z-s)e^{-a\sqrt{\sigma+\lambda}\,\alpha(z,s)}dsdz\\
&+ M(a,\sigma)\lambda^{\alpha+1}\int_0^x\int_z^{2\pi} e^{-a\sqrt{\sigma+\lambda}\,\alpha(x,z)}(s-z)e^{-a\sqrt{\sigma+\lambda}\,\beta(z,s)}dsdz\\
&+ M(a,\sigma)\lambda^{\alpha+1}\int_x^{2\pi}\int_0^z e^{-a\sqrt{\sigma+\lambda}\,\beta(x,z)}(z-s)e^{-a\sqrt{\sigma+\lambda}\,\alpha(z,s)}dsdz\\
&+ M(a,\sigma)\lambda^{\alpha+1}\int_x^{2\pi}\int_z^{2\pi} e^{-a\sqrt{\sigma+\lambda}\,\beta(x,z)}(s-z)e^{-a\sqrt{\sigma+\lambda}\,\beta(z,s)}dsdz\\
&= B_1(x)+B_2(x)+B_3(x)+B_4(x),
\end{aligned}
$$

where

$$B_1(x)=M(a,\sigma)\lambda^{\alpha+1}\int_0^x\int_0^z e^{-a\sqrt{\sigma+\lambda}\,\alpha(x,z)}(z-s)e^{-a\sqrt{\sigma+\lambda}\,\alpha(z,s)}dsdz,$$

$$B_2(x)=M(a,\sigma)\lambda^{\alpha+1}\int_0^x\int_z^{2\pi} e^{-a\sqrt{\sigma+\lambda}\,\alpha(x,z)}(s-z)e^{-a\sqrt{\sigma+\lambda}\,\beta(z,s)}dsdz,$$

$$B_3(x)=M(a,\sigma)\lambda^{\alpha+1}\int_x^{2\pi}\int_0^z e^{-a\sqrt{\sigma+\lambda}\,\beta(x,z)}(z-s)e^{-a\sqrt{\sigma+\lambda}\,\alpha(z,s)}dsdz,$$

$$B_4(x)=M(a,\sigma)\lambda^{\alpha+1}\int_x^{2\pi}\int_z^{2\pi} e^{-a\sqrt{\sigma+\lambda}\,\beta(x,z)}(s-z)e^{-a\sqrt{\sigma+\lambda}\,\beta(z,s)}dsdz.$$

Let us first estimate $B_1(x)$. Using (2.14), we get

$$
\begin{aligned}
B_1(x) &= M(a,\sigma)\lambda^{\alpha+1}\int_0^x\int_0^z e^{-a\sqrt{\sigma+\lambda}\,(x-z)}(z-s)e^{-a\sqrt{\sigma+\lambda}\,(z-s)}dsdz\\
&\leqslant M_1(a,\sigma)\frac{\lambda^{\alpha+1}}{(\sigma+\lambda)^{\frac{3}{2}}}.
\end{aligned}
$$

Since

$$\frac{\lambda^{\alpha+1}\sigma^{\frac{1}{2}-\alpha}}{(\sigma+\lambda)^{\frac{3}{2}}}\leqslant 1 \tag{3.12}$$

for any $\alpha\in[0,\frac{1}{2}]$, we have

$$B_1(x)\leqslant M_2(a,\sigma). \tag{3.13}$$

By (2.14) and (2.15), we get

$$\begin{aligned} B_2(x) &= M(a,\sigma)\lambda^{\alpha+1}\int_0^x\int_z^{2\pi} e^{-a\sqrt{\sigma+\lambda}\,(x-z)}(s-z)e^{-a\sqrt{\sigma+\lambda}\,(s-z)}dsdz \\ &\leqslant M_3(a,\sigma)\frac{\lambda^{\alpha+1}}{(\sigma+\lambda)^{\frac{3}{2}}}. \end{aligned}$$

Using (3.12), we have

$$B_2(x) \leq M_4(a,\sigma). \tag{3.14}$$

From (2.14) and (2.15) it follows

$$\begin{aligned} B_3(x) &= M(a,\sigma)\lambda^{\alpha+1}\int_x^{2\pi}\int_0^z e^{-a\sqrt{\sigma+\lambda}\,(z-x)}(z-s)e^{-a\sqrt{\sigma+\lambda}\,(z-s)}dsdz \\ &\leqslant M_5(a,\sigma)\frac{\lambda^{\alpha+1}}{(\sigma+\lambda)^{\frac{3}{2}}}. \end{aligned}$$

