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AN EXPLORATION OF THE STRUCTURAL PROPERTIES OF MULTI-PARAMETER HYBRID q-SPECIAL POLYNOMIALS RELATED TO APPELL SEQUENCES

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Abstract. In this work, we investigate a two-parameter family of q-Gould–Hopper-based Appell polynomials by means of their generating functions, series representations, determinant expressions, q-derivatives, and related operational identities. Along with these constructions, we discuss the corresponding fundamental properties and the underlying formalism associated with this polynomial family. Furthermore, within the framework of the quasi-monomial extension principle, we define suitable q-multiplicative and q-derivative operators and employ them to derive the differential equation satisfied by these polynomials.

1. Introduction and preliminary results

Over the past few decades, q-calculus has developed into a prominent and actively explored branch of mathematics, attracting considerable attention due to its deep connections with applied mathematics, mechanics, and mathematical physics. Acting as a natural extension of classical calculus, q-calculus reformulates familiar analytical concepts within a q-deformed setting. This transition has paved the way for several important developments, including the introduction of new notational frameworks and the derivation of meaningful results in areas such as combinatorics and number theory.

Central to this theory are q-special polynomials, among which the q-binomial coefficient $\binom{n}{k}_q$ and the q-Pochhammer symbol $(n;q)_n$ play a fundamental role. These objects form the backbone of q-analogues, allowing classical notions to be systematically transferred into the realm of q-calculus. Their utility extends across a wide spectrum of mathematical and physical disciplines, including combinatorial analysis, representation theory, and statistical mechanics. By means of these tools, intricate structures can be examined more effectively, leading to deeper insights and novel analytical techniques. Owing to their rich algebraic and analytic characteristics, q-special polynomials have proven to be indispensable in

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the investigation of various q-deformed phenomena encountered in both pure and applied contexts.

Although many applications of q-calculus originate in advanced physical theories, the theoretical development of the subject depends crucially on clear and consistent notation.

Throughout this work, the symbol $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{N}$ represents the set of natural numbers, and $\mathbb{N}_0$ stands for the set of non-negative integers. The parameter $q \in \mathbb{C}$ is assumed to satisfy the condition $|q| < 1$, ensuring that it lies inside the unit disk of the complex plane. All q-notations and conventions adopted in this paper follow those commonly used in the literature (see [12, 13, 18, 32, 31]).

The q-analogue of a complex number γ is defined by

$$[\gamma]_q = \frac{1-q^\gamma}{1-q}.$$

From a historical perspective, the theory of q-series can be traced back to the work of Christian Heine, who, in the mid-eighteenth century, proposed series expansions in which the classical normalization factor $n!$ was replaced by the polynomial $(q;q)_n$, a polynomial of degree n in q. This idea substantially widened the applicability of series methods, allowing them to capture a broader class of functions and mathematical phenomena.

The q-factorial is given by (see [5, 18])

$$[n]_q! = \frac{(q;q)_n}{(1-q)^n} \; (n \in \mathbb{N}) \text{ and } \; [0]_q! := 1,$$

where

$$(q;q)_\rho := \begin{cases} (1-q)(1-q^2)\cdots(1-q^\rho), & ; \; \rho \geq 1, \\ 1, & ; \; \rho = 0. \end{cases}$$

The two standard q-exponential functions are defined as [19]

$$e_q(x) = \frac{1}{(x(1-q);q)_\infty} = \sum_{l=0}^{\infty} \frac{x^l}{[l]_q!}, \qquad |x| < \frac{1}{1-q}, \tag{1.1}$$

and

$$E_q(x) = (-x(1-q);q)_\infty = \sum_{l=0}^{\infty} q^{\binom{l}{2}} \frac{x^l}{[l]_q!}, \qquad x \in \mathbb{C}.$$

These q-exponential functions are closely related and satisfy the identity (see [5, 18])

$$e_q(x)E_q(-x) = 1, \qquad |x| < \frac{1}{1-q}.$$

The q-derivative of a function f with respect to the variable x is defined by [19]

$$D_{q,x} f(x) = \frac{f(qx) - f(x)}{qx - x}, \qquad x \neq 0,$$

with $\left(\hat{D}_{q,x} f\right)(0) = f'(0)$.

Some useful properties of the q-derivative [1, 30] are recalled below:

$$D_{q,x} \; x^n = [n]_q x^{n-1}, \tag{1.2}$$

$$D_{q,x}e_q(\alpha x) = \alpha e_q(\alpha x), \qquad \alpha \in \mathbb{C}, \tag{1.3}$$

together with the product rule [19]

$$D_{q,x}\big(f(x)g(x)\big) = f(x)D_{q,x}g(x) + g(qx)D_{q,x}f(x). \tag{1.4}$$

The q-Gould–Hopper polynomials form a notable subclass of q-orthogonal polynomials obtained by introducing a deformation parameter q into the classical Gould–Hopper polynomials. They occupy an important position within the family of classical q-polynomials [26]. When $q = 1$, these polynomials reduce to their classical counterparts. Moreover, they arise naturally in the study of q-generalized differential equations, including q-extensions of the heat equation, thereby enhancing their relevance in both mathematical analysis and physical modeling.

