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AN APPLICATION OF THE CAUCHY-SCHWARZ INEQUALITY FOR THE BEREZIN RADIUS IN RKHS

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Abstract. In this study, we present an extension of the Cauchy-Schwarz inequality based on the angle between vectors, and utilize it to establish several new inequalities involving the Berezin radius of bounded linear operators on a reproducing kernel Hilbert space. These results generalize and improve the existing inequalities in the literature.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the C^*-algebra, of all bounded linear operators that act on a nontrivial complex Hilbert space $\mathcal{H}$ with the inner product $\langle\cdot,\cdot\rangle$ and its associated norm $\|\cdot\|$. For $A\in\mathcal{B}(\mathcal{H})$, the symbol A^* denotes the adjoint of A, and $|A|=(A^*A)^{\frac{1}{2}}$. Let $A=U|A|$ be the polar decomposition of A, where $U\in\mathcal{B}(\mathcal{H})$ is a partial isometry. Recall that the operator norm for $A\in\mathcal{B}(\mathcal{H})$ is

$$\|A\|:=\sup\left\{|\langle Ax,y\rangle| : x,y\in\mathcal{H}, \|x\|=\|y\|=1\right\},$$

whereas the numerical radius is given by

$$\omega(A):=\sup\left\{|\langle Ax,x\rangle| : x\in\mathcal{H}, \|x\|=1\right\}.$$

It is well-known that $\omega(\cdot)$ is a norm on $\mathcal{B}(\mathcal{H})$, see [8].

Let Θ be a nonempty set. A functional Hilbert space $\mathcal{H}=\mathcal{H}(\Theta)$ is a Hilbert space containing complex-valued functions on a set Θ. This space has the property that for each $\tau\in\Theta$, the evaluation map $g\mapsto g(\tau)$ is continuous linear functional on $\mathcal{H}$. By the Riesz representation theorem for every $\tau\in\Theta$, there exists a unique vector $k_\tau\in\mathcal{H}$ such that $g(\tau)=\langle g,k_\tau\rangle$ for all $g\in\mathcal{H}$. The set $\{k_\tau:\tau\in\Theta\}$ is known as the reproducing kernel of the space $\mathcal{H}$. If $\{\phi_n\}_{n=0}^{\infty}$ forms an orthonormal basis for $\mathcal{H}$, then the reproducing kernel can be expressed as

$$k_\tau(z)=\sum_{n=0}^{\infty}\overline{\phi_n(\tau)}\phi_n(z),$$

(see [14]). For each $\tau\in\Theta$, define the normalized kernel

$$\hat{k}_\tau=\frac{k_\tau}{\|k_\tau\|}.$$

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Let A be a bounded linear operator on $\mathcal{H}$. Its Berezin symbol, originally introduced by Berezin [7], is a function $\widetilde{A}$ on Θ defined as

$$\widetilde{A}(\tau) := \langle A\hat{k}_\tau, \hat{k}_\tau \rangle.$$

The Berezin set and the Berezin radius(number) associated with A are respectively given by

$$\mathbf{Ber}(A) := \left\{ \langle A\hat{k}_\tau, \hat{k}_\tau \rangle : \tau \in \Theta \right\},$$

and

$$\mathbf{ber}(A) := \sup_{\tau \in \Theta} |\langle A\hat{k}_\tau, \hat{k}_\tau \rangle|.$$

It is apparent that $\widetilde{A}$ is a bounded function on Θ, whose values are contained within the numerical range $W(A)$, and hence for all $A \in \mathcal{B}(\mathcal{H})$,

$$\mathbf{Ber}(A) \subseteq W(A) \quad \text{and} \quad \mathbf{ber}(A) \leq \omega(A).$$

Therefore, the Berezin number has relations with both the numerical radius and the operator norm.

Further properties of the Berezin number of an operator A, discussed in [16], include

(1) $\mathbf{ber}(A) \leq \|A\|$;
(2) $\mathbf{ber}(\alpha A) = |\alpha| \, \mathbf{ber}(A)$ for any $\alpha \in \mathbb{C}$;
(3) $\mathbf{ber}(A + B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$ for all $A, B \in \mathcal{B}(\mathcal{H})$.

It is important to note that, in general, $\mathbf{ber}(\cdot)$ does not define a norm. However, when $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions, on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, then the mapping $\mathbf{ber}(\cdot)$ induces a norm on $\mathcal{B}(\mathcal{H}(\mathbb{D}))$ (see [13, 16]).

An essential attribute of the Berezin symbol is its uniqueness: if $\widetilde{A}(\tau) = \widetilde{B}(\tau)$ for every $\tau \in \Theta$, then $A = B$. This indicates that the Berezin symbol provides a one-to-one mapping between operators and functions. For more details in this area, the reader is encouraged to consult [2, 3, 4, 5, 6, 9, 10, 11, 18, 22]. There exists an analogous relation between the Berezin number and the Berezin norm, which is defined as

$$\|A\|_{\mathbf{ber}} := \sup \left\{ |\langle A\hat{k}_\mu, \hat{k}_\nu \rangle| : \mu, \nu \in \Theta \right\}.$$

From the definition, the Berezin norm satisfies the following properties:

(1) $\mathbf{ber}(A) \leq \|A\|_{\mathbf{ber}}$;
(2) $\|A\|_{\mathbf{ber}} \leq \|A\|$;
(3) $\|A^*\|_{\mathbf{ber}} = \|A\|_{\mathbf{ber}}$.

In [15], Huban et al. and in [14] Hajmohamadi et al. showed that, if $A \in \mathcal{B}(\mathcal{H})$

$$\mathbf{ber}^2(A) \leq \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\|_{\mathbf{ber}}. \tag{1.1}$$

In [1], W. Audeh and M. Al-Labadi, presented some inequalities for numerical radius inequalities for finite sums of operators. Motivated by this article, M. Gurdal and V. Stojiljkovic gave some generalizing inequalities of the inequality (1.1) states that: Let $A_i, B_i \in \mathcal{B}(\mathcal{H})$ $(i = 1, 2, ..., n)$ and f, g are non-negative continuous functions defined on the interval $[0,\infty)$ that satisfy the condition $f(t)\,g(t) = t$ for all $t \in [0,\infty)$ and $a_i \geq 0$, $\sum_{i=1}^{n} a_i = 1$, then

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} a_i(A_i + B_i)\right) \leq 2^{r-2}\,\|Z\|_{\mathbf{ber}} \quad \text{for all } r \geq 2, \tag{1.2}$$

where

$$Z = \sum_{i=1}^{n} a_i\left(f^{2r}(|A_i|) + f^{2r}(|B_i|) + g^{2r}(|A_i^*|) + g^{2r}(|B_i^*|)\right).$$

Considering special cases, they obtained the following results:

(1) If setting $A_i = B_i = 0$, $a_1 = 1$ and $a_i = 0$ for all $i \geq 2$, then

$$\mathbf{ber}^r(A+B) \leq 2^{r-2}\left\|\left(f^{2r}(|A|) + f^{2r}(|B|) + g^{2r}(|A^*|) + g^{2r}(|B^*|)\right)\right\|_{\mathbf{ber}} \tag{1.3}$$

for all $r \geq 2$;

(2) If setting $A_1 = B_1 = A$, $A_i = B_i = 0$, $a_1 = 1$ and $a_i = 0$ for all $i \geq 2$ and $f(t) = g(t) = t^{\frac{1}{2}}$, then

$$\mathbf{ber}^r(A) \leq \frac{1}{2}\,\||A|^r + |A^*|^r\|_{\mathbf{ber}} \quad \text{for all } r \geq 2. \tag{1.4}$$

Also, they proved one generalization of the inequality (1.1) as follows:
Assume $A \in \mathcal{B}(\mathcal{H})$. Then

$$\mathbf{ber}^r(A+B) \leq \frac{1}{2^{3-r}}\,\|Z\|_{\mathbf{ber}} \quad \text{for all } r \geq 2, \tag{1.5}$$

where

$$Z = f^{2r}(|A+B|) + f^{2r}(|A-B|) + g^{2r}(|(A+B)^*|) + g^{2r}(|(A-B)^*|).$$

In particular for the case $A = B$ and $f(t) = g(t) = t^{\frac{1}{2}}$, they gave the following inequality:

$$\mathbf{ber}^r(A) \leq \frac{1}{2^{3-r}}\,\||A|^r + |A^*|^r\|_{\mathbf{ber}} \quad \text{for all } r \geq 2. \tag{1.6}$$

On the other work, Huban et al. [15] proved the following inequality:

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* C_i B_i\right) \leq \frac{n^{r-1}}{\sqrt{2}}\mathbf{ber}\left(\sum_{i=1}^{n}\left(\left[B_i^* f^2(|C_i|)\,B_i\right]^r + i\left[A_i^* g^2(|C_i^*|)\,A_i\right]^r\right)\right). \tag{1.7}$$

By substituting $A_i = B_i = I$, $n = 1$, $r = 1$ and $f(t) = g(t) = t^{\frac{1}{2}}$ into the above inequality, they obtained the following inequality, which represents an improvement of the inequality (1.1):

$$\mathbf{ber}^2(A) \leq \frac{1}{2}\mathbf{ber}^2\left(|A|^2 + i|A^*|^2\right) \leq \frac{1}{2}\left\||A|^2 + i|A^*|^2\right\|_{\mathbf{ber}}. \tag{1.8}$$

For any vectors x and y in an inner product space, the celebrated Cauchy-Schwarz inequality asserts that

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|.$$

This inequality allows us to define the angle between two non-zero vectors x and y as

$$\angle(x,y) = \cos^{-1}\left(\frac{|\langle x, y\rangle|}{\|x\| \cdot \|y\|}\right).$$

In [21, Theorem 2.3], Sababheh et al. established the following refinement: Let $A \in \mathcal{B}(\mathcal{H})$ be a bounded linear operator on a Hilbert space $\mathcal{H}$ with the polar decomposition $A = U|A|$, and let $x, y \in \mathcal{H}$. Then for any $\alpha \in [0,1]$, the inequality

$$|\langle Ax, y\rangle| \le \mu(\theta_{A,x,y})\sqrt{\langle |A|^{2\alpha}x, x\rangle \cdot \langle |A^*|^{2(1-\alpha)}y, y\rangle}$$

holds, where $\theta_{A,x,y} = \angle(|A|^{\alpha}x, |A|^{1-\alpha}U^*y)$, and the function μ is defined by

$$\mu(\theta) := \frac{1}{4}\left(2 + \cos\theta\cot\theta\log\left(\frac{1+\sin\theta}{1-\sin\theta}\right)\right).$$

The domain of μ excludes points $\theta = n\pi$ for $n \in \mathbb{Z}$, but since $\lim_{\theta\to n\pi}\mu(\theta) = 1$, we extend the definition by setting $\mu(n\pi) := 1$. Furthermore, it is shown that μ is monotonically decreasing on $[0, \frac{\pi}{2}]$, and increasing on $[\frac{\pi}{2}, \pi]$, with the bounds

$$\frac{1}{2} \le \mu(\theta) \le 1 \quad \text{for all } \theta \ge 0.$$

For further reading, see [17, 20].