By using (3.12), we obtain

$$B_3(x) \leq M_6(a,\sigma). \tag{3.15}$$

From (2.15) it follows

$$\begin{aligned} B_4(x) &= M(a,\sigma)\lambda^{\alpha+1}\int_x^{2\pi}\int_z^{2\pi} e^{-a\sqrt{\sigma+\lambda}\,(z-x)}(s-z)e^{-a\sqrt{\sigma+\lambda}\,(s-z)}dsdz. \\ &\leqslant M_7(a,\sigma)\frac{\lambda^{\alpha+1}}{(\sigma+\lambda)^{\frac{3}{2}}} \end{aligned}$$

Using (3.12), we have

$$B_4(x) \leq M_8(a,\sigma). \tag{3.16}$$

Applying estimates (3.13)-(3.16), we get estimate (3.11). From estimates (3.10) and (3.11) it follows

$$\max_{x\in[0,2\pi]} |P_3(x)| \leqslant M_9(a,\sigma)\,\|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})}\,. \tag{3.17}$$

Estimates (3.3), (3.8) and (3.17) yield

$$\max_{x\in[0,2\pi]} \left|\lambda^{\alpha}A^x(A^x+\lambda I)^{-1}\varphi(x)\right|$$

$$\leqslant [M(\sigma)+M_9(a,\sigma)]\,\|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})} + M_2(a,\sigma)\Gamma\left(2\alpha+1\right)\|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})}$$

for any $\lambda \geqslant 0$. Thus,

$$\|\varphi\|_{E_\alpha\left(\mathring{C}(\mathbb{R}),A^x\right)} \leqslant M_{10}(a,\sigma)\,\|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})}\,. \tag{3.18}$$

Now, let us prove the reverse inequality. For any positive operator A^x in the Banach space, we can write

$$I = \int_0^\infty A^x(A^x+\lambda I)^{-2}d\lambda,$$

where I is the identity operator. From this relation and formula (2.1) it follows

$$\begin{aligned}\varphi(x) &= \int_0^\infty (A^x+\lambda I)^{-1}A^x(A^x+\lambda I)^{-1}\varphi(x)\,d\lambda \\ &= \int_0^\infty\int_0^{2\pi} G^x(x,s;\lambda)\,A^x(A^x+\lambda I)^{-1}\varphi(s)\,dsd\lambda.\end{aligned}$$

Consequently,

$$\varphi(x_1)-\varphi(x_2)=\int_0^\infty\int_0^{2\pi}\left(G^x(x_1,s;\lambda)-G^x(x_2,s;\lambda)\right)A^x(A^x+\lambda I)^{-1}\varphi(s)dsd\lambda$$

$$=\int_0^\infty \lambda^{-\alpha}\int_0^{2\pi}\left(G^x(x_1,s;\lambda)-G^x(x_2,s;\lambda)\right)\lambda^\alpha A^x(A^x+\lambda I)^{-1}\varphi(s)dsd\lambda.$$

Therefore,

$$|\varphi(x_1)-\varphi(x_2)|\leqslant\int_0^\infty\lambda^{-\alpha}\int_0^{2\pi}|G^x(x_1,s;\lambda)-G^x(x_2,s;\lambda)|\,dsd\lambda\,\|\varphi\|_{E_\alpha(\mathring{C}(\mathbb{R}^1),A^x)}\,.$$

Let

$$T=|x_1-x_2|^{-2\alpha}\left[\int_0^\infty\lambda^{-\alpha}\int_0^{2\pi}|G^x(x_1,s;\lambda)-G^x(x_2,s;\lambda)|\,dsd\lambda\right].$$

Note that for $|x_1-x_2|\geqslant\pi$ we have that

$$\frac{|\varphi(x_1)-\varphi(x_2)|}{|x_1-x_2|^{2\alpha}}\leqslant\frac{|\varphi(x_1)|+|\varphi(x_2)|}{\pi^{2\alpha}}\leqslant\frac{2}{\pi^{2\alpha}}\|\varphi\|_{\mathring{C}(\mathbb{R})}\leqslant M\|\varphi\|_{E_\alpha(\mathring{C}(\mathbb{R}),A^x)}\,.$$