The q-Gould–Hopper polynomials $H_{n,q}^{(m)}(x, y)$ are defined through the generating function [26]

$$e_q(xt)e_q(yt^m) = \sum_{n=0}^{\infty} H_{n,q}^{(m)}(x, y)\frac{t^n}{[n]_q!}.$$

They also admit the operational representation [15, 26]

$$H_{n,q}^{(m)}(x, y) = e_q(yD_{q,x}^m)x^n.$$

Furthermore, Al-Salam introduced the generating function of the q-Appell polynomials $\mathcal{A}_{n,q}(\zeta)$ in the form [3, 4] (see also [28])

$$\mathcal{A}_q(t)\, e_q(\zeta t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q}(\zeta)\frac{t^n}{[n]_q!},$$

where

$$\mathcal{A}_q(t) = \sum_{n=0}^{\infty} \mathcal{A}_{n,q}\frac{t^n}{[n]_q!}, \qquad \mathcal{A}_q(t) \neq 0, \qquad A_{0,q} = 1. \tag{1.5}$$

The study of hybrid special polynomials constructed through q-calculus has received considerable attention in recent years due to their ability to unify various families of classical polynomials within a common operational framework. In particular, Appell-type polynomial sequences play a fundamental role in both classical and modern analysis because of their characteristic derivative property and their connections with generating functions, operational calculus, and combinatorial identities. When these structures are extended into the q-calculus setting, they provide a natural bridge between continuous and discrete analytical models (see [2, 20, 22, 23, 24, 30, 33]).

The q-Gould–Hopper polynomials represent an important generalization of classical Gould–Hopper polynomials and have been investigated in connection with q-difference equations, operational identities, and generalized generating functions. Motivated by these developments, it is natural to explore hybrid constructions in which the structural properties of Appell sequences are combined with the analytical flexibility of q-special functions. Such constructions lead to polynomial families possessing rich algebraic structures, including recurrence relations, operational identities, and determinant representations.

The purpose of the present work is therefore to introduce and investigate a multi-parameter hybrid class of q-special polynomials related to Appell sequences.

The proposed family extends several known polynomial systems and provides a unified framework for deriving structural identities, q-differential relations, and operational representations. These properties contribute to the broader theory of q-special functions and open possibilities for further applications in difference equations, combinatorial analysis, and mathematical physics.

The structure of this article is organized as follows. The paper begins with the introduction of a two-parameter family of bivariate q-Gould–Hopper–Appell polynomials and discusses their connections with well-known classes of q-polynomials. In Section 2, several fundamental properties are established, including series representations, determinant forms, and q-recurrence relations. Section 3 investigates these polynomials from the perspective of the monomiality principle, shedding light on their algebraic framework. A collection of summation identities is presented in Section 4. Section 5 is devoted to numerical experiments illustrating the zero distributions and mesh behavior of the polynomials. Finally, the concluding section summarizes the main findings and outlines possible directions for future research.

2. q-Gould-Hopper-based Appell polynomials with two parameters

In this section, we introduce the q-Gould–Hopper based Appell polynomials with two parameters (qGHAP2P) ${}_HA^{(m)}_{n,q}(x,y;s_1,s_2)$ by means of an appropriate generating function. We then present their series form and the associated q-derivative relations. After establishing these foundational facts, we derive the corresponding properties for the q-Gould–Hopper based Appell polynomials with two parameters (qGHAP2P) ${}_HA^{(m)}_{n,q}(x,y;s_1,s_2)$.

Let $x,y\in\mathbb{R}$ and $s_1,s_2\in\mathbb{C}$. We define q-Gould-Hopper-based Appell polynomials with two parameters (qGHAP2P) ${}_HA^{(m)}_{n,q}(x,y;s_1,s_2)$ as:

$$A_q(t)e_q(xs_1t)e_q(ys_2t^m)=\sum_{n=0}^{\infty}{}_HA^{(m)}_{n,q}(x,y;s_1;s_2)\frac{t^n}{[n]_q!}. \tag{2.1}$$

The generating function introduced above provides a convenient analytical tool for investigating the structural properties of the proposed polynomial family. Through this representation, one can systematically derive explicit series expansions, recurrence relations, and operational identities associated with the sequence. Moreover, the presence of multiple parameters allows the proposed polynomials to encompass several previously studied families as particular cases, thereby revealing deeper structural connections within the theory of q-special functions.

For $A_q(t)=1$ in (2.1), we get the q-Gould-Hopper polynomials with two parameters as:

$$e_q(xs_1t)e_q(ys_2t^m)=\sum_{n=0}^{\infty}H^{(m)}_{n,q}(x,y;s_1;s_2)\frac{t^n}{[n]_q!}. \tag{2.2}$$

By employing equation (1.5) to expand the generating relation in (2.1), and then matching coefficients of equal powers of t on both sides, we obtain the following series representation for the qGHAP2P polynomials ${}_HA^{(m)}_{n,q}(x,y;s_1;s_2)$:

$$ {}_{H}A^{(m)}_{n,q}(x,y;s_1;s_2) = \sum_{k=0}^{n} \binom{n}{k}_q A_{k,q} H^{(m)}_{n-k,q}(x,y;s_1,s_2). \tag{2.3} $$

Determinant representations are particularly significant in the study of q-special polynomials, since they provide a concise and structurally transparent description of these families. Such formulations often serve as an efficient way to encode key information, including recurrence schemes, generating mechanisms, and other structural traits that are central to both theoretical investigations and symbolic computations. In addition, the determinantal viewpoint reveals natural links with wider algebraic and combinatorial settings, thereby offering a deeper understanding of the underlying structure of q-special functions. Alongside computational convenience, these representations also contribute meaningfully to the broader development of the theory of special functions.

Keleshteri and Mahmudov [21], building upon the approaches described in [7, 8], derived a determinant form for the q-Appell polynomials. Motivated by the usefulness of such expressions in analysis and applications, they extended their technique to obtain the determinantal representation for the q-Gould–Hopper–Appell polynomials, denoted by ${}_{H}A^{(m)}_{n,q}(x,y;s_1;s_2)$. Their derivation relies on a fundamental theorem that yields a clear and systematically arranged determinant expression, thereby strengthening both the applicability and the theoretical understanding of these polynomial sequences.

