Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

NODAL SOLUTIONS OF SOME NONLINEAR PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS OF FOURTH ORDER WITH COMPLETELY REGULAR BOUNDARY CONDITIONS

ZIYATKHAN S. ALIYEV AND ELCHIN A. MAMMADOV

Abstract. In this paper, we consider a nonlinear problem for ordinary differential equations of fourth order with completely regular boundary conditions. We establish sufficient conditions for the existence of nodal (sign-changing) solutions to this problem.

1. Introduction

We consider the following nonlinear fourth order ordinary differential equation

$$\ell(u)(x) \equiv (p\,(x)u''(x))'' - (q(x)u'(x))' = r\,\tau(x)h\,(u\,(x)),\ x \in (0,l), \tag{1.1}$$

subject to the completely regular boundary conditions

$$u(0)\cos\alpha - (pu'')(0)\sin\alpha = 0, \tag{1.2}$$

$$u\,(0)\cos\beta + Tu(0)\sin\beta = 0, \tag{1.3}$$

$$u\,(l)\cos\gamma + (pu'')(l)\sin\gamma = 0, \tag{1.4}$$

$$u\,(l)\cos\delta + Tu(l)\sin\delta = 0, \tag{1.5}$$

where p is a twice positive continuously differentiable function on $[0,l]$, q is a non-negative continuously differentiable function on $[0,l]$, τ is a positive continuous function on $[0,l]$, r is a real parameter, α, β, γ and δ are real constants such that $0 \le \alpha,\beta,\gamma,\delta \le \pi/2$ (except for the case $\beta = \delta = \pi/2$). The function h has the form $f+g$, where f and g are real-valued continuous functions on $\mathbb{R}$ that satisfy the following conditions:

(H_1) there exist positive constants C_0 and C_∞, a sufficiently small positive constant χ_0 and a sufficiently large positive constant χ_∞ such that

$$\left|\frac{f(s)}{s}\right| \le C_0 \text{ for any } s \in \mathbb{R},\ 0 < |s| < \chi_0, \tag{1.6}$$

$$\left|\frac{f(s)}{s}\right| \le C_\infty \text{ for any } s \in \mathbb{R},\ |s| > \chi_\infty; \tag{1.7}$$

2010 *Mathematics Subject Classification.* 34B15, 34B24, 34C23, 47J10, 47J15.
Key words and phrases. linearizable eigenvalue problem, indefinite weight, bifurcation point, global bifurcation, simple zero.

(H_2) there exist real constants g_0 and g_∞ such that $g_0 g_\infty > 0$ and

$$\lim_{|s|\to 0+} \frac{g(s)}{s} = g_0, \tag{1.8}$$

$$\lim_{|s|\to +\infty} \frac{g(s)}{s} = g_\infty. \tag{1.9}$$

The study of the existence of nodal solutions to nonlinear problems of the type (1.1)-(1.5) for the Sturm-Liouville operator has been the goal of many authors (see [4, 8, 10, 12, 13, 16] and the references therein). Using various methods, these authors determine the intervals of the parameter r, in which there exist nodal solutions of the boundary value problems considered by them. Similar results on the existence of nodal solutions to the nonlinear problem (1.1)-(1.5) were obtained in [2, 5, 9, 11, 14]. It should be noted that in all the listed papers, with the exception of [4] and [8], the nonlinear term g satisfies the conditions $sg(s) > 0$ for $s \in \mathbb{R}$, $s \neq 0$, and $g_0, g_\infty \in (0, +\infty)$.

The purpose of this paper is to determine the values of the parameter r, for which there exist nodal solutions of the nonlinear problem (1.1)-(1.5).

The rest of this paper is organized as follows. Section 2 first presents the oscillatory properties of eigenfunctions of a completely regular Sturmian system of fourth order established in [1], and then presents classes of functions constructed in [1] that have oscillatory properties of eigenfunctions of this system and their derivatives. Nonlinear eigenvalue problem is also introduced, from which problem (1.1)-(1.5) is obtained by equating the spectral parameter to unity. In Section 3, we study the global bifurcation of solutions from zero and infinity of this nonlinear eigenvalue problem. Here, we show that the global components of the set of solutions of this nonlinear eigenvalue problem, contained in the classes presented in Section 2 and emanating from the bifurcation intervals of the line of trivial solutions and the line $\mathbb{R} \times \{\infty\}$ coincide. In Section 4, using this result, we find the values of the parameter r, for which there exist nodal solutions of problem (1.1)-(1.5).