Therefore, no loss of generality we can put $x_2>x_1$ and x_2-x_1 is the number small than π. For any $x_1,x_2\in[0,2\pi]$ such that $x_2>x_1$, we have

$$\frac{|\varphi(x_1)-\varphi(x_2)|}{|x_1-x_2|^{2\alpha}}\leqslant T\|\varphi\|_{E_\alpha(\mathring{C}(\mathbb{R}),A^x)}\,.$$

Now, we will prove that

$$T\leqslant\frac{M(\sigma)}{2\alpha(1-2\alpha)}. \tag{3.19}$$

We have

$$T=|x_1-x_2|^{-2\alpha}\int_0^\infty\lambda^{-\alpha}\left[\int_0^{x_1}|G^x(x_1,s;\lambda)-G^x(x_2,s;\lambda)|\,ds\right.$$

$$\left.+\int_{x_1}^{x_2}|G^x(x_1,s;\lambda)-G^x(x_2,s;\lambda)|\,ds+\int_{x_2}^{2\pi}|G^x(x_1,s;\lambda)-G^x(x_2,s;\lambda)|\,ds\right]$$

$$=T_1+T_2+T_3,$$

where

$$T_1 = |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_0^{x_1} |G^x(x_1, s; \lambda) - G^x(x_2, s; \lambda)|\, ds\, d\lambda,$$

$$T_2 = |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_{x_1}^{x_2} |G^x(x_1, s; \lambda) - G^x(x_2, s; \lambda)|\, ds\, d\lambda,$$

$$T_3 = |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_{x_2}^{2\pi} |G^x(x_1, s; \lambda) - G^x(x_2, s; \lambda)|\, ds\, d\lambda,$$

Let us estimate T_1. Using estimate (2.38), we get

$$T_1 \leqslant |x_1 - x_2|^{-2\alpha} \int_0^\infty \lambda^{-\alpha} \int_0^{x_1} \int_0^{x_2 - x_1} |G_z^x(z + x_1, s; \lambda)|\, dz\, ds\, d\lambda$$

$$\leqslant M(a, \sigma)\, |x_1 - x_2|^{-2\alpha} \int_0^{x_1} \int_0^{x_2 - x_1} \int_0^\infty \lambda^{-\alpha} e^{-a\sqrt{\lambda}\,(z + x_1 - s)} d\lambda\, dz\, ds$$

Using the substitution $p = a\sqrt{\lambda}\,(z + x_1 - s)$ and identities

$$\int_0^\infty \lambda^{-\alpha} e^{-a\sqrt{\lambda}\,(z + x_1 - s)} d\lambda = 2a^{2\alpha - 2}\,(z + x_1 - s)^{2\alpha - 2}\, \Gamma(2 - 2\alpha),$$

$$\int_0^{x_1} \int_0^{x_2 - x_1} (z + x_1 - s)^{2\alpha - 2}\, dzds = \frac{\left[(x_2 - x_1)^{2\alpha} + x_1^{2\alpha} - x_2^{2\alpha}\right]}{2\alpha\,(1 - 2\alpha)},$$

we obtain

$$T_1 \leqslant \frac{M_1(a, \sigma)}{2\alpha\,(1 - 2\alpha)}. \tag{3.20}$$

Now, let us estimate T_2. Using the triangle inequality and estimate (2.37), we get

$$T_2 \leqslant M_1(a, \sigma)\, |x_1 - x_2|^{-2\alpha} \int_{x_1}^{x_2} \int_0^\infty \lambda^{-\alpha - \frac{1}{2}} \left[e^{-a\sqrt{\lambda}\,(s - x_1)} + e^{-a\sqrt{\lambda}\,(x_2 - s)}\right] d\lambda ds$$

$$\leqslant M_2(a, \sigma) 4\Gamma(1 - 2\alpha).$$

From this it follows that

$$T_2 \leq M_3(a, \sigma). \tag{3.21}$$

Finally, we will estimate T_3. By estimate (2.38), we obtain

$$T_3 \leqslant M(a, \sigma)\, |x_1 - x_2|^{-2\alpha} \int_{x_2}^{2\pi} \int_0^{x_2 - x_1} \int_0^\infty \lambda^{-\alpha} e^{-a\sqrt{\lambda}\,(s - z - x_1)} d\lambda dz ds.$$