Theorem 2.1. *The determinant representation of q-Gould-Hopper based Appell polynomials with two parameter ${}_{H}A^{(m)}_{n,q}(x,y;s_1;s_2)$ of degree n is*

$$ {}_{H}A^{(m)}_{0,q}(x,y;s_1;s_2) = \frac{1}{\beta_{0,q}}, $$

$$ {}_{H}A^{(m)}_{n,q}(x,y;s_1;s_2) = \frac{(-1)^n}{(\beta_{0,q})^{n+1}} $$

$$ \times \begin{vmatrix} 1 & H^{(m)}_{1,q}(x,y;s_1;s_2) & H^{(m)}_{2,q}(x,y;s_1;s_2) & \cdots & H^{(m)}_{n-1,q}(x,y;s_1;s_2) & H^{(m)}_{n,q}(x,y;s_1;s_2) \\ \beta_{0,q} & \beta_{1,q} & \beta_{2,q} & \cdots & \beta_{n-1,q} & \beta_{n,q} \\ 0 & \beta_{0,q} & \binom{2}{1}_q \beta_{1,q} & \cdots & \binom{n-1}{1}_q \beta_{n-2,q} & \binom{n}{1}_q \beta_{n-1,q} \\ 0 & 0 & \beta_{0,q} & \cdots & \binom{n-1}{1}_q \beta_{n-3,q} & \binom{n}{2}_q \beta_{n-2,q} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \\ 0 & 0 & 0 & \cdots & \beta_{0,q} & \binom{n}{n-1}_q \beta_{1,q} \end{vmatrix}, $$

$$ \beta_{n,q} = -\frac{1}{A_{0,q}} \left(\sum_{k=1}^{n} \binom{n}{k}_q A_{k,q} \beta_{n-k,q} \right), \quad n = 0,1,2,\ldots, $$

where $\beta_{0,q} \neq 0$, $\beta_{0,q} = \frac{1}{A_{0,q}}$ and $H^{(m)}_{n,q}(x,y;s_1;s_2)$, $n = 0,1,2,\ldots,$ are the q-Gould-Hopper polynomials with two parameter defined by equation (2.3).

Proof. Inserting the series expansions of the q-Gould-Hopper polynomials into the generating function of the q-Gould-Hopper-Appell polynomials, we obtain

$$ A_q(t) \sum_{n=0}^{\infty} H^{(m)}_{n,q}(x,y;s_1;s_2) \frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} {}_{H}A^{(m)}_{n,q}(x,y;s_1;s_2) \frac{t^n}{[n]_q!}. $$

Multiplying both sides by

$$\frac{1}{A_q(t)}=\sum_{k=0}^{\infty}\beta_{k,q}\frac{t^k}{[k]_q!},$$

we arrive at

$$\sum_{n=0}^{\infty}H_{n,q}^{(m)}(x,y;s_1;s_2)\frac{t^n}{[n]_q!}=\sum_{k=0}^{\infty}\beta_{k,q}\frac{t^k}{[k]_q!}\sum_{n=0}^{\infty}{}_HA_{n,q}^{(m)}(x,y;s_1;s_2)\frac{t^n}{[n]_q!}. \tag{2.4}$$

Using the Cauchy product formula in (2.4), we get

$$H_{n,q}^{(m)}(x,y;s_1;s_2)=\sum_{k=0}^{n}\binom{n}{k}_q\beta_{k,q}\ {}_HA_{n-k,q}(x,y;s_1;s_2).$$

This relation yields a system of n linear equations in the unknowns ${}_HA_{n,q}(x,y;s_1;s_2)$, $n=0,1,2,\cdots$. Applying Cramer's rule to this system, and observing that the denominator is the determinant of a lower triangular matrix equal to $(\beta_{0,q})^{,n+1}$, we proceed by transposing the numerator and then shifting the i^{th} row to the $(i+1)$-th position for $i=1,2,\ldots,n-1$. Carrying out these steps produces the desired determinant representation. □

Theorem 2.2. *For qGHAP2P ${}_HA_{n,q}^{(m)}(x,y;s_1;s_2)$, the following q-partial derivative relations hold:*

$$D_{q,x}{}_HA_{n,q}^{(m)}(x,y;s_1;s_2)=s_1{}_HA_{n-1,q}^{(m)}(x,y;s_1;s_2),\qquad n\geq 1, \tag{2.5}$$

$$D_{q,y}{}_HA_{n,q}^{(m)}(x,y;s_1;s_2)=s_2{}_HA_{n-m,q}(x,y;s_1;s_2),\qquad n\geq m, \tag{2.6}$$

$$D_{q,s_1}{}_HA_{n,q}^{(m)}(x,y;s_1;s_2)=x{}_HA_{n-1,q}(x,y;s_1;s_2),\qquad n\geq 1, \tag{2.7}$$

$$D_{q,s_2}{}_HA_{n,q}^{(m)}(x,y;s_1;s_2)=y{}_HA_{n-m,q}(x,y;s_1;s_2),\qquad n\geq m. \tag{2.8}$$

Proof. Taking the q-partial derivative of both sides of (2.1) with respect to x and using equation (1.3), we obtain

$$\sum_{n=0}^{\infty}D_{q,x}{}_HA_{n,q}^{(m)}(x,y;s_1;s_2)t^n=s_1tA_q(t)e_q(xs_1t)e_q(ys_2t^m). \tag{2.9}$$

Applying (2.1) to the right-hand side of (2.9), we get

$$\sum_{n=0}^{\infty}D_{q,x}{}_HA_{n,q}^{(m)}(x,y;s_1;s_2)\frac{t^n}{[n]_q!}=s_1\sum_{n=0}^{\infty}{}_HA_{n,q}^{(m)}(x,y;s_1;s_2)\frac{t^{n+1}}{[n]_q!}.$$

Comparing coefficients of like powers of t on both sides leads directly to (2.5).