2. Preliminaries

For the study of existence of nodal solutions of problem (1.1)-(1.5) we consider the following linear eigenvalue problem

$$\begin{cases} \ell(u)(x) = \lambda\, \tau(x) u\,(x),\ x \in (0, l), \\ u \in B.C., \end{cases} \tag{2.1}$$

where $B.C.$ is the set of functions satisfying the boundary conditions (1.2)-(1.5). Note that the linear eigenvalue problem (2.1) is called a completely regular Sturmian system of fourth order, as defined by S. A. Janczewsky [7]. It follows from [6, Theorems 5.4 and 5.5] that the eigenvalues of problem (2.1) are positive and simple and form an infinitely increasing sequence $\{\lambda_k\}_{k=1}^{\infty}$. Moreover, for each $k \in \mathbb{N}$ the eigenfunction $y_k(x)$ corresponding to the eigenvalue λ_k has exactly $k-1$ simple nodal zeros in $(0, l)$. In [1] the oscillatory properties of the derivatives of eigenfunctions of problem (2.1) were also studied.

Let E be the Banach space $C^3[0,l] \cap B.C.$ with the usual norm $||u||_3 = \max_{x\in[0,l]} |u(x)| + \max_{x\in[0,l]} |u'(x)| + \max_{x\in[0,l]} |u''(x)| + \max_{x\in[0,l]} |u'''(x)|$.

By S we denote the subset of E defined as follows:

$$S = \{u \in E \ : \ |u(x)| + |u'(x)| + |u''(x)| + |u'''(x)| > 0, \, x \in [0, \, l]\}.$$

In [1, § 3.1], the classes S_k^ν of functions $u \in S$ that satisfy the oscillatory properties of eigenfunctions of problem (2.1) and their derivatives were constructed. Note that for each $k \in \mathbb{N}$ and each $\nu \in \{+, -\}$ the sets S_k^ν are open subsets of E.

By (1.8) and (1.9) we get

$$g(s) = g_0 s + \phi_0(s) \ \text{ and } \ g(s) = g_\infty s + \phi_\infty(s), \ \ s \in R, \ \ s \neq 0, \tag{2.2}$$

where

$$\phi_0(s) = o(|s|) \text{ as } |s| \to 0, \text{ and } \phi_\infty(s) = o(|s|) \text{ as } |s| \to +\infty. \tag{2.3}$$

By the first relation of (2.3) for any $\varepsilon > 0$ there exists $\varrho_\varepsilon > 0$ such that

$$\left| \frac{\phi_0(s)}{s} \right| < \varepsilon \text{ for any } s \in \mathbb{R}, \ 0 < |s| < \varrho_\varepsilon. \tag{2.4}$$

By the second relation of (2.3) there exists $\sigma_\varepsilon > 0$ such that

$$\left| \frac{\phi_\infty(s)}{s} \right| < \varepsilon \text{ for any } s \in \mathbb{R}, \ |s| > \sigma_\varepsilon. \tag{2.5}$$

It follows from (1.8) that

$$g(s) \to 0 \ \text{ as } \ |\, s\,| \to 0,$$

whence, by $g \in C(\mathbb{R})$, we get $g(0) = 0$, and consequently, by (2.2) we obtain $\phi_0(0) = 0$. Then, by relation $\phi_0 \in C(\mathbb{R})$, there exists a positive constant M_ε such that

$$|\phi_0(s)| \leq M_\varepsilon \ \text{ for any } \ s \in \mathbb{R}, \ |s| \leq \sigma_\varepsilon. \tag{2.6}$$

By (2.2) we have

$$\phi_\infty(s) = (g_0 - g_\infty)\, s + \phi_0(s), \ \ s \in \mathbb{R}, \tag{2.7}$$

which implies that

$$|\phi_\infty(s)| \ \leq \ |g_0 - g_\infty|\, \sigma_\varepsilon + M_\varepsilon, \ s \in \mathbb{R}, \ |s| \leq \sigma_\varepsilon. \tag{2.8}$$

Let

$$N_\varepsilon = |g_0 - g_\infty|\, \sigma_\varepsilon + M_\varepsilon, \tag{2.9}$$

and let $\rho_\varepsilon > \sigma_\varepsilon$ be a sufficiently large positive number such that

$$\frac{N_\varepsilon}{\rho_\varepsilon} < \varepsilon. \tag{2.10}$$

Remark 2.1. Since the eigenvalues of problem (2.1) are positive, problem (1.1)-(1.5) for $r = 0$ has no nontrivial solutions.

In what follows we will assume that r is a real fixed non-zero number.