In this paper, we present an extension of the Cauchy-Schwarz inequality in terms of the angle between vectors. Also, by using this extension we present new extensions and sharper bounds for Berezin-type inequalities concerning bounded linear operators acting on reproducing kernel Hilbert spaces.

2. Results

In this section, we present an extension of the Cauchy-Schwarz inequality in terms of the angle between vectors. Moreover, by using this extension, we give some general bounds for certain inequalities related to the Berezin number.

We begin by recalling the following well-known lemmas, which play a fundamental role in the subsequent analysis, and are crucial to establish our main results.

Lemma 2.1. *Let a, b be two real numbers and $r \ge 2$. Then*

$$|a+b|^r + |a-b|^r \ge 2(|a|^r + |b|^r).$$

Lemma 2.2. *(Minkowski's inequality) Let $a_i, b_i > 0$ for $i = 1, 2, ..., n$, and suppose that $r > 1$. Then*

$$\left(\sum_{i=1}^{n}(a_i+b_i)^r\right)^{\frac{1}{r}} \le \left(\sum_{i=1}^{n}a_i^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^{n}b_i^r\right)^{\frac{1}{r}}.$$

Lemma 2.3. *Let $a, b \ge 0$ and $r \ge 1$. Then*

$$a^r + b^r \le (a+b)^r \le 2^{r-1}(a^r + b^r).$$

Lemma 2.4. *Let a_i be a positive real number $(i = 1, 2, ..., n)$. Then for all $r \ge 1$*

$$\left(\sum_{i=1}^{n}a_i\right)^r \le n^{r-1}\sum_{i=1}^{n}a_i^r.$$

Lemma 2.5. [19] *Suppose that $A \in \mathcal{B}(\mathcal{H})$ is positive and $x \in \mathcal{H}$ is a norm one vector. Then*

$$\langle Ax, x\rangle^r \le \langle A^r x, x\rangle \quad \textit{for all } r \ge 1.$$

Lemma 2.6. [3, 20] *Assume* $A \in \mathcal{B}(\mathcal{H})$ *have the polar decomposition* $A = U|A|$, *and* f, g *are non-negative continuous functions defined on the interval* $[0, \infty)$ *that satisfy the condition* $f(t)\, g(t) = t$ *for all* $t \in [0, \infty)$. *Then*

$$\left|\langle A\hat{k}_\lambda, \hat{k}_\zeta\rangle\right| \leq \mu\left(\theta_{A,\hat{k}_\lambda,\hat{k}_\zeta}\right)\sqrt{\langle f^2(|A|)\,\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle g^2(|A^*|)\,\hat{k}_\zeta, \hat{k}_\zeta\rangle},$$

where $\hat{k}_\lambda, \hat{k}_\zeta \in \mathcal{H}$ *and* $\theta_{A,\hat{k}_\lambda,\hat{k}_\zeta} = \angle_{f(|A|)\hat{k}_\lambda, g(|A|)U^*\hat{k}_\zeta}$.

Theorem 2.1. *Let* $\mathcal{H} = \mathcal{H}(\Theta)$, $A_i, B_i \in \mathcal{B}(\mathcal{H})$ *with the polar decompositions* $A_i = U_i|A_i|$, $B_i = V_i|B_i|$ $(i = 1, 2, ..., n)$, *and let* f, g *be as in Lemma 2.6,* $\theta_{A_i,\hat{k}_\lambda} = \angle_{f(|A_i|)\hat{k}_\lambda, g(|A_i|)U_i^*\hat{k}_\lambda}$, $\theta_{B_i,\hat{k}_\lambda} = \angle_{f(|B_i|)\hat{k}_\lambda, g(|B_i|)V_i^*\hat{k}_\lambda}$, $a_i \geq 0$ $(i = 1, 2, ..., n)$ *with* $\sum_{i=1}^n a_i = 1$. *If either* $0 \leq \theta_i^A < \theta_{A_i,\hat{k}_\lambda} \leq \frac{\pi}{2}$, $0 \leq \theta_i^B < \theta_{B_i,\hat{k}_\lambda} \leq \frac{\pi}{2}$ *for all* $\lambda \in \Theta$ *or* $\frac{\pi}{2} \leq \theta_{A_i,\hat{k}_\lambda} < \theta_i^A \leq \pi$, $\frac{\pi}{2} \leq \theta_{B_i,\hat{k}_\lambda} < \theta_i^B \leq \pi$ *for all* $\lambda \in \Theta$ $(i = 1, 2, ..., n)$, *then for any* $r \geq 2$

$$\begin{aligned}
&\mathbf{ber}^r\left(\sum_{i=1}^n a_i(A_i + B_i)\right)\\
&\leq 2^{r-2}\mu^r(\theta)\left\|\sum_{i=1}^n a_i\left(f^{2r}(|A_i|) + f^{2r}(|B_i|) + g^{2r}(|A_i^*|) + g^{2r}(|B_i^*|)\right)\right\|_{\mathbf{ber}},
\end{aligned}$$

where $\theta' = \min_{1\leq i\leq n}\{\theta_i^A, \theta_i^B\}$, $\theta'' = \max_{1\leq i\leq n}\{\theta_i^A, \theta_i^B\}$, *and*

$$\mu(\theta) = \max\{\mu(\theta'), \mu(\theta'')\}.$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Employing the triangle inequality and the Minkowski inequality, we get

$$\begin{aligned}
&\left|\langle\sum_{i=1}^n a_i(A_i + B_i)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^r\\
&\leq \sum_{i=1}^n a_i\left|\left\langle(A_i + B_i)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \text{ (by the convexity of } f(t) = t^r)\\
&\leq \sum_{i=1}^n\left(a_i^{\frac{1}{r}}\left|\left\langle A_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| + a_i^{\frac{1}{r}}\left|\left\langle B_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\right)^r\\
&\leq \left[\left(\sum_{i=1}^n a_i\left|\left\langle A_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r\right)^{\frac{1}{r}} + \left(\sum_{i=1}^n a_i\left|\left\langle B_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r\right)^{\frac{1}{r}}\right]^r.
\end{aligned}$$

Utilizing Lemma 2.6, the arithmetic-geometric mean inequality, and Lemma 2.5, we deduce that

$$\begin{aligned}
&\left|\langle\sum_{i=1}^{n} a_i(A_i+B_i)\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|^r\\
&\le\left[\left(\sum_{i=1}^{n} a_i\left|\left\langle A_i\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\right)^{\frac{1}{r}}+\left(\sum_{i=1}^{n} a_i\left|\left\langle B_i\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\right)^{\frac{1}{r}}\right]^r\\
&\le 2^{r-1}\left[\sum_{i=1}^{n} a_i\left|\left\langle A_i\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r+\sum_{i=1}^{n} a_i\left|\left\langle B_i\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\right]\\
&\le 2^{r-1}\left[\sum_{i=1}^{n} a_i\mu^r\left(\theta_{A_i,\hat{k}_\lambda}\right)\langle f^2\left(|A_i|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{r}{2}}\langle g^2\left(|A_i^*|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{r}{2}}\right.\\
&\quad\left.+\sum_{i=1}^{n} a_i\mu^r\left(\theta_{B_i,\hat{k}_\lambda}\right)\langle f^2\left(|B_i|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{r}{2}}\langle g^2\left(|B_i^*|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle^{\frac{r}{2}}\right]\\
&\le 2^{r-2}\left[\sum_{i=1}^{n} a_i\mu^r\left(\theta_{A_i,\hat{k}_\lambda}\right)\left(\langle f^2\left(|A_i|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle^r+\langle g^2\left(|A_i^*|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle^r\right)\right.\\
&\quad\left.+\sum_{i=1}^{n} a_i\mu^r\left(\theta_{B_i,\hat{k}_\lambda}\right)\left(\langle f^2\left(|B_i|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle^r+\langle g^2\left(|B_i^*|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle^r\right)\right].
\end{aligned}$$