Next, by taking the q-partial derivatives of (2.1) with respect to y, s_1, and s_2, and repeating the same coefficient-comparison procedure used for (2.5), we obtain (2.6) and (2.7) (and similarly (2.8)).

Theorem 2.2 is proved. □

Remark 2.1. We obtain the generating function for qGHAP1P ${}_HA_{n,q}^{(m)}(x,y;s_1)$ by taking $s_2=1$ in equation (2.1) as follows:

$$A_q(t)e_q(xs_1t)e_q(yt^m)=\sum_{n=0}^{\infty}{}_HA_{n,q}^{(m)}(x,y;s_1)\frac{t^n}{[n]_q!}. \tag{2.10}$$

Expanding the above identity via (1.1) and then equating coefficients of identical powers of t on both sides yields the series representation of qGHAP1P ${}_HA^{(m)}_{n,q}(x,y;s_1)$:

$$ {}_HA^{(m)}_{n,q}(x,y;s_1)=[n]_q!\sum_{r=0}^{[n/m]}\frac{y^{n-mr}A_{r,q}(xs_1)}{[r]_q![n-mr]_q!}. \tag{2.11}$$

Moreover, differentiating (2.5) with respect to x, y, and s_1, we obtain the following q-derivative relations for the qGHAP1P family ${}_HA^{(m)}_{n,q}(x,y;s_1)$:

$$D_{q,x}\,{}_HA^{(m)}_{n,q}(x,y;s_1)=s_1\,{}_HA^{(m)}_{n-1,q}(x,y;s_1),\qquad n\geq 1,$$

$$D_{q,y}\,{}_HA^{(m)}_{n,q}(x,y;s_1)={}_HA^{(m)}_{n-2,q}(x,y;s_1),\qquad n\geq 2$$

and

$$D_{q,s_1}\,{}_HA^{(m)}_{n,q}(x,y;s_1)=x\,{}_HA_{n-1,q}(x,y;s_1),\qquad n\geq 1.$$

From the discussion above, it is evident that operational identities provide an effective route for analyzing q-special polynomials, since they lead to explicit operational descriptions for both the qGHAP1P family ${}_HA_{n,q}(x,y;s_1)$ and the qGHAP2P family ${}_HA_{n,q}(x,y;s_1,s_2)$.

We now state the following result:

Theorem 2.3. *The q-Gould-Hopper based Appell polynomials with two-parameter ${}_HA^{(m)}_{n,q}(x,y;s_1;s_2)$ and q-Gould-Hopper based Appell polynomials with 1 parameter ${}_HA_{n,q}(x,y;s_1)$ satisfy the following respective operational identities:*

$$ {}_HA^{(m)}_{n,q}(x,y;s_1;s_2)=e_q(s_2yD^m_{q,x})\ \{A_{n,q}(s_1x)\} \tag{2.12}$$

and

$$ {}_HA^{(m)}_{n,q}(x,y;s_1;)=e_q(yD^m_{q,x})\ \{A_{n,q}(s_1x)\}, \tag{2.13}$$

or

$$ {}_HA^{(m)}_{n,q}(x,y;s_1)=e_q(s_1yD^m_{q,x})\ \{A_{n,q}(x)\},$$

where $D^m_{q,x}$ is the m^{th} q-derivative operators.

Proof. In view of equation (1.2), we have

$$D^{mr}_{q,x}\ x^n=\frac{[n]_q!}{[n-mr]_q!}x^{n-mr}.$$

Using this identity in conjunction with (2.2), we obtain

$$ {}_HA^{(m)}_{n,q}(x,y;s_1;s_2)=A_q(t)\sum_{r=0}^{\infty}\frac{\left(s_2yD^m_{q,x}\right)^r s_1^nx^n}{[r]_q![n]_q!}.$$

Invoking expression (1.5) on the right-hand side gives (2.12). Again, applying the same argument as in (2.12) and using (1.5), we obtain (2.13).

Theorem 2.3 is proved. □

The operational descriptions (2.10) and (2.11) make the investigation of qGHAP2P ${}_HA^{(m)}_{n,q}(x,y;s_1;s_2)$ and qGHAP1P ${}_HA^{(m)}_{n,q}(x,y,z;s_1)$ considerably more direct, and they also provide a convenient starting point for studying their extensions and related families. With these definitions in hand, we can establish parallel operational relations for qGHAP2P ${}_HA^{(m)}_{n,q}(x,y;s_1;s_2)$ and qGHAP1P ${}_HA^{(m)}_{n,q}(x,y,z;s_1)$.

Theorem 2.4. *The qGHAP2P ${}_HA^{(m)}_{n,q}(x,y;s_1;s_2)$ and qGHAP1P ${}_HA^{(m)}_{n,q}(x,y,z;s_1)$ satisfy the following equivalent operational identities:*

$$E_q(-s_2yD^m_{q,x})_HA^{(m)}_{n,q}(x,y;s_1;s_2) = A_{n,q}(s_1x). \tag{2.14}$$

$$E_q(-s_1yD^m_{q,x})_HA^{(m)}_{n,q}(x,y;s_1) = A_{n,q}(x). \tag{2.15}$$

Proof. By the operational identities (2.12) and (2.13), we immediately obtain (2.14) and (2.15). □

The monomiality principle has proved to be an effective analytical tool in the study of polynomial sequences associated with operational calculus. In recent years, several authors have extended this principle to the framework of q-calculus in order to analyze hybrid polynomial families and their operator structures. In particular, the q-Gould–Hopper-based polynomial families have been investigated in connection with generalized Appell sequences, operational identities, and differential-type equations. These developments motivate the extension of the monomiality framework to the present class of multi-parameter hybrid q-special polynomials.