To study the existence of nodal solutions to problem (1.1)-(1.5) we consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(u) = \lambda r\, \tau(x)\, g_0 u \ + r\, \tau(x) f(u) + r\, \tau(x)\, \phi_0(u), \; x \in (0, l), \\ u \in B.C., \end{cases} \tag{2.11}$$

which by (2.7) can be rewritten in the form:

$$\begin{cases} \ell(u) = (\lambda\, g_0 + g_\infty - g_0)\, r\, \tau(x)\, u + r\, \tau(x) f\,(u) + \\ \qquad\qquad\qquad\qquad r\, \tau(x)\, \phi_\infty(u),\ x \in (0,l), \\ u \in B.C. \end{cases} \tag{2.12}$$

3. Global bifurcation from zero and infinity of solutions of problem (2.11)

Since the eigenvalues of problem (2.1) are positive, then $\lambda = 0$ is not eigenvalue of the linear problem

$$\begin{cases} \ell(u)(x) = \lambda r g_0 \tau(x) u\,(x),\ \ x \in (0,l), \\ u \in B.C. \end{cases} \tag{3.1}$$

Then nonlinear eigenvalue problems (2.11) and (2.12) are equivalent to the following nonlinear integral equations

$$\begin{gathered} u(x) = \lambda r g_0 \int_0^l K(x,t)\tau(t)u\,(t)dt + r\int_0^l K(x,t)\tau(t)f(u\,(t))dt + \\ r\int_0^l K(x,t)\tau(t)\,\phi_0\,(u\,(t))dt, \end{gathered} \tag{3.2}$$

and

$$\begin{gathered} u(x) = (\lambda g_0 + g_\infty - g_0) r\int_0^l K(x,t)\tau(t)u\,(t)dt + \\ r\int_0^l K(x,t)\tau(t)f(u\,(t))dt + r\int_0^l K(x,t)\tau(t)\,\phi_\infty\,(u\,(t))dt, \end{gathered} \tag{3.3}$$

respectively, where $K(x,t)$, $(x,t) \in [0,l;0,l]$, is the Green's function for the differential expression $\ell(u)$ with boundary conditions (1.2)-(1.5).

Let

$$Lu(x) = \int_0^l K(x,t)\tau(t)u\,(t)dt, \tag{3.4}$$

$$F(u)(x) = \int_0^l K(x,t)\tau(t)f(u\,(t))dt, \tag{3.5}$$

$$G_0(u)(x) = \int_0^l K(x,t)\tau(t)\phi_0(u\,(t))dt, \tag{3.6}$$

$$G_\infty(u)(x) = \int_0^l K(x,t)\tau(t)\phi_\infty(u\,(t))dt, \tag{3.7}$$

Since $\tau(t) \in C[0,l]$ it follows from the properties of the Green's function $K(x,t)$ that the operator acts in E and is completely continuous. The operator F can be represented as the composition of the operator L and the superposition operator $\mathfrak{f}(u)(t) = f(u(t))$. Since f is continuous and bounded on $\mathbb{R}$ it follows that $\mathfrak{f} : E \to C[0,l]$ is continuous and bounded. Hence the operator F acts in E and is completely continuous. Similarly, the operators G_0 and G_∞ act in E and are completely continuous. Moreover, by (2.4), for any $u \in E$ such that $||u||_1 < \varrho_\varepsilon$ we get

$$||G_0(u)\,||_3 = \max_{x\in[0,l]} \left| \int_0^l K(x,t)\tau(t)\,\phi_0\,(u\,(t))dt \right| +$$

$$\max_{x\in[0,l]} \left| \int_0^l K'_x(x,t)\tau(t)\,\phi_0\,(u\,(t))dt \right| + \max_{x\in[0,l]} \left| \int_0^l K''_{xx}(x,t)\tau(t)\,\phi_0\,(u\,(t))dt \right| +$$

$$\max_{x\in[0,l]} \left| \int_0^l K'''_{xxx}(x,t)\tau(t)\,\phi_0\,(u\,(t))dt \right| < \varepsilon\,\kappa\tau_1 l||u||_\infty \le \varepsilon\,\kappa\tau_1 l||u||_3,$$

where

$$\kappa = \max\{ \max_{(x,t)\in[0,\,l;0,\,l]} |K(x,t)|, \max_{(x,t)\in[0,\,l;\,0,\,l]} |K'_x(x,t)|, \max_{(x,t)\in[0,\,l;\,0,\,l]} |K''_{xx}(x,t)|,$$