Using the substitution $p = a\sqrt{\lambda}\,(s-z-x_1)$, and identities

$$\int_0^\infty \lambda^{-\alpha} e^{-a\sqrt{\lambda}(s-z-x_1)} d\lambda = 2a^{2\alpha-2}(s-z-x_1)^{2\alpha-2}\Gamma(2-2\alpha),$$

$$\int_{x_2}^{2\pi}\int_0^{x_2-x_1}(s-z-x_1)^{2\alpha-2}dzds \leqslant \frac{(x_2-x_1)^{2\alpha}}{2\alpha(1-2\alpha)},$$

we have

$$T_3 \leqslant M(a,\sigma)\,(x_2-x_1)^{-2\alpha}\, 2\Gamma(2-2\alpha)a^{2\alpha-2}\frac{(x_2-x_1)^{2\alpha}}{2\alpha\,(1-2\alpha)}. \tag{3.22}$$

From estimate (3.22) it follows

$$T_3 \leqslant \frac{M_4(a,\sigma)}{2\alpha(1-2\alpha)}. \tag{3.23}$$

Combining estimates (3.20)-(3.23), we conclude for $0 \leqslant x_1 < x_2 \leqslant 2\pi$ and $x_2 - x_1 < \pi$ that

$$T \leqslant \frac{M(a,\sigma)}{2\alpha(1-2\alpha)}.$$

Thus, (3.19) is proved. Thus, for any $x_1, x_2 \in [0, 2\pi]$ we have

$$|x_1-x_2|^{-2\alpha}\,|\varphi(x_1)-\varphi(x_2)| \leqslant \frac{M(a,\sigma)}{2\alpha(1-2\alpha)}\,\|\varphi\|_{E_\alpha(\mathring{C}(\mathbb{R}),A^x)}\,.$$

This means that the following inequality holds:

$$\|\varphi\|_{\mathring{C}^{2\alpha}(\mathbb{R})} \leqslant \frac{M(\sigma,a)}{2\alpha(1-2\alpha)}\,\|\varphi\|_{E_\alpha(\mathring{C}(\mathbb{R}),A^x).} \tag{3.24}$$

Estimates (3.18) and (3.24) finish the proof of Theorem 3.2. □

Since A^x is a positive operator in the fractional space $E_\alpha(\mathring{C}\,(\mathbb{R})\,, A^x)$, from the result of Theorem 3.2 it follows also that it is positive operator in the Hölder space $\mathring{C}^{2\alpha}\,(\mathbb{R})\,.$

4. Applications

Now, we will consider the applications of Theorems 3.1, 3.2. First, we consider the boundary value problem

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(t,x)}{\partial x}\right) + \sigma u(t,x) = f(t,x), 0<t<T, x\in\mathbb{R}, \\ u(0,x) = \varphi(x), u(T,x) = \psi(x), x\in\mathbb{R}, \\ u(t,x) = u(t,x+2\pi), \int_0^{2\pi} u(t,s)ds = 0, 0\leqslant t\leqslant T, x\in\mathbb{R}. \end{cases} \tag{4.1}$$

Here, $\varphi(x), \psi(x), a(x)$ and $f(t,x)$ are sufficiently smooth 2π-periodic functions in x and they satisfy any compatibility conditions which guarantee problem (4.1) has a smooth solution $u(t,x)$.

Theorem 4.1. *Let $0 < \alpha < \frac{1}{2}$. Then, for the solution of the boundary value problem (4.1), we have the following coercive stability inequality*

$$\|u_{tt}\|_{C\left([0,T],\mathring{C}^{2\alpha}(\mathbb{R})\right)} + \|u\|_{C\left([0,T],\mathring{C}^{2\alpha+2}(\mathbb{R})\right)}$$

$$\leqslant M(\alpha)\left[\|\varphi\|_{\mathring{C}^{2\alpha+2}(\mathbb{R})} + \|\psi\|_{\mathring{C}^{2\alpha+2}(\mathbb{R})} + \|f\|_{C\left([0,T],\mathring{C}^{2\alpha}(\mathbb{R})\right)}\right].$$