3. Monomiality principle

The concept of monomiality dates back to 1941, when Steffenson introduced the idea of poweroids [29], which was later further developed and systematized by Dattoli [9, 10, 11]. Over time, the monomiality principle has emerged as a fundamental pillar in the theory of polynomials, particularly in the study of special functions. It provides a decisive framework for guaranteeing orthogonality and completeness within polynomial families, both of which are essential for a thorough theoretical understanding and effective practical implementation. Verifying that a given family of polynomials satisfies the monomiality principle is therefore of central importance, as it underpins their applicability in diverse mathematical and computational problems such as approximation theory, numerical analysis, and the solution of differential equations. Beyond ensuring structural consistency, this principle plays a key role in deriving recurrence relations, operational identities, and explicit representations. As a result, polynomial sequences that adhere to monomiality become more manageable and robust tools for addressing intricate mathematical problems, facilitating efficient computations and offering a coherent pathway for developing systematic solutions across a wide spectrum of scientific applications.

In recent years, several authors [6, 25, 27] have extended the monomiality principle to the setting of q-polynomials, thereby revealing the quasi-monomial nature of a number of important q-special polynomial families. This generalized approach has significantly enriched the understanding of their algebraic and operational structures. In particular, it has proved highly effective in the analysis of q-Hermite polynomials, as discussed in [15, 16, 17], as well as general Appell polynomials, as shown in [28]. Within this framework, the operators $\hat{\mathcal{M}}_q$ and $\hat{\mathcal{D}}_q$ act as the multiplicative and derivative operators, respectively, on the polynomial sequence $\{\Phi_{m,q}(x)\}_{m\in\rho}$. These polynomials are characterized by the relations

$$\Phi_{m+1,q}(x) = \hat{\mathcal{M}}_q\{\Phi_{m,q}(x)\} \tag{3.1}$$

and

$$[m]_q\ \Phi_{m-1,q}(x) = \hat{\mathcal{D}}_q\{\Phi_{m,q}(x)\}. \tag{3.2}$$

The operators governing the quasi-monomial sequence $\{\Phi_{m,q}(x)\}_{m\in\rho}$ are required to satisfy the commutation relation

$$[\hat{\mathcal{D}}_q, \hat{\mathcal{M}}_q] = \hat{\mathcal{D}}_q\hat{\mathcal{M}}_q - \hat{\mathcal{M}}_q\hat{\mathcal{D}}_q.$$

The defining characteristics of the quasi-monomial family $\{\Phi_{m,q}(x)\}_{m\in\rho}$ are closely tied to the intrinsic properties of the operators $\hat{\mathcal{M}}_q$ and $\hat{\mathcal{D}}_q$. These operators determine how the polynomials behave under algebraic and differential actions, ensuring that the sequence satisfies a prescribed set of axioms. Such axioms form the structural backbone of the theory, dictating the response of the polynomial family to various transformations. Consequently, the mathematical properties exhibited by $\{\Phi_{m,q}(x)\}$ arise naturally from the interaction between $\hat{\mathcal{M}}_q$ and $\hat{\mathcal{D}}_q$, reflecting an underlying algebraic coherence described below:

(i) The polynomial $\Phi_{m,q}(x)$ satisfies the following differential-type relations:

$$\hat{\mathcal{M}}_q\hat{\mathcal{D}}_q\{\Phi_{m,q}(x)\} = [m]_q\ \Phi_{m,q}(x) \tag{3.3}$$

and

$$\hat{\mathcal{D}}_q\hat{\mathcal{M}}_q\{\Phi_{m,q}(x)\} = [m+1]_q\ \Phi_{m,q}(x), \tag{3.4}$$

whenever $\hat{\mathcal{M}}_q$ and $\hat{\mathcal{D}}_q$ admit differential realizations.

(ii) The explicit representation of $\Phi_{m,q}(x)$ is given by

$$\Phi_{m,q}(x) = \hat{\mathcal{M}}_q^{\ m}\ \{1\}, \tag{3.5}$$

with the normalization $\Phi_{0,q}(x) = 1$.

(iii) The exponential generating function of $\Phi_{m,q}(x)$ takes the form

$$\mathfrak{E}_q\Big\{t\hat{\mathcal{M}}_q\Big\}\{1\} = \sum_{m=0}^{\infty} \Phi_{m,q}(x)\frac{t^m}{[m]_q!},\ \ |t| < \infty\ ,$$

which follows directly from the identity in (3.5).

The q-dilatation operator $\mathcal{T}_\nu$ acts on functions of the complex variable γ according to [14]

$$\mathcal{T}_\nu^m(h(\gamma)) = h(q^m\gamma) \tag{3.6}$$

and satisfies the identity

$$\mathcal{T}_\nu^{-1}\mathcal{T}_\nu^1(h(\gamma)) = h(\gamma),$$

where q denotes a fixed complex parameter. Through this action, the operator rescales the argument of the function by a factor of q, thereby modifying its analytical behavior in a controlled manner.

The following equality holds [6]:

$$\widehat{D}_{q,t}e_q(\phi t^m) = \phi t^{m-1}T_{(\phi;m)}e_q(\phi t^m).$$

where

$$T_{(\phi;m)} = \frac{1-q^mT_\phi^m}{1-qT_\phi} = 1+qT_\phi+\cdots+q^{m-1}T_\phi^{m-1}.$$

The monomiality principle serves as a central mechanism for defining both raising and lowering operators within this theoretical setting. To illustrate its effectiveness, we now present key results that clarify the operational structure and behavior of the polynomial family under consideration.