Now, we have two cases

(i) If $0\le\theta_i^A<\theta_{A_i,\hat{k}_\lambda}\le\frac{\pi}{2}$, $0\le\theta_i^B<\theta_{B_i,\hat{k}_\lambda}\le\frac{\pi}{2}$ for all $\lambda\in\Theta$ $(i=1,2,...,n)$, considering $\theta'=\min\{\theta_i^A,\theta_i^B\}$, we obtain

$$\begin{aligned}
&\left|\langle\sum_{i=1}^{n} a_i(A_i+B_i)\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|^r\\
&\le 2^{r-2}\left[\sum_{i=1}^{n} a_i\mu^r\left(\theta'\right)\left(\langle f^{2r}\left(|A_i|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle g^{2r}\left(|A_i^*|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle\right)\right.\\
&\quad\left.+\sum_{i=1}^{n} a_i\mu^r\left(\theta'\right)\left(\langle f^{2r}\left(|B_i|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle g^{2r}\left(|B_i^*|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle\right)\right]\\
&\le 2^{r-2}\left[\sum_{i=1}^{n} a_i\mu^r\left(\theta'\right)\left(\langle f^{2r}\left(|A_i|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle g^{2r}\left(|A_i^*|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle\right.\right.\\
&\quad\left.\left.+\langle f^{2r}\left(|B_i|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle g^{2r}\left(|B_i^*|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle\right)\right]\\
&\le 2^{r-2}\mu^r\left(\theta'\right)\left[\sum_{i=1}^{n} a_i\langle f^{2r}\left(|A_i|\right)+g^{2r}\left(|A_i^*|\right)+f^{2r}\left(|B_i|\right)+g^{2r}\left(|B_i^*|\right)\hat{k}_\lambda,\hat{k}_\lambda\rangle\right].
\end{aligned}$$

(ii) If $\frac{\pi}{2} \leq \theta_{A_i,\hat{k}_\lambda} < \theta_i^A \leq \pi$, $\frac{\pi}{2} \leq \theta_{B_i,\hat{k}_\lambda} < \theta_i^B \leq \pi$ for all $\lambda \in \Theta$ $(i = 1, 2, ..., n)$, considering $\theta'' = \max\{\theta_i^A, \theta_i^B\}$, we obtain

$$\begin{aligned}
&\left|\langle\sum_{i=1}^n a_i(A_i + B_i)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^r \\
&\leq 2^{r-2}\left[\sum_{i=1}^n a_i\mu^r\left(\theta''\right)\left(\langle f^{2r}\left(|A_i|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle g^{2r}\left(|A_i^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)\right. \\
&\quad \left. + \sum_{i=1}^n a_i\mu^r\left(\theta''\right)\left(\langle f^{2r}\left(|B_i|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle g^{2r}\left(|B_i^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)\right] \\
&\leq 2^{r-2}\left[\sum_{i=1}^n a_i\mu^r\left(\theta''\right)\left(\langle f^{2r}\left(|A_i|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle g^{2r}\left(|A_i^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right.\right. \\
&\quad \left.\left. + \langle f^{2r}\left(|B_i|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle g^{2r}\left(|B_i^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right)\right] \\
&\leq 2^{r-2}\mu^r\left(\theta''\right)\left[\sum_{i=1}^n a_i\langle f^{2r}\left(|A_i|\right) + g^{2r}\left(|A_i^*|\right) + f^{2r}\left(|B_i|\right) + g^{2r}\left(|B_i^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right].
\end{aligned}$$

Since $\frac{1}{2} \leq \mu(\theta'), \mu(\theta'') \leq 1$, and considering $\mu(\theta) = \max\{\mu(\theta'), \mu(\theta'')\}$, we have

$$\begin{aligned}
&\left|\langle\sum_{i=1}^n a_i(A_i + B_i)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^r \\
&\leq 2^{r-2}\mu^r\left(\theta\right)\left[\sum_{i=1}^n a_i\langle f^{2r}\left(|A_i|\right) + g^{2r}\left(|A_i^*|\right) + f^{2r}\left(|B_i|\right) + g^{2r}\left(|B_i^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right].
\end{aligned}$$

Therefore, taking the supremum over all $\lambda \in \Theta$, we get the desired bound as follows:

$$\begin{aligned}
&\mathbf{ber}^r\left(\sum_{i=1}^n a_i(A_i + B_i)\right) \\
&\leq 2^{r-2}\mu^r(\theta)\left\|\sum_{i=1}^n a_i\left(f^{2r}\left(|A_i|\right) + f^{2r}\left(|B_i|\right) + g^{2r}\left(|A_i^*|\right) + g^{2r}\left(|B_i^*|\right)\right)\right\|_{\mathbf{ber}}.
\end{aligned}$$

The second case is easily established in a similar way. The proof is complete. □

Remark 2.1. Since $\frac{1}{2} \leq \mu(\theta) \leq 1$ for all $\theta \geq 0$, we obtain

$$\begin{aligned}
&\mathbf{ber}^r\left(\sum_{i=1}^n a_i(A_i + B_i)\right) \\
&\leq 2^{r-2}\mu^r(\theta)\left\|\sum_{i=1}^n a_i\left(f^{2r}\left(|A_i|\right) + f^{2r}\left(|B_i|\right) + g^{2r}\left(|A_i^*|\right) + g^{2r}\left(|B_i^*|\right)\right)\right\|_{\mathbf{ber}} \\
&\leq 2^{r-2}\left\|\sum_{i=1}^n a_i\left(f^{2r}\left(|A_i|\right) + f^{2r}\left(|B_i|\right) + g^{2r}\left(|A_i^*|\right) + g^{2r}\left(|B_i^*|\right)\right)\right\|_{\mathbf{ber}}.
\end{aligned}$$

This shows that Theorem 2.1 is stronger than (1.2).

If we set $A_i = B_i = 0$, $a_1 = 1$, and $a_i = 0$ for all $i \geq 2$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.1. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ and $A, B \in \mathcal{B}(\mathcal{H})$ with the polar decompositions $A = U|A|$, $B = V|B|$. Also, let f, g be as in Lemma 2.6, $\theta_{A,\hat{k}_\lambda} = \angle_{f(|A|)\hat{k}_\lambda, g(|A|)U^*\hat{k}_\lambda}$, and $\theta_{B,\hat{k}_\lambda} = \angle_{f(|B|)\hat{k}_\lambda, g(|B|)V^*\hat{k}_\lambda}$. If either $0 \leq \theta_1^A < \theta_{A,\hat{k}_\lambda} \leq \frac{\pi}{2}$, $0 \leq \theta_1^B < \theta_{B,\hat{k}_\lambda} \leq \frac{\pi}{2}$ for all $\lambda \in \Theta$ or $\frac{\pi}{2} \leq \theta_{A,\hat{k}_\lambda} < \theta_1^A \leq \pi$, $\frac{\pi}{2} \leq \theta_{B,\hat{k}_\lambda} < \theta_1^B \leq \pi$ for all $\lambda \in \Theta$, then for any $r \geq 2$*

$$\mathbf{ber}^r(A+B) \leq 2^{r-2}\mu^r(\theta)\left\|f^{2r}(|A|) + f^{2r}(|B|) + g^{2r}(|A^*|) + g^{2r}(|B^*|)\right\|_{\mathbf{ber}},$$

where $\theta' = \min\{\theta_1^A, \theta_1^B\}$, $\theta'' = \max\{\theta_1^A, \theta_1^B\}$, and $\mu(\theta) = \max\{\mu(\theta'), \mu(\theta'')\}$.

Remark 2.2. Since $\frac{1}{2} \leq \mu(\theta) \leq 1$ for all $\theta \geq 0$, then for any $r \geq 2$

$$\begin{aligned}\mathbf{ber}^r(A+B) &\leq 2^{r-2}\mu^r(\theta)\left\|f^{2r}(|A|) + f^{2r}(|B|) + g^{2r}(|A^*|) + g^{2r}(|B^*|)\right\|_{\mathbf{ber}} \\ &\leq 2^{r-2}\left\|f^{2r}(|A|) + f^{2r}(|B|) + g^{2r}(|A^*|) + g^{2r}(|B^*|)\right\|_{\mathbf{ber}}.\end{aligned}$$

This demonstrates that Corollary 2.1 is stronger than (1.3).

If we set $A = A_1 = B_1 = B$, $a_1 = 1$ and $a_i = 0$ for all $i \geq 2$, and let $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, where $0 \leq \alpha \leq 1$ in Theorem 2.1, then we obtain the following result.

Corollary 2.2. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ and $A \in \mathcal{B}(\mathcal{H})$ with the polar decomposition $A = U|A|$. Also, let f, g be as in Lemma 2.6, and $\theta_{A,\hat{k}_\lambda} = \angle_{f(|A|)\hat{k}_\lambda, g(|A|)U^*\hat{k}_\lambda}$. If either $0 \leq \theta' < \theta_{A,\hat{k}_\lambda} \leq \frac{\pi}{2}$ for all $\lambda \in \Theta$ or $\frac{\pi}{2} \leq \theta_{A,\hat{k}_\lambda} < \theta'' \leq \pi$ for all $\lambda \in \Theta$, then for any $r \geq 2$*

$$\mathbf{ber}^r(A) \leq \frac{\mu^r(\theta)}{2}\left\|f^{2r}(|A|) + g^{2r}(|A^*|)\right\|_{\mathbf{ber}},$$

where $\mu(\theta) = \max\{\mu(\theta'), \mu(\theta'')\}$.

Remark 2.3. Consider all the assumptions outlined in Corollary 2.2. It follows from $\frac{1}{2} \leq \mu(\theta) \leq 1$ for all $\theta \geq 0$ that for any $r \geq 2$, we have

$$\mathbf{ber}^r(A) \leq \frac{\mu^r(\theta)}{2}\left\|f^{2r}(|A|) + g^{2r}(|A^*|)\right\|_{\mathbf{ber}} \leq \frac{1}{2}\left\|f^{2r}(|A|) + g^{2r}(|A^*|)\right\|_{\mathbf{ber}}.$$

By substituting $f(t) = g(t) = t^{1/2}$ into the inequality provided above, we deduce that

$$\mathbf{ber}^r(A) \leq \frac{\mu^r(\theta)}{2}\||A|^r + |A^*|^r\|_{\mathbf{ber}} \leq \frac{1}{2}\||A|^r + |A^*|^r\|_{\mathbf{ber}},$$

which for $r = 2$, we get

$$\mathbf{ber}^2(A) \leq \frac{\mu^2(\theta)}{2}\left\||A|^2 + |A^*|^2\right\|_{\mathbf{ber}} \leq \frac{1}{2}\left\||A|^2 + |A^*|^2\right\|_{\mathbf{ber}}.$$

We present an improvement of the associated inequalities (1.1) and (1.4).