$$\max_{(x,t)\in[0,\,l;\,0,\,l]} |K'''_{xxx}(x,t)|\}.$$

This relation show that

$$G_0(u) = o\,(||u||_3) \text{ as } ||u||_3 \to 0. \tag{3.8}$$

Next, by (2.5), for any $u \in E$ such that $||u||_3 > \rho_\varepsilon$ we get

$$||G_\infty(u)\,||_3 = \max_{x\in[0,l]} \left| \int_0^l K(x,t)\tau(t)\,\phi_\infty\,(u\,(t))dt \right| +$$

$$\max_{x\in[0,l]} \left| \int_0^l K'_x(x,t)\tau(t)\,\phi_\infty\,(u\,(t))dt \right| + \max_{x\in[0,l]} \left| \int_0^l K''_{xx}(x,t)\tau(t)\,\phi_\infty\,(u\,(t))dt \right| +$$

$$\max_{x\in[0,l]} \left| \int_0^l K'''_{xxx}(x,t)\tau(t)\,\phi_\infty\,(u\,(t))dt \right| \le \kappa\tau_1 \left\{ \int_{|u(t)|\,\le\,\Delta_\varepsilon} \phi_\infty\,(u\,(t))|dt + \right.$$

$$\left. \int_{|u(t)|\,>\,\Delta_\varepsilon} \phi_\infty\,(u\,(t))|dt \right\} \le \kappa\tau_1 l\,\{N_\varepsilon + \varepsilon||u||_\infty\} <$$

$$< \kappa\tau_1 l\,\{\varepsilon\,\rho_\varepsilon + \varepsilon||u||_3\} < \kappa\tau_1 l\,\{\varepsilon\,||u||_3 + \varepsilon||u||_3\} =$$

$$= 2\,\varepsilon\,\kappa\tau_1 l\;||u||_3.$$

The last relation shows that

$$G_\infty(u) = o\,(||u||_3) \text{ as } ||u||_3 \to +\infty. \tag{3.9}$$

According to (3.4)-(3.7), equations (3.2) and (3.3) can be rewritten as follows:

$$u = \lambda r g_0 L u + r\,F(u) + r\,G_0(u), \tag{3.10}$$

and

$$u = (\lambda g_0 + g_\infty - g_0)\,r L u + r\,F(u) + r\,G_\infty(u), \tag{3.11}$$

respectively.

It follows from (2.1) and (3.1) that for each $k \in \mathbb{N}$ the kth eigenvalue $\tilde{\lambda}_k$ of the linear problem (3.1) is defined by the formula

$$\tilde{\lambda}_k = \frac{\lambda_k}{r g_0}. \tag{3.12}$$

Moreover, by (2.1), for each $k \in \mathbb{N}$ the kth eigenvalue $\hat{\lambda}_k$ of the linear problem

$$\begin{cases} \ell(u)\,(x) = (\lambda\, g_0 \,+ g_\infty - g_0)\, r\, \tau(x)\, u\,(x)\, x \in (0,l), \\ u \in B.C. \end{cases} \tag{3.13}$$

is determined by the relation

$$\tilde{\lambda}_k = \frac{\lambda_k}{r g_0} + 1 - \frac{g_\infty}{g_0}. \tag{3.14}$$

Remark 3.1. As a norm in the space $\mathbb{R} \times E$, we take $||(\lambda, u)|| = \{|\lambda|^2 + ||u||_3^2\}^{\frac{1}{2}}$.

Remark 3.2. We add the points at infinity (λ, ∞), $\lambda \in \mathbb{R}$, to our space $\mathbb{R} \times E$ and define an appropriate topology on the resulting set.

In view of [15, Theorem 3.3], by [1, Theorem 1.1] and [3, Theorem 3.1] for equations (3.10) and (3.11) in the case $f \equiv 0$ we get the following results.

Theorem 3.1. *Let $f \equiv 0$. Then for every $k \in \mathbb{N}$ and every $\nu \in \{+\,,\,-\}$ there exists a continuum $C^{\nu}_{k,0}$ of solutions of equation* (3.10)*, which contains $(\tilde{\lambda}_k, 0)$, is contained in $(\mathbb{R} \times S^{\nu}_k) \cup \{(\tilde{\lambda}_k, 0)\}$ and either meets $(\hat{\lambda}_k, \infty)$ or its projection onto $\mathbb{R} \times \{0\}$ is unbounded.*