We proceed to establish the q-multiplicative and q-derivative operators associated with ${}_HA_{n,q}^{(m)}(x,y;s_1;s_2)$, as stated in the following theorem.

Theorem 3.1. *The q-Gould–Hopper based Appell polynomials with two parameters* ${}_HA_{n,q}^{(m)}(x,y;s_1;s_2)$ *form a quasi-monomial sequence with respect to the following q-multiplicative and q-derivative operators:*

$$\widehat{M}_{2VqGHAP} = \left(xs_1T_y + ys_2\widehat{D}_{q,x}^{m-1}T_{(y;m)}\right)\frac{A_q(q\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})} + \frac{A_q'(\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})}, \tag{3.7}$$

or, equivalently,

$$\widehat{M}_{2VqGHAP} = \left(xs_1 + ys_2\widehat{D}_{q,x}^{m-1}T_{(y;m)}T_x\right)\frac{A_q(q\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})} + \frac{A_q'(\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})}, \tag{3.8}$$

and

$$\widehat{P}_{2VqGHAP} = \widehat{D}_{q,x}, \tag{3.9}$$

respectively, where T_x and T_y denote the q-dilatation operators defined in (3.6).

Proof. Taking the q-derivative of both sides of equation (2.1) with respect to t and applying (1.4), we obtain

$$\sum_{n=1}^{\infty} {}_HA_{n,q}^{(m)}(x,y;s_1;s_2)\widehat{D}_{q,t}\frac{t^n}{[n]_q!}$$

$$= A_q(qt)e_q\widehat{D}_{q,t}\left(e_q(xs_1t)e_q(ys_2t^m)\right) + e_q(xs_1t)e_q(ys_2t^m)A_q'(t). \tag{3.10}$$

Using (1.4) with $f(t)=e_q(xs_1t)e_q(ys_2t^m)$ and $g(t)=A_q(t)$, and simplifying the resulting expression by means of (3.1), (3.2), and (3.6), the left-hand side reduces to

$$\left(\left(xs_1T_y + ys_2t^{m-1}T_{(y;m)}\right)\frac{A_q(qt)}{A_q(t)} + \frac{A_q'(t)}{A_q(t)}\right)A_q(t)e_q(\zeta t)e_q(\eta t^m)$$

$$= \sum_{n=1}^{\infty} {}_HA_{n,q}^{(m)}(x,y;s_1;s_2)\frac{t^{n-1}}{[n-1]_q!}.$$

Since

$$\widehat{D}_{q,x}A_q(t)e_q(xs_1t)e_q(ys_2t^m) = ts_1A_q(t)e_q(xs_1t)e_q(ys_2t^m),$$

and $\frac{A_q'(t)}{A_q(t)}$ admits a q-power series expansion in t due to the invertibility of $A_q(t)$, it follows from (2.1) that

$$\left(\left(xs_1T_y + ys_2\widehat{D}_{q,x}^{m-1}T_{(y;m)}\right)\frac{A_q(q\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})} + \frac{A_q'(\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})}\right)$$
$$\times\sum_{n=0}^{\infty}{}_HA_{n,q}^{(m)}(x,y;s_1;s_2)\frac{t^n}{[n]_q!} = \sum_{n=1}^{\infty}{}_HA_{n,q}^{(m)}(x,y;s_1;s_2)\frac{t^{n-1}}{[n-1]_q!}.$$

By equating the coefficients of like powers of t on both sides of (3.8) and invoking (3.1), assertion (3.7) follows.

Similarly, by applying (3.10) with $f_q(t) = e_q(ys_2t^m)$ and $g_q(t) = e_q(xs_1t)$ and proceeding along the same lines, assertion (3.8) is obtained.

Finally, assertions (3.3) and (3.4) immediately yield (3.9). □

Theorem 3.2. *The polynomial ${}_HA_{n,q}^{(m)}(x,y;s_1;s_2)$ satisfies the following q-differential equations:*

$$\left(xs_1\widehat{D}_{q,x}\frac{A_q(q\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})}T_y + ys_2\widehat{D}_{q,x}^{m}T_{(\eta;m)}\frac{A_q(q\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})} + \frac{A_q'(\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})}\widehat{D}_{q,x} - [n]_q\right)$$
$$\times\ {}_HA_{n,q}^{(m)}(x,y;s_1;s_2) = 0, \quad (3.11)$$

and

$$\left(xs_1\widehat{D}_{q,x}\frac{A_q(q\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})} + ys_2\widehat{D}_{q,x}^{m}T_{(y;m)}\frac{A_q(q\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})}T_x + \frac{A_q'(\widehat{D}_{q,x})}{A_q(\widehat{D}_{q,x})}\widehat{D}_{q,x} - [n]_q\right)$$
$$\times\ {}_HA_{n,q}^{(m)}(x,y;s_1;s_2) = 0. \quad (3.12)$$

Proof. Substituting (3.7), (3.8), and (3.9) into (3.3) directly yields the desired results (3.11) and (3.12). □

4. Summation formulae

Summation formulas occupy a central position in the theory of polynomials, as they offer concise representations that simplify analysis and enhance computational efficiency. Beyond their compact form, such formulas uncover intrinsic structural features of polynomial families and serve as effective tools for deriving recurrence relations, generating functions, and important special cases. As a consequence, they provide deeper insight into related combinatorial identities and differential equations. Moreover, summation identities often reveal connections with other well-known polynomial systems, thereby broadening their scope of applications in areas such as mathematical physics, number theory, and approximation theory. From a practical viewpoint, these formulas also facilitate both analytical manipulations and numerical computations, making them indispensable in a wide range of mathematical and scientific studies.