Example 2.1. Let $\Theta = \{\tau_1, \tau_2\}$ be a nonempty set. Consider $\mathcal{H} = \mathbb{C}^2$ and functional Hilbert space $\mathcal{H}(\Theta)$ is a Hilbert space containing complex-valued functions on a set Θ. This space has the property that each vector $v = \begin{bmatrix} x \\ y \end{bmatrix}$ can be defined as a function g_v on Θ as follow:

$$g_v(\tau_1) = x, \quad \text{and} \quad g_v(\tau_2) = y.$$

Then

$$\hat{k}_{\tau_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \hat{k}_{\tau_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Let $A = \begin{bmatrix} 1 & 3 \\ -2 & 5 \end{bmatrix}$. For $\hat{k}_{\tau_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have

$$|A|^{\frac{1}{2}}\hat{k}_{\tau_1} = \begin{bmatrix} 1.4117 & -0.2358 \\ -0.2358 & 2.3887 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1.4117 \\ -0.2358 \end{bmatrix},$$

and

$$|A|^{\frac{1}{2}}U^*\hat{k}_{\tau_1} = \begin{bmatrix} 1.4117 & -0.2358 \\ -0.2358 & 2.3887 \end{bmatrix} \begin{bmatrix} 0.7682 & -0.6425 \\ 0.6402 & 0.7682 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.933509 \\ 1.3481 \end{bmatrix}.$$

To calculate the angle $\theta_{A,\hat{k}_{\tau_1}}$, we have the following:

$$\begin{aligned} \cos(\theta_{A,\hat{k}_{\tau_1}}) &= \frac{\langle |A|^{\frac{1}{2}}\hat{k}_{\tau_1}, |A|^{\frac{1}{2}}U^*\hat{k}_{\tau_1}\rangle}{\||A|^{\frac{1}{2}}\hat{k}_{\tau_1}\|\||A|^{\frac{1}{2}}U^*\hat{k}_{\tau_1}\|} \\ &= \frac{(1.4117)(0.933509) + (-0.2358)(1.3481)}{\sqrt{(1.4117)^2 + (-0.2358)^2}\sqrt{(0.933509)^2 + (1.3481)^2}} \approx 0.43. \end{aligned}$$

Thus, $\theta_{A,\hat{k}_{\tau_1}} = 64.53°$, and so $\mu(\theta_{A,\hat{k}_{\tau_1}}) \approx 0.6523$.

One the other hand for $\hat{k}_{\tau_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we deduce that

$$|A|^{\frac{1}{2}}\hat{k}_{\tau_2} = \begin{bmatrix} 1.4117 & -0.2358 \\ -0.2358 & 2.3887 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.2358 \\ 2.3887 \end{bmatrix},$$

and

$$|A|^{\frac{1}{2}}U^*\hat{k}_{\tau_2} = \begin{bmatrix} 1.4117 & -0.2358 \\ -0.2358 & 2.3887 \end{bmatrix} \begin{bmatrix} 0.7682 & -0.6425 \\ 0.6402 & 0.7682 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1.08491 \\ 1.98596 \end{bmatrix}.$$

To calculate the angle $\theta_{A,\hat{k}_{\tau_2}}$, we have the following:

$$\begin{aligned} \cos(\theta_{A,\hat{k}_{\tau_2}}) &= \frac{\langle |A|^{\frac{1}{2}}\hat{k}_{\tau_2}, |A|^{\frac{1}{2}}U^*\hat{k}_{\tau_2}\rangle}{\||A|^{\frac{1}{2}}\hat{k}_{\tau_2}\|\||A|^{\frac{1}{2}}U^*\hat{k}_{\tau_2}\|} \\ &= \frac{(-0.2358)(-1.08491) + (2.3887)(1.98596)}{\sqrt{(-0.2358)^2 + (2.3887)^2}\sqrt{(-1.08491)^2 + (1.98596)^2}} \approx 0.920439. \end{aligned}$$

Thus $\theta_{A,\hat{k}_{\tau_2}} = 23.01°$. It follows from $0 \leq \theta_{A,\hat{k}_{\tau_1}}, \theta_{A,\hat{k}_{\tau_2}} \leq \frac{\pi}{2}$ that we set $\theta = \min\{\theta_{A,\hat{k}_{\tau_1}}, \theta_{A,\hat{k}_{\tau_2}}\} = 23.01°$. Therefore, we have $\mu(\theta) \approx 0.9474$. Consequently $\mu^2(\theta) \approx 0.89756676$. Hence

$$\begin{aligned} \mathbf{ber}^2(A) = 25 &\lneq \frac{\mu^2(\theta)}{2} \left\| |A|^2 + |A^*|^2 \right\|_{\mathbf{ber}} = 28.27335294 \\ &\lneq \frac{1}{2} \left\| |A|^2 + |A^*|^2 \right\|_{\mathbf{ber}} = 31.5. \end{aligned}$$

Theorem 2.2. *Let $\mathcal{H}=\mathcal{H}(\Theta)$ and $A,B\in\mathcal{B}(\mathcal{H})$. Also, let f, g be as in Lemma 2.6, $A+B$, $A-B$ have the polar decompositions $A+B=U|A+B|$, $A-B=V|A-B|$, $\theta_{A+B,\hat{k}_\lambda}=\angle_{f(|A+B|)\hat{k}_\lambda,g(|A+B|)U^*\hat{k}_\lambda}$, and $\theta_{A-B,\hat{k}_\lambda}=\angle_{f(|A-B|)\hat{k}_\lambda,g(|A-B|)V^*\hat{k}_\lambda}$. If either $0\leq\theta_1^{A+B}<\theta_{A+B,\hat{k}_\lambda}\leq\frac{\pi}{2}$, $0\leq\theta_1^{A-B}<\theta_{A-B,\hat{k}_\lambda}\leq\frac{\pi}{2}$ for all $\lambda\in\Theta$ or $\frac{\pi}{2}\leq\theta_{A+B,\hat{k}_\lambda}<\theta_2^{A+B}\leq\pi$, $\frac{\pi}{2}\leq\theta_{A-B,\hat{k}_\lambda}<\theta_2^{A-B}\leq\pi$ for all $\lambda\in\Theta$, then for any $r\geq 2$*

$$\begin{aligned}&\mathbf{ber}^r(A+B)\\&\leq\frac{\mu^r(\theta)}{2^{3-r}}\left\|f^{2r}(|A+B|)+f^{2r}(|A-B|)+g^{2r}(|(A+B)^*|)+g^{2r}(|(A-B)^*|)\right\|_{\mathbf{ber}},\end{aligned}$$

where $\theta'=\min\{\theta_1^{A+B},\theta_1^{A-B}\}$, $\theta''=\max\{\theta_2^{A+B},\theta_2^{A-B}\}$ and

$$\mu(\theta)=\max\{\mu(\theta'),\mu(\theta'')\}.$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Now, we have two cases:

(i) If $0\leq\theta_1^{A+B}<\theta_{A+B,\hat{k}_\lambda}\leq\frac{\pi}{2}$, and $0\leq\theta_1^{A-B}<\theta_{A-B,\hat{k}_\lambda}\leq\frac{\pi}{2}$ for all $\lambda\in\Theta$, then for any $r\geq 2$, we have

$$\begin{aligned}&\left|\langle(A+B)\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|^r\leq\left|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle B\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|^r\\&\leq 2^{r-1}\left(\left|\langle A\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|^r+\left|\langle B\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|^r\right)\quad\text{(by Lemma 2.3)}\\&\leq 2^{r-2}\left(\left|\langle(A+B)\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|^r+\left|\langle(A-B)\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|^r\right)\\&\leq 2^{r-2}\Big[\mu^r\left(\theta_{A+B,\hat{k}_\lambda}\right)\langle f^r(|A+B|)\hat{k}_\lambda,\hat{k}_\lambda\rangle\langle g^r(|(A+B)^*|)\hat{k}_\lambda,\hat{k}_\lambda\rangle\\&\quad+\mu^r\left(\theta_{A-B,\hat{k}_\lambda}\right)\langle f^r(|A-B|)\hat{k}_\lambda,\hat{k}_\lambda\rangle\langle g^r(|(A-B)^*|)\hat{k}_\lambda,\hat{k}_\lambda\rangle\Big]\\&\leq 2^{r-2}\Big[\mu^r(\theta')\langle f^r(|A+B|)\hat{k}_\lambda,\hat{k}_\lambda\rangle\langle g^r(|(A+B)^*|)\hat{k}_\lambda,\hat{k}_\lambda\rangle\\&\quad+\mu^r(\theta')\langle f^r(|A-B|)\hat{k}_\lambda,\hat{k}_\lambda\rangle\langle g^r(|(A-B)^*|)\hat{k}_\lambda,\hat{k}_\lambda\rangle\Big]\\&\leq 2^{r-3}\Big[\mu^r(\theta')\Big(\langle f^r(|A+B|)\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle g^r(|(A+B)^*|)\hat{k}_\lambda,\hat{k}_\lambda\rangle\Big)\\&\quad+\mu^r(\theta')\Big(\langle f^r(|A-B|)\hat{k}_\lambda,\hat{k}_\lambda\rangle+\langle g^r(|(A-B)^*|)\hat{k}_\lambda,\hat{k}_\lambda\rangle\Big)\Big]\\&\leq\frac{\mu(\theta')}{2^{3-r}}\Big\langle f^r(|A+B|)+g^r(|(A+B)^*|)+f^r(|A-B|)+g^r(|(A-B)^*|)\hat{k}_\lambda,\hat{k}_\lambda\Big\rangle.\end{aligned}$$