Theorem 3.2. *Let $f \equiv 0$. Then for every $k \in \mathbb{N}$ and every $\nu \in \{+\,,\,-\}$ there exists a continuum $C^{\nu}_{k,\infty}$ of solutions of equation* (3.10) *(or* (3.11)*), which contains $(\hat{\lambda}_k, \infty)$, is contained in $(\mathbb{R} \times S^{\nu}_k) \cup \{(\hat{\lambda}_k, \infty)\}$, and either meets $(\tilde{\lambda}_k, 0)$ or its projection onto $\mathbb{R} \times \{0\}$ is unbounded.*

Using Theorems 3.1 and 3.2, and by following the arguments in Lemmas 5.3, 5.4, Corollaries 5.3, 5.4 and Theorem 1.3 of [1], and in Lemmas 5.1, 5.2, 5.4-5.6, Corollary 5.7 and Theorem 5.9 of [3] for problem (3.10) in the case $f \not\equiv 0$ we have the following global bifurcation results.

Lemma 3.1. *For every $k \in \mathbb{N}$ and every $\nu \in \{+\,,\,-\}$ the set of bifurcation points of equation* (3.10) *with respect to the set $\mathbb{R} \times S^{\nu}_k$ is nonempty. Furthermore, if $(\lambda, 0)$ is a bifurcation point of equation* (3.10) *with respect to $\mathbb{R} \times S^{\nu}_k$, then $\lambda \in \tilde{I}_k = \left[\frac{\lambda_k}{g_0 r} - \frac{C_0}{|g_0|}, \frac{\lambda_k}{g_0 r} + \frac{C_0}{|g_0|}\right]$.*

For every $k \in \mathbb{N}$ and every $\nu \in \{+\,,\,-\}$, let $\tilde{D}^{\nu}_{k,0}$ be the union of all components of the set of solutions of equation (3.10) that meet the bifurcation interval $\tilde{I}_k \times \{0\}$ with respect to the set $\mathbb{R} \times S^{\nu}_k$. This set may not be connected in $\mathbb{R} \times E$, but the set $D^{\nu}_{k,0} = \tilde{D}^{\nu}_{k,0} \cup (\tilde{I}_k \times \{0\})$ is connected in $\mathbb{R} \times E$.

Theorem 3.3. *For every $k \in \mathbb{N}$ and every $\nu \in \{+, -\}$ the component $D^{\nu}_{k,0}$ of the set of solutions of equation* (3.10) *lies in $(\mathbb{R} \times S^{\nu}_k) \cup (\tilde{I}_k \times \{0\})$ and either meets $\hat{I}_k \times \{\infty\}$ or its projection onto $\mathbb{R} \times \{0\}$ is unbounded, where $\hat{I}_k = \left[\frac{\lambda_k}{g_0 r} + 1 - \frac{g_\infty}{g_0} - \frac{C_\infty}{|g_0|}, \frac{\lambda_k}{g_0 r} + 1 - \frac{g_\infty}{g_0} + \frac{C_\infty}{|g_0|}\right]$.*

Lemma 3.2. *For every $k \in \mathbb{N}$ and every $\nu \in \{+, -\}$ the set of asymptotic bifurcation points of equation* (3.10) *(or* (3.11)*) with respect to the set $\mathbb{R} \times S_k^\nu$ is nonempty. Furthermore, if (λ, ∞) is an asymptotic bifurcation point of equation* (3.10) *(or* (3.11)*) with respect to $\mathbb{R} \times S_k^\nu$, then $\lambda \in \hat{I}_k$.*

For every $k \in \mathbb{N}$ and every $\nu \in \{+, -\}$, let $\tilde{D}_{k,\infty}^\nu$ be the union of all components of the set of solutions of equation (3.10) (or (3.11)) which meet asymptotic bifurcation interval $\hat{I}_k \times \{\infty\}$ with respect to the set $\mathbb{R} \times S_k^\nu$. This set may not be connected in $\mathbb{R} \times E$, but the set $D_{k,\infty}^\nu = \tilde{D}_{k,\infty}^\nu \cup (\hat{I}_k \times \{\infty\})$ is connected in $\mathbb{R} \times E$.

Theorem 3.4. *For every $k \in \mathbb{N}$ and every $\nu \in \{+, -\}$ the component $D_{k,\infty}^\nu$ of the set of solutions of equation* (3.10) *(or* (3.11)*) lies in $(\mathbb{R} \times S_k^\nu) \cup (\hat{I}_k \times \{\infty\})$ and either meets $\tilde{I}_k \times \{0\}$ or its projection onto $\mathbb{R} \times \{0\}$ is unbounded.*

Remark 3.3. From the definitions of the sets $D_{k,0}^\nu$ and $D_{k,\infty}^\nu$ it follows that if their projections onto $\mathbb{R} \times \{0\}$ are bounded, then they coincide.