In this section, we derive implicit summation representations for the q-Gould–Hopper polynomials with two parameter, denoted by ${}_HA_{n,q}^{(m)}(x,y;s_1;s_2)$. The detailed derivations are presented below.

Theorem 4.1. *The summation formula for the q-Gould–Hopper based Appell polynomials with two parameters* ${}_HA^{(m)}_{n,q}(x,y;s_1;s_2)$ *is given by*

$$ {}_HA^{(m)}_{n,q}(x+u,y;s_1;s_2) = \sum_{k=0}^{n} \binom{n}{k}_q u^k s_1^k {}_HA^{(m)}_{n-k,q}(x,y;s_1;s_2). \tag{4.1} $$

Proof. Replacing x by $x+u$ in (2.1), we obtain

$$ \sum_{n=0}^{\infty} {}_HA^{(m)}_{n,q}(x+u,y;s_1;s_2)\frac{t^n}{[n]_q!} = A_q(t)e_q((x+u)s_1t)e_q(ys_2t^m), $$

which can be rewritten as

$$ = A_q(t)e_q(xs_1t)e_q(ys_2t^m)e_q(us_1t). $$

Expanding the second exponential term on the right-hand side and invoking (2.1), we arrive at

$$ \sum_{n=0}^{\infty} {}_HA^{(m)}_{n,q}(x+u,y;s_1;s_2)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_HA^{(m)}_{n,q}(x,y;s_1;s_2)u^k s_1^k \frac{t^{n+k}}{[n]_q![k]_q!}. $$

By applying the series rearrangement technique, the above expression becomes

$$ \sum_{n=0}^{\infty} {}_HA^{(m)}_{n,q}(x+u,y;s_1;s_2)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}_q u^k s_1^k {}_HA^{(m)}_{n-k,q}(x,y;s_1;s_2)\right)\frac{t^n}{[n]_q!}. $$

Finally, comparing the coefficients of identical powers of t on both sides yields the desired result in (4.1). □

Theorem 4.2. *The following summation formula holds for the q-Gould–Hopper based Appell polynomials with two parameters* ${}_HA^{(m)}_{n,q}(x,y;s_1;s_2)$*:*

$$ {}_HA^{(m)}_{n,q}(x+v,y+w;s_1;s_2) = \sum_{k=0}^{n}\binom{n}{k}_q {}_HA^{(m)}_{n-k,q}(x,y;s_1;s_2)H^{(m)}_{k,q}(v,w;s_1;s_2). \tag{4.2} $$

Proof. Substituting x by $x+v$ and y by $y+w$ in (2.1), we find

$$ \sum_{n=0}^{\infty} {}_HA^{(m)}_{n,q}(x+v,y+w;s_1;s_2)\frac{t^n}{[n]_q!} = A_q(t)e_q((x+v)s_1t)e_q((y+w)s_2t^m). $$

Using the series representation, this leads to

$$ = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} {}_HA^{(m)}_{n-k,q}(x,y;s_1;s_2)H^{(m)}_{k,q}(v,w;s_1;s_2)\frac{t^{n+k}}{[n]_q![k]_q!}. $$

Rearranging the series yields

$$ = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}_q {}_HA^{(m)}_{n-k,q}(x,y;s_1;s_2)H^{(m)}_{k,q}(v,w;s_1;s_2)\frac{t^n}{[n]_q!}. $$

Finally, equating the coefficients of equal powers of t on both sides establishes the identity given in (4.2). □

5. Examples

In this section, we examine some particular instances of the family of q-Gould–Hopper–based Appell polynomials with two parameters defined by (2.1). The discussion below highlights appropriate choices that lead to well-known subclasses within the broader class of q-Appell polynomials.

Table 1.1 Several q-Appell polynomial families

S. No.	q-Appell Polynomials	Generating function	$\mathcal{A}_q(t)$
I.	**The q-Bernoulli Polynomials** [4, 12]	$\frac{t}{e_q(t)-1}e_q(xt)=\sum_{n=0}^{\infty}\mathcal{B}_{n,q}(x)\frac{t^n}{[n]_q!}$	$\mathcal{A}_q(t)=\frac{t}{e_q(t)-1}$
II.	**The q-Euler Polynomials** [4, 12]	$\frac{[2]_q}{e_q(t)+1}e_q(xt)=\sum_{n=0}^{\infty}\mathcal{E}_{n,q}(x)\frac{t^n}{[n]_q!}$	$\mathcal{A}_q(t)=\frac{[2]_q}{e_q(t)+1}$
III.	**The q-Genocchi Polynomials** [4, 12]	$\frac{[2]_q t}{e_q(t)+1}e_q(xt)=\sum_{n=0}^{\infty}\mathcal{G}_{n,q}(x)\frac{t^n}{[n]_q!}$	$\mathcal{A}_q(t)=\frac{[2]_q t}{e_q(t)+1}$

When the variable is specialized to $x=0$, the elements of the q-Appell polynomial family $\mathcal{A}_{n,q}(x)$ naturally reduce to the corresponding q-numbers $\mathcal{A}_{n,q}$. In Table 1.2, we collect the initial values of several classical sequences arising in this framework, namely the q-Bernoulli numbers $\mathcal{B}_{n,q}$ [4, 12], the q-Euler numbers $\mathcal{E}_{n,q}$ [4, 12, 13], and the q-Genocchi numbers $\mathcal{G}_{n,q}$ [4, 12].