(ii) If $0\leq\theta_1^{A-B}<\theta_{A-B,\hat{k}_\lambda}\leq\frac{\pi}{2}$ and $\frac{\pi}{2}\leq\theta_{A-B,\hat{k}_\lambda}<\theta_2^{A-B}\leq\pi$ for all $\lambda\in\Theta$, then for any $r\geq 2$, we have

$$\begin{aligned}\left|\langle(A+B)\hat{k}_\lambda,\hat{k}_\lambda\rangle\right|^r\leq\frac{\mu(\theta'')}{2^{3-r}}&\big[(\langle f^r(|A+B|)+g^r(|(A+B)^*|)+f^r(|A-B|)\\&+g^r(|(A-B)^*|)\hat{k}_\lambda,\hat{k}_\lambda\big\rangle\big)\big].\end{aligned}$$

Consider $\mu(\theta) = \max\{\mu(\theta'), \mu(\theta'')\}$, we obtain

$$\left|\langle (A+B)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|^r \leq \frac{\mu^r(\theta)}{2^{3-r}} \langle f^r\left(|A+B|\right) + g^r\left(|(A+B)^*|\right) + f^r\left(|A-B|\right) + g^r\left(|(A-B)^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle.$$

Therefore, taking the supremum over all $\lambda \in \Theta$, we get the desired bound. □

Substituting B with A and $f(t) = g(t) = t^{\frac{1}{2}}$ in Theorem 2.2, we get the following result.

Corollary 2.3. *Let $\mathcal{H} = \mathcal{H}(\Theta)$, $A \in \mathcal{B}(\mathcal{H})$ with the polar decomposition $A = U|A|$, and $\theta_{A,\hat{k}_\lambda} = \angle_{|A|^{\frac{1}{2}}\hat{k}_\lambda, |A|^{\frac{1}{2}}U^*\hat{k}_\lambda}$. If either $0 \leq \theta_1^A < \theta_{A,\hat{k}_\lambda} \leq \frac{\pi}{2}$ for all $\lambda \in \Theta$ or $\frac{\pi}{2} \leq \theta_{A,\hat{k}_\lambda} < \theta_2^A \leq \pi$ for all $\lambda \in \Theta$, then for any $r \geq 2$*

$$\mathbf{ber}^r(A) \leq \frac{\mu^r(\theta)}{2^{3-r}} \left\||A|^r + |A^*|^r\right\|_{\mathbf{ber}},$$

where $\mu(\theta) = \max\{\mu(\theta_1^A), \mu(\theta_2^A)\}$.

Remark 2.4. Consider all the assumptions outlined in Corollary 2.3. It follows from $\frac{1}{2} \leq \mu(\theta) \leq 1$ for all $\theta \geq 0$ that for any $r \geq 2$, we have

$$\begin{aligned}\mathbf{ber}^r(A) &\leq \frac{\mu^r(\theta)}{2^{3-r}} \left\||A|^r + |A^*|^r\right\|_{\mathbf{ber}} \\ &\leq \frac{1}{2^{3-r}} \left\||A|^r + |A^*|^r\right\|_{\mathbf{ber}}.\end{aligned}$$

It is evident that Corollary 2.2 improves upon Corollary 2.3 for $r > 2$, indicating that Corollary 2.3 is better than (1.6).

Remark 2.5. Consider all the assumptions outlined in Corollary 2.2. Since $\frac{1}{2} \leq \mu(\theta) \leq 1$ for all $\theta \geq 0$, then for any $r \geq 2$

$$\begin{aligned}&\mathbf{ber}^r(A+B) \\ &\leq \frac{\mu^r(\theta)}{2^{3-r}} \left\|f^{2r}\left(|A+B|\right) + f^{2r}\left(|A-B|\right) + g^{2r}\left(|(A+B)^*|\right) + g^{2r}\left(|(A-B)^*|\right)\right\|_{\mathbf{ber}} \\ &\leq \frac{1}{2^{3-r}} \left\|f^{2r}\left(|A+B|\right) + f^{2r}\left(|A-B|\right) + g^{2r}\left(|(A+B)^*|\right) + g^{2r}\left(|(A-B)^*|\right)\right\|_{\mathbf{ber}}.\end{aligned}$$

This indicates that Theorem 2.2 is better than (1.5).

Another application of Theorem 2.2 is the following inequality.

Theorem 2.3. *Let A, B be self-adjoint operators in $\mathcal{B}(\mathcal{H})$, and let $\theta_{A+B,\hat{k}_\lambda} = \angle_{f(|A+B|)\hat{k}_\lambda, g(|A+B|)U^*\hat{k}_\lambda}$, $\theta_{A-B,\hat{k}_\lambda} = \angle_{f(|A-B|)\hat{k}_\lambda, g(|A-B|)V^*\hat{k}_\lambda}$. If either $0 \leq \theta_1^{A+B} < \theta_{A+B,\hat{k}_\lambda} \leq \frac{\pi}{2}$, $0 \leq \theta_1^{A-B} < \theta_{A-B,\hat{k}_\lambda} \leq \frac{\pi}{2}$ for all $\lambda \in \Theta$ or $\frac{\pi}{2} \leq \theta_{A+B,\hat{k}_\lambda} < \theta_2^{A+B} \leq \pi$, $\frac{\pi}{2} \leq \theta_{A-B,\hat{k}_\lambda} < \theta_2^{A-B} \leq \pi$ for all $\lambda \in \Theta$, then for any $r \geq 2$*

$$\mathbf{ber}^r(A+B) \leq \frac{\mu^r(\theta)}{2^{2-r}} \left\||A+B|^r + |(A-B)^*|^r\right\|_{\mathbf{ber}},$$

where $\theta'' = \max\{\theta_2^{A+B}, \theta_2^{A-B}\}$, $\theta' = \min\{\theta_1^{A+B}, \theta_1^{A-B}\}$, and

$$\mu(\theta) = \max\{\mu(\theta'), \mu(\theta'')\}.$$

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$, and let $f(t) = g(t) = t^{\frac{1}{2}}$ in Theorem 2.2. If either $0 \leq \theta_1^{A+B} < \theta_{A+B,\hat{k}_\lambda} \leq \frac{\pi}{2}$, $0 \leq \theta_1^{A-B} < \theta_{A-B,\hat{k}_\lambda} \leq \frac{\pi}{2}$ or $\frac{\pi}{2} \leq \theta_{A+B,\hat{k}_\lambda} < \theta_2^{A+B} \leq \pi$, $\frac{\pi}{2} \leq \theta_{A-B,\hat{k}_\lambda} < \theta_2^{A-B} \leq \pi$ for all $\lambda \in \Theta$, then for any $r \geq 2$, we have

$$\begin{aligned}\mathbf{ber}^r(A+B) &\leq \frac{\mu^r(\theta)}{2^{3-r}} \left\| |A+B|^r + |A-B|^r + |(A+B)^*|^r + |(A-B)^*|^r \right\|_{\mathbf{ber}} \\ &\leq \frac{\mu^r(\theta)}{2^{3-r}} \left\| 2|A+B|^r + 2|A-B|^r \right\|_{\mathbf{ber}} \\ &\leq \frac{\mu^r(\theta)}{2^{2-r}} \left\| |A+B|^r + |A-B|^r \right\|_{\mathbf{ber}}.\end{aligned}$$

This completes the proof. □

Remark 2.6. Since $\frac{1}{2} \leq \mu(\theta) \leq 1$ for all $\theta \geq 0$, then for any $r \geq 2$

$$\begin{aligned}\mathbf{ber}^r(A+B) &\leq \frac{\mu^r(\theta)}{2^{2-r}} \left\| |A+B|^r + |A-B|^r \right\|_{\mathbf{ber}} \\ &\leq \frac{1}{2^{2-r}} \left\| |A+B|^r + |A-B|^r \right\|_{\mathbf{ber}}.\end{aligned}$$

This indicates that Theorem 2.3 improves upon the inequality presented in [12, inequality (3.8)].

By substituting B with A in the inequality mentioned above, we obtain the following result.

Corollary 2.4. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ and $A \in \mathcal{B}(\mathcal{H})$ be self-adjoint, and let $\theta_{A,\hat{k}_\lambda} = \angle_{|A|^{\frac{1}{2}}\hat{k}_\lambda, |A|^{\frac{1}{2}}U^*\hat{k}_\lambda}$. If either $0 \leq \theta_1^A < \theta_{A,\hat{k}_\lambda} \leq \frac{\pi}{2}$ for all $\lambda \in \Theta$ or $\frac{\pi}{2} \leq \theta_{A,\hat{k}_\lambda} < \theta_2^A \leq \pi$ for all $\lambda \in \Theta$, then for any $r \geq 2$*

$$\mathbf{ber}^r(A) \leq \frac{\mu^r(\theta)}{2^{2-r}} \left\| |A|^r \right\|_{\mathbf{ber}},$$

where $\mu(\theta) = \max\{\mu(\theta_1^A), \mu(\theta_2^A)\}$.