Theorem 3.5. *For every $k \in \mathbb{N}$ and every $\nu \in \{+, -\}$ the projections of the sets $D_{k,0}^\nu$ and $D_{k,\infty}^\nu$ onto $\mathbb{R} \times \{0\}$ are bounded.*

Proof. Let $(\lambda^*, u^*) \in D_{k,0}^\nu$ for some $k \in \mathbb{N}$ and $\nu \in \{+, -\}$. Then (λ^*, u^*) is a solution of the following linear problem

$$\begin{cases} \ell(u) + r\tau(x)\phi^*(x)u + r\tau(x)\psi^*(x)u = \lambda r\tau(x)g_0 u, x \in (0, l), \\ u \in B.C, \end{cases} \tag{3.15}$$

where

$$\phi^*(x) = \begin{cases} -\frac{f(u^*(x))}{u^*(x)} & \text{if } u^*(x) \neq 0, \\ 0 & \text{if } u^*(x) = 0, \end{cases} \tag{3.16}$$

$$\psi_0^*(x) = \begin{cases} -\frac{\phi_0(u^*(x))}{u^*(x)} & \text{if } u^*(x) \neq 0, \\ 0 & \text{if } u^*(x) = 0. \end{cases} \tag{3.17}$$

In view of (1.7), by (3.16), we get

$$|\phi^*(x)| \leq C_{0,\infty}^*, \ x \in [0, l], \tag{3.18}$$

where

$$C_{0,\infty}^* = \max\left\{C_0, C_\infty, \max_{\chi_0 \leq |s| \leq \chi_\infty} \left|\frac{f(s)}{s}\right|\right\}.$$

It follows from (2.7) that

$$\frac{\varphi_0(s)}{s} = g_\infty - g_0 + \frac{\varphi_\infty(s)}{s}, \ s \in \mathbb{R}, \ s \neq 0, \tag{3.19}$$

whence, by (2.5), we get

$$\left|\frac{\varphi_0(s)}{s}\right| < |g_\infty - g_0| + 1 \ \text{ for any } s \in \mathbb{R}, \ |s| > \sigma_1. \tag{3.20}$$

Since $\frac{\phi_0(s)}{s} \in C[\varrho_1, \sigma_1]$ it follows that there exists a positive constant $\tilde{M}_0$ such that

$$\left|\frac{\varphi_0(s)}{s}\right| < \tilde{M}_0 \ \text{ for any } s \in \mathbb{R}, \ \varrho_1 \leq |s| \leq \sigma_1. \tag{3.21}$$

Then by (2.4), (3.20) and (3.21) we get

$$\left|\frac{\varphi_0(s)}{s}\right| < M_0^* \text{ for any } s \in \mathbb{R},\, s \neq 0, \tag{3.22}$$

where

$$M_0^* = \max\{|g_\infty - g_0| + 1, \tilde{M}_0\}.$$

Hence, by (3.22), from (3.17) we obtain

$$|\psi_0^*(x)| < M_0^*,\, x \in [0, l]. \tag{3.23}$$

Therefore, by relations (3.12), (3.18) and (3.23), it follows from [1, formula (4.2)] that

$$\frac{\lambda_k}{r g_0} - \frac{C_{0,\infty}^* + M_0^*}{|g_0|} \leq \lambda^* \leq \frac{\lambda_k}{r g_0} + \frac{C_{0,\infty}^* + M_0^*}{|g_0|},$$

which implies that the projection of the set $D_{k,0}^\nu$ onto $\mathbb{R} \times \{0\}$ is bounded.

In a similar way we can show that the projection of the set $D_{k,\infty}^\nu$ onto $\mathbb{R} \times \{0\}$ is bounded. The proof of this theorem is complete.

Corollary 3.1. *For every $k \in \mathbb{N}$ and every $\nu \in \{+, -\}$ the following relation holds:*

$$D_{k,0}^\nu = D_{k,\infty}^\nu.$$

4. Existence of nodal solutions of problem (1.1)-(1.5)

In this section, using Theorems 3.3, 3.5, relations (3.12), (3.18) and Corollary 3.1, we establish sufficient conditions for the existence of nodal solutions to problem (1.1)-(1.5).