The proof of Theorem 4.1 is based on Theorem 3.2 on the structure of the fractional spaces $E_\alpha = E_\alpha(C(\mathbb{R}), A^x)$, Theorem 3.1 on the positivity of the operator A^x, on the following theorems on coercive stability of boundary value problem for the abstract elliptic equation and on the structure of the fractional space $E'_\alpha = E_\alpha(E, A^{1/2})$ which is the Banach space consisting of those $v \in E$ for which the norm

$$\|v\|_{E'_\alpha} = \sup_{\lambda>0} \lambda^\alpha \left\|A^{1/2}(\lambda + A^{1/2})^{-1}v\right\|_{E_\alpha} + \|v\|_E$$

is finite.

Theorem 4.2. [11] *The spaces $E_\alpha(E, A)$ and $E'_{2\alpha}(E, A^{1/2})$ coincide for any $0 < \alpha < \frac{1}{2}$, and their norms are equivalent.*

Theorem 4.3. [11] *Let $f \in C([0,T], E'_\alpha), 0 < \alpha < 1$. Then, for the solution of the nonlocal boundary value problem*

$$-u'' + Au(t) = f(t), 0 < t < T, u(0) = \varphi, u(T) = \psi \tag{4.2}$$

in a Banach space E with positive operator A the coercive inequality holds:

$$\|u''\|_{C([0,T],E'_\alpha)} + \|Au\|_{C([0,T],E'_\alpha)}$$

$$\leqslant M\left[\|A\varphi\|_{E'_\alpha} + \|A\psi\|_{E'_\alpha} + \frac{M}{\alpha(1-\alpha)}\|f\|_{C([0,T],E'_\alpha)}\right].$$

Second, we consider the nonlocal boundary value problem for the elliptic equation

$$\begin{cases} -\frac{\partial^2 u(t,x)}{\partial t^2} - \frac{\partial}{\partial x}\left(a(x)\frac{\partial u(t,x)}{\partial x}\right) + \sigma u(t,x) = f(t,x), 0 < t < T, x \in \mathbb{R}, \\ u(0,x) = u(T,x), u_t(0,x) = u_t(T,x), x \in \mathbb{R}, \\ u(t,x) = u(t,x+2\pi), \int\limits_0^{2\pi} u(t,s)ds = 0, 0 \leqslant t \leqslant T, x \in \mathbb{R}. \end{cases} \tag{4.3}$$

Here, $a(x)$ and $f(t,x)$ are sufficiently smooth 2π-periodic functions in x and they satisfy any compatibility conditions which guarantee problem (4.3) has a smooth solution $u(t,x)$.

Theorem 4.4. *Let $0 < \alpha < \frac{1}{2}$. Then, for the solution of boundary value problem (4.3), we have the following coercive stability inequality*

$$\|u_{tt}\|_{C\left([0,T],\mathring{C}^{2\alpha}(\mathbb{R})\right)} + \|u\|_{C\left([0,T],\mathring{C}^{2\alpha+2}(\mathbb{R})\right)} \leqslant M(\alpha)\|f\|_{C\left([0,T],\mathring{C}^{2\alpha}(\mathbb{R})\right)}.$$

The proof of Theorem 4.4 is based on Theorem 3.2 on the structure of the fractional spaces $E_\alpha = E_\alpha(\overset{\circ}{C}(\mathbb{R}), A^x)$, Theorem 3.1 on the positivity of the operator A^x, Theorem 4.2 on the structure of the fractional space $E'_\alpha = E_\alpha(E, A^{1/2})$ and on the following theorem on coercive stability of the nonlocal boundary value problem for the abstract elliptic equation.

Theorem 4.5. [12] *Let* $f \in C([0,T], E'_\alpha)$, $0 < \alpha < 1$. *Then, for the solution of the nonlocal boundary value problem*

$$-u'' + Au(t) = f(t), 0 < t < T, u(0) = u(T), u'(0) = u'(T) \tag{4.4}$$

in a Banach space E *with positive operator* A, *the coercive inequality*

$$\left\| u'' \right\|_{C([0,T],E'_\alpha)} + \left\| Au \right\|_{C([0,T],E'_\alpha)} \leqslant \frac{M}{\alpha(1-\alpha)} \left\| f \right\|_{C([0,T],E'_\alpha)}$$

holds.

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