Theorem 2.4. *Let $\mathcal{H} = \mathcal{H}(\Theta)$ and $A, B \in \mathcal{B}(\mathcal{H})$. Also, let f, g be as in Lemma 2.6, AB have the polar decomposition $AB = U|AB|$, and $\theta_{AB,\hat{k}_\lambda} = \angle_{f(|AB|)\hat{k}_\lambda, g(|AB|)U^*\hat{k}_\lambda}$. If either $0 \leq \theta_1^{AB} < \theta_{AB,\hat{k}_\lambda} \leq \frac{\pi}{2}$ for all $\lambda \in \Theta$ or $\frac{\pi}{2} \leq \theta_{AB,\hat{k}_\lambda} < \theta_2^{AB} \leq \pi$ for all $\lambda \in \Theta$, then for any $r \geq 2$*

$$\mathbf{ber}(AB) \leq \frac{\mu(\theta)}{2} \left\| f^2(|AB|) + g^2(|(AB)^*|) \right\|_{\mathbf{ber}},$$

where $\mu(\theta) = \max\{\mu(\theta_1^{AB}), \mu(\theta_2^{AB})\}$.

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. If $0 \leq \theta_1^{AB} < \theta_{AB,\hat{k}_\lambda} \leq \frac{\pi}{2}$ for all $\lambda \in \Theta$, then for any $r \geq 2$, we have

$$\begin{aligned}
\left|\langle (AB)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right| &\leq \mu\left(\theta_{AB,\hat{k}_\lambda}\right) \langle f^2\left(|AB|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}} \langle g^2\left(|(AB)^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}} \\
&\quad \text{(by Lemma 2.6)} \\
&\leq \frac{\mu\left(\theta_{AB,\hat{k}_\lambda}\right)}{2} \left(\langle f^2\left(|AB|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle g^2\left(|(AB)^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right) \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{\mu\left(\theta_{AB,\hat{k}_\lambda}\right)}{2} \left\langle \left(f^2\left(|AB|\right)\hat{k}_\lambda, \hat{k}_\lambda + g^2\left(|(AB)^*|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&\leq \frac{\mu\left(\theta_1^{AB}\right)}{2} \left\langle \left(f^2\left(|AB|\right)\hat{k}_\lambda, \hat{k}_\lambda + g^2\left(|(AB)^*|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle. \\
&\quad \text{(by the monotonicity of } \mu)
\end{aligned}$$

If $\frac{\pi}{2} \leq \theta_{AB,\hat{k}_\lambda} < \theta_1^{AB} \leq \pi$, then for any $r \geq 2$, we have

$$\begin{aligned}
\left|\langle (AB)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right| &\leq \mu\left(\theta_{AB,\hat{k}_\lambda}\right) \langle f^2\left(|AB|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}} \langle g^2\left(|(AB)^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle^{\frac{1}{2}} \\
&\quad \text{(by Lemma 2.6)} \\
&\leq \frac{\mu\left(\theta_{AB,\hat{k}_\lambda}\right)}{2} \left(\langle f^2\left(|AB|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle g^2\left(|(AB)^*|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right) \\
&\quad \text{(by the arithmetic-geometric mean inequality)} \\
&= \frac{\mu\left(\theta_{AB,\hat{k}_\lambda}\right)}{2} \left\langle \left(f^2\left(|AB|\right)\hat{k}_\lambda, \hat{k}_\lambda + g^2\left(|(AB)^*|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle \\
&\leq \frac{\mu\left(\theta_2^{AB}\right)}{2} \left\langle \left(f^2\left(|AB|\right)\hat{k}_\lambda, \hat{k}_\lambda + g^2\left(|(AB)^*|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle. \\
&\quad \text{(by the monotonicity of } \mu)
\end{aligned}$$

By considering $\mu(\theta) = \max\{\mu(\theta_1^{AB}), \mu(\theta_2^{AB})\}$ for two cases, we have

$$\left|\langle (AB)\hat{k}_\lambda, \hat{k}_\lambda\rangle\right| \leq \frac{\mu\left(\theta\right)}{2} \left\langle \left(f^2\left(|AB|\right)\hat{k}_\lambda, \hat{k}_\lambda + g^2\left(|(AB)^*|\right)\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle.$$

By taking the supremum over all $\lambda \in \Theta$, we achieve the desired result. □

Remark 2.7. Since $\frac{1}{2} \leq \mu(\theta) \leq 1$, then for any $r \geq 2$

$$\begin{aligned}
\mathbf{ber}\left(AB\right) &\leq \frac{\mu(\theta)}{2} \left\|f^2\left(|AB|\right) + g^2\left(|(AB)^*|\right)\right\|_{\mathbf{ber}} \\
&\leq \frac{1}{2} \left\|f^2\left(|AB|\right) + g^2\left(|(AB)^*|\right)\right\|_{\mathbf{ber}}.
\end{aligned}$$

This indicates that Theorem 2.4 provides a sharper inequality than the inequality presented in [12, inequality (3.11)].

Theorem 2.5. *Let $\mathcal{H} = \mathcal{H}(\Theta)$, $A_i, B_i \in \mathcal{B}(\mathcal{H})$, and $C_i \in \mathcal{B}(\mathcal{H})$ have the polar decomposition $C_i = U_i|C_i|$ $(i = 1, 2, ..., n)$. Also, let f, g be as in Lemma 2.6, and $\theta_{C_i, B_i\hat{k}_\lambda, A_i\hat{k}_\lambda} = \angle_{f(|C_i|)B_i\hat{k}_\lambda, g(|C_i|)U_i^* A_i\hat{k}_\lambda}$. If either $0 \leq \theta_i < \theta_{C_i, B_i\hat{k}_\lambda, A_i\hat{k}_\lambda} \leq \frac{\pi}{2}$*

for all $\lambda \in \Theta$ *or* $\frac{\pi}{2} \leq \theta_{C_i, B_i\hat{k}_\lambda, A_i\hat{k}_\lambda} < \theta_i \leq \pi$ *for all* $\lambda \in \Theta$ $(i = 1, 2, ..., n)$*, then for any* $r \geq 1$

$$\begin{aligned}&\mathbf{ber}^r\left(\sum_{i=1}^n A_i^* C_i B_i\right)\\ &\leq \frac{n^{r-1}\mu^r(\theta)}{\sqrt{2}}\mathbf{ber}\left(\sum_{i=1}^n \left(\left[B_i^* f^2\left(|C_i|\right) B_i\right]^r + i\left[A_i^* g^2\left(|C_i^*|\right) A_i\right]^r\right)\right),\end{aligned}$$

where $\theta' = \min_{1\leq i\leq n}\theta_i$*,* $\theta'' = \max_{1\leq i\leq n}\theta_i$ *and* $\mu(\theta) = \max\{\mu(\theta'), \mu(\theta'')\}$*.*

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. Now, we have two cases: (i) If $0 \leq \theta_i < \theta_{C_i, B_i\hat{k}_\lambda, A_i\hat{k}_\lambda} \leq \frac{\pi}{2}$ for all $i = 1, 2, ..., n$ and for all $\lambda \in \Theta$, then put $\theta' = \min\{\theta_1, \theta_2, ..., \theta_n\}$. It follows from the monotonicity of μ, $\mu(\theta_{C_i, B_i\hat{k}_\lambda, A_i\hat{k}_\lambda}) \leq \mu(\theta_i) \leq \mu(\theta')$, and so

$$\begin{aligned}&\left|\left\langle\left(\sum_{i=1}^n A_i^* C_i B_i\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \leq \left(\sum_{i=1}^n \left|\langle A_i^* C_i B_i\hat{k}_\lambda, \hat{k}_\lambda\rangle\right|\right)^r\\
&\leq n^{r-1}\left(\sum_{i=1}^n \left|\left\langle C_i B_i\hat{k}_\lambda, A_i\hat{k}_\lambda\right\rangle\right|^r\right) \quad \text{(by Lemma 2.4)}\\
&\leq n^{r-1}\left(\sum_{i=1}^n \left(\mu^r\left(\theta_{C_i, B_i\hat{k}_\lambda, A_i\hat{k}_\lambda}\right)\left\langle B_i^* f^2\left(|C_i|\right) B_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{r}{2}}\left\langle A_i^* g^2\left(|C_i|\right) A_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{r}{2}}\right)\right).\\
&\quad\text{(by Lemma 2.6)}\\
&\leq n^{r-1}\mu^r\left(\theta'\right)\sum_{i=1}^n \left\langle B_i^* f^2\left(|C_i|\right) B_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{r}{2}}\left\langle A_i^* g^2\left(|C_i|\right) A_i\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{r}{2}}\\
&\leq n^{r-1}\mu^r\left(\theta'\right)\sum_{i=1}^n \left\langle\left(B_i^* f^2\left(|C_i|\right) B_i\right)^r\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left\langle\left(A_i^* g^2\left(|C_i^*|\right) A_i\right)^r\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\\
&\quad\text{(by Lemma 2.5)}\\
&\leq \frac{n^{r-1}\mu^r\left(\theta'\right)}{2}\sum_{i=1}^n \left(\left\langle\left(B_i^* f^2\left(|C_i|\right) B_i\right)^r\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + \left\langle\left(A_i^* g^2\left(|C_i^*|\right) A_i\right)^r\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right)\\
&\quad\text{(by the arithmetic-geometric mean inequality)}\\
&\leq \frac{n^{r-1}\mu^r\left(\theta'\right)}{\sqrt{2}}\left|\sum_{i=1}^n \left\langle\left(B_i^* f^2\left(|C_i|\right) B_i\right)^r\hat{k}_\lambda, \hat{k}_\lambda\right\rangle + i\sum_{i=1}^n \left\langle\left(A_i^* g^2\left(|C_i^*|\right) A_i\right)^r\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|\\
&\quad(\text{as } |a+b| \leq \sqrt{2}|a+ib|)\\
&\leq \frac{n^{r-1}\mu^r\left(\theta'\right)}{\sqrt{2}}\left|\sum_{i=1}^n \left\langle\left(\left(B_i^* f^2\left(|C_i|\right) B_i\right)^r + i\left(A_i^* g^2\left(|C_i^*|\right) A_i\right)^r\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|.\end{aligned}$$