Remark 4.1. By Corollary 3.1 for every $k \in \mathbb{N}$ and every $\nu \in \{+, -\}$ the component $D_{k,0}^\nu$ of the set of solutions of problem (3.10) meets the intervals $\tilde{I}_k \times \{0\}$ and $\hat{I}_k \times \{\infty\}$. Moreover, the set $D_{k,0}^\nu \backslash \{(\tilde{I}_k \times \{0\}) \cup (\hat{I}_k \times \{\infty\})\}$ lies in $\mathbb{R} \times S_k^\nu$. Since $D_{k,0}^\nu$ is connected it follows that if for some $k \in \mathbb{N}$ the right end of the interval $\tilde{I}_k$ lies to the left of 1 and the left end of the interval $\hat{I}_k$ lies to the right of 1 or if the right end of the interval $\hat{I}_k$ lies to the left of 1 and the left end of the interval $\tilde{I}_k$ lies to the right of 1 on the real axis, then the set $D_{k,0}^\nu$ intersects the hyperplane $\{1\} \times E$. In this case for every $\nu \in \{+, -\}$ there exists a solution $(1, y) \in \mathbb{R} \times S_k^\nu$ of problem (3.10), which implies that there exists a solution $y \in S_k^\nu$ of problem (1.1)-(1.5).

Let $g_0 > 0$. Then it follows from the condition $g_0\, g_\infty > 0$ that $g_\infty > 0$. If the right end of the interval $\tilde{I}_k$ lies to the left of 1 and the left end of the interval $\hat{I}_k$ lies to the right of 1 on the real axis, then we have

$$\frac{\lambda_k}{g_0 r} + \frac{C_0}{g_0} < 1 < \frac{\lambda_k}{g_0 r} + 1 - \frac{g_\infty}{g_0} - \frac{C_\infty}{g_0}. \tag{4.1}$$

Due to the right-hand side of relation (4.1), we obtain

$$\frac{g_\infty + C_\infty}{g_0} < \frac{\lambda_k}{g_0 r},$$

whence we get $r > 0$, and consequently, the following relation holds:

$$r < \frac{\lambda_k}{g_\infty + C_\infty}\,. \tag{4.2}$$

By the left-hand side of relation (4.1), we get

$$\frac{\lambda_k}{r} < g_0 - C_0,$$

which implies that $g_0 > C_0$ and

$$\frac{\lambda_k}{g_0 - C_0} < r. \tag{4.3}$$

Then it follows from (4.2) and (4.3) that

$$\frac{\lambda_k}{g_0 - C_0} < r < \frac{\lambda_k}{g_\infty + C_\infty}. \tag{4.4}$$

Now the right end of the interval $\hat{I}_k$ lies to the left of 1 and the left end of the interval $\tilde{I}_k$ lies to the right of 1 on the real axis. Then we have

$$\frac{\lambda_k}{g_0 r} + 1 - \frac{g_\infty}{g_0} + \frac{C_\infty}{g_0} < 1 < \frac{\lambda_k}{g_0 r} - \frac{C_0}{g_0}. \tag{4.5}$$

By the right hand-side of (4.5) we get

$$g_0 + C_0 < \frac{\lambda_k}{r},$$

and consequently, $r > 0$ and

$$r < \frac{\lambda_k}{g_0 + C_0}. \tag{4.6}$$

By the left hand-side of (4.5), we have

$$\frac{\lambda_k}{r} < g_\infty - C_\infty \tag{4.7}$$

It follows from (4.7) that $g_\infty > C_\infty$ and

$$\frac{\lambda_k}{g_\infty - C_\infty} < r. \tag{4.8}$$

Then, by (4.6) and (4.8), we obtain

$$\frac{\lambda_k}{g_\infty - C_\infty} < r < \frac{\lambda_k}{g_0 + C_0}. \tag{4.9}$$

Thus, we have established the conditions under which problem (1.1)-(1.5) has nodal solutions.

Theorem 4.1. *Let $g_0 > C_0$ and for some $k \in \mathbb{N}$ condition (4.4) holds. Then there are solutions y_k^+ and y_k^- of problem (1.1)-(1.5) such that $y_k^+ \in S_k^+$ and $y_k^- \in S_k^-$. In this case the function y_k^+ has exactly $k-1$ simple nodal zeros in $(0, l)$ and is positive near $x = 0$, and the function y_k^- has also exactly $k-1$ simple nodal zeros in $(0, l)$ and is negative near $x = 0$.*

Theorem 4.2. *Let $g_\infty > C_\infty$ and for some $k \in \mathbb{N}$ condition (4.9) holds. Then there are solutions y_k^+ and y_k^- of problem (1.1)-(1.5) such that $y_k^+ \in S_k^+$ and $y_k^- \in S_k^-$.*

Similarly, we can show that the following results hold.