Therefore

$$\begin{aligned}&\left|\left\langle\left(\sum_{i=1}^n A_i^* C_i B_i\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r\\
&\leq \frac{n^{r-1}\mu^r\left(\theta'\right)}{\sqrt{2}}\left|\sum_{i=1}^n \left\langle\left(\left(B_i^* f^2\left(|C_i|\right) B_i\right)^r + i\left(A_i^* g^2\left(|C_i^*|\right) A_i\right)^r\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|.\end{aligned}$$

(ii) If $\frac{\pi}{2} \leq \theta_{C_i, B_i\hat{k}_\lambda, A_i\hat{k}_\lambda} < \theta_i \leq \pi$ for all $i = 1, 2, ..., n$ and for all $\lambda \in \Theta$, then put

$$\theta'' = \max\{\theta_1, \theta_2, ..., \theta_n\}.$$

It follows from by the monotonicity of μ, $\mu(\theta_{C_i,B_i\hat{k}_\lambda,A_i\hat{k}_\lambda}) \le \mu(\theta_i) \le \mu(\theta'')$. Similar to the first case, we get

$$\begin{aligned}&\left|\left\langle\left(\sum_{i=1}^{n} A_i^* C_i B_i\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\\&\le \frac{n^{r-1}\mu^r\left(\theta''\right)}{\sqrt{2}}\left|\sum_{i=1}^{n}\left\langle\left(\left(B_i^* f^2\left(|C_i|\right)B_i\right)^r + i\left(A_i^* g^2\left(|C_i^*|\right)A_i\right)^r\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|.\end{aligned}$$

By considering $\mu(\theta) = \max\{\mu(\theta'),\mu(\theta'')\}$, if either $0 \le \theta_i < \theta_{C_i,B_i\hat{k}_\lambda,A_i\hat{k}_\lambda} \le \frac{\pi}{2}$ or $\frac{\pi}{2} \le \theta_{C_i,B_i\hat{k}_\lambda,A_i\hat{k}_\lambda} < \theta_i \le \pi$ for all $i = 1,2,...,n$ and for all $\lambda \in \Theta$, then for all $r \ge 1$, we have

$$\begin{aligned}&\left|\left\langle\left(\sum_{i=1}^{n} A_i^* C_i B_i\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\\&\le \frac{n^{r-1}\mu^r\left(\theta\right)}{\sqrt{2}}\left|\sum_{i=1}^{n}\left\langle\left(\left(B_i^* f^2\left(|C_i|\right)B_i\right)^r + i\left(A_i^* g^2\left(|C_i^*|\right)A_i\right)^r\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|.\end{aligned}$$

By taking the supremum over all $\lambda \in \Theta$, we achieve the desired result. □

Remark 2.8. Since $\frac{1}{2} \le \mu(\theta) \le 1$, then for any $r \ge 1$

$$\begin{aligned}&\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* C_i B_i\right)\\&\le \frac{n^{r-1}\mu^r(\theta)}{\sqrt{2}}\mathbf{ber}\left(\sum_{i=1}^{n}\left(\left[B_i^* f^2\left(|C_i|\right)B_i\right]^r + \left[A_i^* g^2\left(|C_i^*|\right)A_i\right]^r\right)\right)\\&\le \frac{n^{r-1}}{\sqrt{2}}\mathbf{ber}\left(\sum_{i=1}^{n}\left(\left[B_i^* f^2\left(|C_i|\right)B_i\right]^r + \left[A_i^* g^2\left(|C_i^*|\right)A_i\right]^r\right)\right).\end{aligned}$$

It demonstrates an improvement of (1.7) .

For the functions $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, where $0 \le \alpha \le 1$, we obtain the following result.

Corollary 2.5. *Assume $A_i, B_i \in \mathcal{B}(\mathcal{H})$, and $C_i \in \mathcal{B}(\mathcal{H})$ with the polar decomposition $C_i = U|C_i|$ $(i = 1,2,...,n)$ and $\theta_{C_i,B_i\hat{k}_\lambda,A_i\hat{k}_\lambda} = \angle_{|C_i|^{2\alpha}B_i\hat{k}_\lambda,|C_i|^{2(1-\alpha)}U_i^*A_i\hat{k}_\lambda}$. If either $0 \le \theta_i < \theta_{C_i,B_i\hat{k}_\lambda,A_i\hat{k}_\lambda} \le \frac{\pi}{2}$ for all $\lambda \in \Theta$ or $\frac{\pi}{2} \le \theta_{C_i,B_i\hat{k}_\lambda,A_i\hat{k}_\lambda} < \theta_i \le \pi$ for all $\lambda \in \Theta$ $(i = 1,2,...,n)$, then for any $r \ge 1$*

$$\begin{aligned}&\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* C_i B_i\right)\\&\le \frac{n^{r-1}\mu^r(\theta)}{\sqrt{2}}\mathbf{ber}\left(\sum_{i=1}^{n}\left(\left[B_i^*\,|C_i|^{2\alpha}\,B_i\right]^r + i\left[A_i^*\,|C_i^*|^{2(1-\alpha)}\,A_i\right]^r\right)\right),\end{aligned}$$

where $\theta' = \min_{1\le i\le n}\theta_i$, $\theta'' = \max_{1\le i\le n}\theta_i$, and $\mu(\theta) = \max\{\mu(\theta'),\mu(\theta'')\}$.

For $A_i = B_i = I(i = 1,2,\ldots,n)$ in Theorem 2.5, we obtain the following corollary.

Corollary 2.6. *Let $\mathcal{H}=\mathcal{H}(\Theta)$, and $C_i\in\mathcal{B}(\mathcal{H})$ with the polar decomposition $C_i=U_i|C_i|(i=1,2,...,n)$. Also, let f, g be as in Lemma 2.6, and $\theta_{C_i,\hat{k}_\lambda}=\angle_{f(|C_i|)\hat{k}_\lambda,g(|C_i|)U_i^*\hat{k}_\lambda}$. If either $0\leq\theta_i<\theta_{C_i,\hat{k}_\lambda}\leq\frac{\pi}{2}$ for all $\lambda\in\Theta$ or $\frac{\pi}{2}\leq\theta_{C_i,\hat{k}_\lambda}<\theta_i\leq\pi$ for all $\lambda\in\Theta$, then for any $r\geq1$*

$$\mathbf{ber}^r\left(\sum_{i=1}^n C_i\right)\leq\frac{n^{r-1}\mu^r(\theta)}{\sqrt{2}}\mathbf{ber}\left(\sum_{i=1}^n\left(f^{2r}\left(|C_i|\right)+ig^{2r}\left(|C_i^*|\right)\right)\right),$$

where $\theta'=\min_{1\leq i\leq n}\theta_i$, $\theta''=\max_{1\leq i\leq n}\theta_i$ and $\mu(\theta)=\max\{\mu(\theta'),\mu(\theta'')\}$.

Specifically, by taking $n=1$, $r=1$, and $f(t)=g(t)=t^{\frac{1}{2}}$ in Corollary 2.6, we obtain the following inequality.

Corollary 2.7. *Assume $\mathcal{H}=\mathcal{H}(\Theta)$ and $A\in\mathcal{B}(\mathcal{H})$ with the polar decomposition $A=U|A|$, and also $\theta_{A,\hat{k}_\lambda}=\angle_{|A|^{\frac{1}{2}}\hat{k}_\lambda,|A|^{\frac{1}{2}}U^*\hat{k}_\lambda}$. If either $0\leq\theta_1<\theta_{A,\hat{k}_\lambda}\leq\frac{\pi}{2}$ for all $\lambda\in\Theta$ or $\frac{\pi}{2}\leq\theta_{C_i,\hat{k}_\lambda}<\theta_2\leq\pi$ for all $\lambda\in\Theta$, then*

$$\mathbf{ber}\left(A\right)\leq\frac{\mu(\theta)}{\sqrt{2}}\mathbf{ber}\left(|A|+i\left|A^*\right|\right),$$

where $\mu(\theta)=\max\{\mu(\theta_1),\mu(\theta_2)\}$.

Remark 2.9. Since $\frac{1}{2}\leq\mu(\theta)\leq1$ for all $\theta\geq0$, then

$$\begin{aligned}\mathbf{ber}\left(A\right)&\leq\frac{\mu(\theta)}{\sqrt{2}}\mathbf{ber}\left(|A|+i\left|A^*\right|\right)\\&\leq\frac{1}{\sqrt{2}}\mathbf{ber}\left(|A|+i\left|A^*\right|\right).\end{aligned}$$

It is obvious that $\mathbf{ber}^2\left(A\right)\leq\left\||A|^2+|A^*|^2\right\|_{\mathbf{ber}}$. Therefore

$$\begin{aligned}\mathbf{ber}^2\left(A\right)&\leq\frac{\mu^2(\theta)}{2}\mathbf{ber}\left(|A|+i\left|A^*\right|\right)\leq\frac{1}{2}\mathbf{ber}^2\left(|A|^2+i|A^*|^2\right)\\&\leq\frac{1}{2}\left\||A|^2+i|A^*|^2\right\|_{\mathbf{ber}}.\end{aligned}$$

Therefore, the inequality stated in Corollary 2.7 is stronger than the inequality given in (1.8).

Theorem 2.6. *Let $\mathcal{H}=\mathcal{H}(\Theta)$, $A,B\in\mathcal{B}(\mathcal{H})$, and A have the polar decomposition $A=U|A|$. Also, let $\theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda}=\angle_{f(|B|)\hat{k}_\lambda,g(|B|)U^*A\hat{k}_\lambda}$, and f, g be as in Lemma 2.6. If either $0\leq\theta_1<\theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda}\leq\frac{\pi}{2}$ for all $\lambda\in\Theta$ or $\frac{\pi}{2}\leq\theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda}<\theta_2\leq\pi$ for all $\lambda\in\Theta$, then for all $r\geq1$*

$$\mathbf{ber}^r(A^*B)\leq\frac{\mu^r\left(\theta\right)}{2}\left\|f^{2r}\left(|B|\right)+\left(A^*g^2\left(|B^*|\right)A\right)^r\right\|_{\mathbf{ber}},$$

where $\mu(\theta)=\max\{\mu(\theta_1),\mu(\theta_2)\}$.

Proof. Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$. By the Schwarz inequality, we have

$$\begin{aligned}
\left|\left\langle A^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| &= \left|\left\langle B\hat{k}_\lambda, A\hat{k}_\lambda\right\rangle\right| \\
&\leq \mu\left(\theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda}\right)\sqrt{\langle f^2\left(|B|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle g^2\left(|B^*|\right)A\hat{k}_\lambda, A\hat{k}_\lambda\rangle} \\
&= \mu\left(\theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda}\right)\sqrt{\langle f^2\left(|B|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle\langle A^*g^2\left(|B^*|\right)A\hat{k}_\lambda, \hat{k}_\lambda\rangle}.
\end{aligned}$$

Applying the arithmetic-geometric mean inequality and then the convexity of function $f(t) = t^r$, $r \geq 1$, we get

$$\begin{aligned}
\left|\left\langle A^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right| &\leq \mu\left(\theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda}\right)\left(\frac{\langle f^2\left(|B|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle A^*g^2\left(|B^*|\right)A\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right) \\
&\leq \mu\left(\theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda}\right)\left(\frac{\langle f^2\left(|B|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle^r + \langle A^*g^2\left(|B^*|\right)A\hat{k}_\lambda, \hat{k}_\lambda\rangle^r}{2}\right)^{\frac{1}{r}} \\
&\leq \mu\left(\theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda}\right)\left(\frac{\langle f^{2r}\left(|B|\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle + \langle (A^*g^2\left(|B^*|\right)A)^r\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right)^{\frac{1}{r}} \\
&\leq \mu\left(\theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda}\right)\left(\frac{\langle \left(f^{2r}\left(|B|\right) + (A^*g^2\left(|B^*|\right)A)^r\right)\hat{k}_\lambda, \hat{k}_\lambda\rangle}{2}\right)^{\frac{1}{r}}.
\end{aligned}$$

Now, we have two cases:

(i) If $0 \leq \theta_1 < \theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda} \leq \frac{\pi}{2}$ for all $\lambda \in \Theta$, it follows from the monotonicity of μ that $\mu(\theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda}) \leq \mu(\theta_1)$, and so

$$\left|\left\langle A^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \leq \frac{\mu^r\left(\theta_1\right)}{2}\left\langle\left(f^{2r}\left(|B|\right) + \left(A^*g^2\left(|B^*|\right)A\right)^r\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle$$

(ii) If $\frac{\pi}{2} \leq \theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda} < \theta_2 \leq \pi$ for all $\lambda \in \Theta$, it follows from the monotonicity of μ that $\mu(\theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda}) \leq \mu(\theta_2)$, and so

$$\left|\left\langle A^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \leq \frac{\mu^r\left(\theta_2\right)}{2}\left\langle\left(f^{2r}\left(|B|\right) + \left(A^*g^2\left(|B^*|\right)A\right)^r\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle.$$

By considering $\mu(\theta) = \max\{\mu(\theta_1), \mu(\theta_2)\}$, if either $0 \leq \theta_1 < \theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda} \leq \frac{\pi}{2}$ or $\frac{\pi}{2} \leq \theta_{B,\hat{k}_\lambda,A\hat{k}_\lambda} < \theta_2 \leq \pi$, then for all $r \geq 1$

$$\left|\left\langle A^*B\hat{k}_\lambda, \hat{k}_\lambda\right\rangle\right|^r \leq \frac{\mu^r\left(\theta\right)}{2}\left\langle\left(f^{2r}\left(|B|\right) + \left(A^*g^2\left(|B^*|\right)A\right)^r\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle.$$

By taking the supremum over all $\lambda \in \Theta$, we achieve the desired result. □

Remark 2.10. For $f\left(t\right) = t^\alpha$ and $g\left(t\right) = t^{1-\alpha}$, where $0 \leq \alpha \leq 1$, in Theorem 2.6, we get

$$\mathbf{ber}^r(A^*B) \leq \frac{\mu^r\left(\theta\right)}{2}\left\||B|^{2\alpha r} + \left(A^*\left|B^*\right|^{2(1-\alpha)}A\right)^r\right\|_{\mathbf{ber}}.$$

Putting $\alpha = 1$, we obtain

$$\begin{aligned}\mathbf{ber}^r(A^*B) &\leq \frac{\mu^r(\theta)}{2}\left\| |B|^{2r} + |A|^{2r} \right\|_{\mathbf{ber}} \\ &\leq \frac{1}{2}\left\| |B|^{2r} + |A|^{2r} \right\|_{\mathbf{ber}}.\end{aligned}$$

It demonstrates an improvement for [15, inequality (2.6)].

3. Conclusion

In this paper, we introduce an extension of the Cauchy-Schwarz inequality based on the angle between vectors. Our results build upon the recent inequality presented in [21, Theorem 2.3], allowing us to improve several existing Berezin-type inequalities. Using this extension, we derive new extensions and sharper bounds for Berezin-type inequalities related to bounded linear operators acting on reproducing kernel Hilbert spaces. We highlight how our extension of the Cauchy-Schwarz inequality can be utilized to establish several new inequalities involving the Berezin number of bounded linear operators in a reproducing kernel Hilbert space. These findings not only generalize but also improve existing inequalities within the literature.

References

[1] W. Audeh, M. Al-Labadi, Numerical radius inequalities for finite sums of operators, *Complex Anal. Oper. Theory* **17** (2023), no. 8, Paper No. 128, 10 pp.

[2] M. Bakherad, A new mean-Berezin norm for operators in reproducing kernel Hilbert spaces, *J. Inequal. Appl.* (2025), Paper No. 139.

[3] M. Bakherad, Generalized A-numerical radius inequalities in semi-Hilbert spaces through the angle between two vectors, *Quaest. Math.* **48** (2025), no. 10, 1491–1507.

[4] M. Bakherad, U. Yamancı, M. W. Alomari, Improved Berezin radius inequalities for functional Hilbert space operators, *Turkish J. Math.* **49** (2025), no. 5, 545–561.

[5] M. Bakherad, M. T. Garayev, Berezin number inequalities for operators, *Concr. Oper.* **6** (2019), no. 1, 33–43.

[6] M. Bakherad, M. T. Garayev, The Berezin radius and the Berezin norm associated with the tensor product, *Oper. Matrices* **19** (2025), no. 3, 321–337.

[7] F. A. Berezin, Covariant and contravariant symbols for operators, *Math. USSR-Izv.* **6** (1972), 1117–1151.

[8] P. Bhunia, S. S. Dragomir, M. S. Moslehian, K. Paul, *Lectures on Numerical Radius Inequalities*, Infosys Science Foundation Series in Mathematical Sciences, Springer, Cham, 2022.

[9] M. T. Garayev, Berezin symbols, Hölder-McCarthy and Young inequalities and their applications, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **43** (2017), no. 2, 287–295.

[10] M. Gürdal, M.W. Alomari, H. Başaran, Berezin radius inequalities via Orlicz function, *Palest. J. Math.* **14** (2025), no. 1, 928–945.

[11] M. Gürdal, G. Gül Erkan, M. Garayev, Berezin norm and Berezin radius inequalities of products and sums with Selberg operator, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **50** (2024), no. 2, 258–273.

[12] M. Gürdal, V. Stojiljkovic, Berezin radius inequalities for finite sums of functional Hilbert space operators, *Gulf J. Math.* **17** (2024), no. 1, 101–109.

[13] M. Gürdal, R. Tapdigoglu, New Berezin radius upper bounds, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **49** (2023), no. 2, 210–218.

[14] M. Hajmohamadi, R. Lashkaripour, M. Bakherad, Improvements of Berezin number inequalities, *Linear Multilinear Algebra* **68** (2020), no. 6, 1218–1229.

[15] M. B. Huban, H. Başaran, M. Gürdal, New upper bounds related to the Berezin number inequalities, *J. Inequal. Spec. Funct.* **12** (2021), no. 3, 1–12.

[16] M. T. Karaev, Berezin symbol and invertibility of operators on the functional Hilbert spaces, *J. Funct. Anal.* **238** (2006), no. 1, 181–192.

[17] M. T. Karaev, S. Saltan, Some results on Berezin symbols, *Complex Var. Theory Appl.* **50** (2005), no. 3, 185–193.

[18] S. Khan, M. S. Akram, M. Gürdal, Improving bounds for generalized Berezin number of operators, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **51** (2025), no. 2, 171–191.

[19] F. Kittaneh, Notes on some inequalities for Hilbert space operators. *Publ. Res. Inst. Math. Sci.* **24** (1988), no. 2, 283–293.

[20] R. Rahimi, R. Lashkaripour, M. Bakherad, Numerical radius and norm inequalities via a Cauchy-Schwarz extension, *Math. Slovaca* (to appear).

[21] M. Sababheh, S. Furuichi, H. R. Moradi, Operator inequalities via the triangle inequality, *J. Math. Inequal.* **18** (2024), no. 2, 631–642.

[22] V. Stojiljković, M. Gürdal, 11 Integral-type Berezin radius inequalities, *Fixed Point Theory and Functional Analysis: Metric Spaces, Banach Spaces, Hilbert Spaces*, edited by Pradip Debnath, Hari Mohan Srivastava and Yeol Je Cho, De Gruyter, 2026, 215–233.

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