Theorem 4.3. *Let* $g_0 < -C_0$ *and for some* $k \in \mathbb{N}$ *condition* (4.9) *holds. Then there are solutions* y_k^+ *and* y_k^- *of problem* (1.1)-(1.5) *such that* $y_k^+ \in S_k^+$ *and* $y_k^- \in S_k^-$.

Theorem 4.4. *Let* $g_\infty < -C_\infty$ *and for some* $k \in \mathbb{N}$ *condition* (4.4) *holds. Then there are solutions* y_k^+ *and* y_k^- *of problem* (1.1)-(1.5) *such that* $y_k^+ \in S_k^+$ *and* $y_k^- \in S_k^-$.

References

[1] Z.S. Aliyev, Global bifurcation of solutions of certain nonlinear eigenvalue problems for ordinary differential equations of fourth order, *Sb. Math.* **207** (2016), no. 12, 1625–1649.
[2] Z.S. Aliyev, Y.N. Aliyeva, Existence of nodal solutions to some nonlinear boundary value problems for ordinary differential equations of fourth order, *Electron. J. Qual. Theory Differ. Equ.* (2024), no. 25, 1–13.
[3] Z.S. Aliyev, N.A. Mustafayeva, Bifurcation of solutions from infinity for certain nonlinear eigenvalue problems of fourth-order ordinary differential equations, *Electron. J. Differ. Equ.* (2018), no. 98, 1–19.
[4] Z.S. Aliyev, K.R. Rahimova, Existence of nodal solutions of some nonlinear Sturm-Liouville problems with a parameter in the boundary condition, *Azerb. J. Math* **15** (2025), no. 1, 243–256.
[5] Y.N. Aliyeva, Nodal solutions of some nonlinear fourth-order boundary value problems, *Proc. Inst. Math. Mech. Natl. Acad. Sci. Azerb.* **50** (2024), no. 1, 104–114.
[6] D.O. Banks, G. J. Kurowski, A Prüfer transformation for the equation of a vibrating beam subject to axial forces, *J. Differential Equations* **24** (1977), no. 1, 57–74.
[7] S. A. Janczewsky, Oscillation theorems for the differential boundary value problems of the fourth order, *Ann. Math.* **29** (1928), no. 2, 521–542.
[8] Y. Cui, J. Sun, Y. Zou, Global bifurcation and multiple results for Sturm-Liouville problems, *J. Comput. Appl. Math.* **235** (2011), no. 8, 2185–2192.
[9] R. Ma, Nodal solutions for a fourth-order two-point boundary value problem, *J. Math. Anal. Appl.* **314** (2006), no. 1, 254–265.
[10] R. Ma, B. Thompson, Nodal solutions for nonlinear eigenvalue problems, *Nonlinear Anal.* **59** (2004), no. 5, 707–718.
[11] R. Ma, B. Thompson, Nodal solutions for a nonlinear fourth-order eigenvalue problem, *Acta Math. Sin. Engl. Ser.* **24** (2008), no. 1, 27–34.
[12] R. Ma, G. Dai, Global bifurcation and nodal solutions for a Sturm-Liouville problem with a nonsmooth nonlinearity, *J. Funct. Anal.* **265** (2013), no. 8, 1443–1459.
[13] Y. Naito, S. Tanaka. On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations, *Nonlinear Anal.* **56** (2004), no. 4, 919–935.
[14] N.A. Neymatov, R.G. Poladov, Nodal solutions of certain boundary value problem for fourth order ordinary differential equations, *Caspian J. Appl. Math. Ecol. Econ.* **10** (2022), no. 2, 3–10.
[15] B.P. Rynne, Bifurcation from zero or infinity in Sturm-Liouville problems which are not linearizable, *J. Math. Anal. Appl.* **228** (1998), no. 1, 141–156.
[16] Tanaka S. On the uniqueness of solutions with prescribed numbers of zeros for a two-point boundary value problem. *Differential Integral Equations* **20** (2007), no. 1, 93–104.

Ziyatkhan S. Aliyev
Baku State University, Baku AZ1148, Azerbaijan
E-mail address: z_aliyev@mail.ru

Elchin A. Mammadov
Baku State University, Baku AZ1148, Azerbaijan
E-mail address: elchinmamedov88@mail.ru

Received: December 8, 2025; Revised: March 9, 2026; Accepted: March 16, 2026