Table 1.2. The first five q-numbers $\mathcal{B}_{n,q}$, $\mathcal{E}_{n,q}$ and $\mathcal{G}_{n,q}$

n	0	1	2	3	4
$\mathcal{B}_{n,q}$ [12, 13]	1	$-(1+q)^{-1}$	$\frac{q^2}{[3]_q!}$	$\frac{(1-q)q^3}{[2]_q[4]_q}$	$\frac{q^4(1-q^2-2q^3-q^4+q^6)}{[2]_q^2[3]_q[5]_q}$
$\mathcal{E}_{n,q}$ [12, 13]	1	$-\frac{1}{2}$	$\frac{-1+q}{4}$	$\frac{-1+2q+q^2-q^3}{8}$	$\frac{(q-1)[3]_q!(q^2-4q+1)}{16}$
$\mathcal{G}_{n,q}$ [12, 13]	0	$\frac{2-q}{1+q}$	$\frac{q(q-5)}{(1+q)^2}$	$-\frac{3q^2(q-5)}{(1+q)^3}-\frac{3q(2-q)}{(1+q)^2}-\frac{q}{1+q}$	$-\frac{3q}{1+q}\left(\frac{3q^3+10q^2-28q+7}{(1+q)^3}\right)$

By choosing an appropriate expression for the function $A_q(t)$ from Table 1.1 and substituting it into equation (2.1), we arrive at the corresponding generating functions of the q-Gould–Hopper–based Bernoulli polynomials ${}_H\mathbb{B}_{n,q}^{(m)}(x,y;s_1;s_2)$, the Euler polynomials ${}_H\mathbb{E}_{n,q}^{(m)}(x,y;s_1;s_2)$, and the Genocchi polynomials ${}_H\mathbb{G}_{n,q}^{(m)}(x,y;s_1;s_2)$, which are given as follows:

$$\frac{t}{e_q(t)-1}e_q(xs_1t)e_q(ys_2t^m)=\sum_{n=0}^{\infty}{}_H\mathbb{B}_{n,q}^{(m)}(x,y;s_1;s_2)\frac{t^n}{[n]_q!},$$

$$\frac{2}{e_q(t)+1}e_q(xs_1t)e_q(ys_2t^m)=\sum_{n=0}^{\infty}{}_H\mathbb{E}_{n,q}^{(m)}(x,y;s_1;s_2)\frac{t^n}{[n]_q!},$$

and

$$\frac{2t}{e_q(t)+1}e_q(xs_1t)e_q(ys_2t^m)=\sum_{n=0}^{\infty}{}_H\mathbb{G}_{n,q}^{(m)}(x,y;s_1;s_2)\frac{t^n}{[n]_q!}.$$

Moreover, by making use of relation (2.2), it follows that the polynomials ${}_H\mathbb{B}_{n,q}^{(m)}(x,y;s_1;s_2)$, ${}_H\mathbb{E}_{n,q}^{(m)}(x,y;s_1;s_2)$, and ${}_H\mathbb{G}_{n,q}^{(m)}(x,y;s_1;s_2)$ admit the explicit representations:

$${}_H\mathbb{B}_{n,q}^{(m)}(x,y;s_1;s_2)=\sum_{k=0}^{n}\binom{n}{k}_q\ \mathbb{B}_{k,q}\ H_{n-k,q}^{(m)}(x,y;s_1;s_2),$$

$${}_H\mathbb{E}_{n,q}^{(m)}(x,y;s_1;s_2)=\sum_{k=0}^{n}\binom{n}{k}_q\ \mathbb{E}_{k,q}\ H_{n-k,q}^{(m)}(x,y;s_1;s_2),$$

and

$${}_H\mathbb{G}_{n,q}^{(m)}(x,y;s_1;s_2)=\sum_{k=0}^{n}\binom{n}{k}_q\ \mathbb{G}_{k,q}\ H_{n-k,q}^{(m)}(x,y;s_1;s_2).$$

Finally, in light of expressions (2.3), the above polynomial families also possess determinant representations. These are summarized below for each case.

Here, $H_{n,q}^{(m)}(x,y)$ $(n=0,1,2,\ldots)$ denote the q-Gould–Hopper–based Appell polynomials of degree n.

6. Concluding remarks and future observations

In this work, we have carried out a comprehensive investigation of a two-parameter family of q-Gould–Hopper–based Appell polynomials by systematically combining generating function techniques with the tools of q-calculus and operational methods. Starting from an appropriate generating function framework, we derived explicit series representations, determinant forms, and q-derivative relations, thereby establishing a coherent algebraic and analytic structure for this polynomial family. The use of the extension of monomiality (quasi-monomial) principle proved to be particularly effective, as it allowed us to identify suitable q-multiplicative and q-derivative operators and to deduce the corresponding q-differential equations satisfied by these polynomials. In addition, several summation identities were obtained, highlighting the internal consistency of the theory and revealing transparent connections with classical and well-known q-Appell-type sequences.

The examples discussed in the final section further demonstrate the flexibility of the proposed framework. By choosing specific forms of the generating function $\mathcal{A}_q(t)$, we showed that the general construction naturally yields q-Bernoulli, q-Euler, and q-Genocchi type polynomials as special cases. This unifying viewpoint not only clarifies the relationships among these classical families but also emphasizes the role of the q-Gould–Hopper structure as a common backbone underlying a wide class of q-special polynomials.

From a broader perspective, the results obtained here open several promising directions for future research. One natural continuation is the detailed study of orthogonality properties and moment representations of these polynomials, which may lead to new classes of q-orthogonal systems. Another important direction concerns the investigation of their integral transforms and connections with fractional and q-fractional calculus, where operational representations are expected to play a crucial role. It would also be of interest to explore further extensions within the framework of multivariable and matrix-valued polynomials, as well as q-analogues in non-commutative or quantum algebraic settings.

On the applied side, the explicit operational and summation formulas derived in this paper suggest potential applications in mathematical physics, particularly in the study of q-deformed differential equations, discrete models, and signal processing problems. Numerical aspects, including zero distributions and asymptotic behavior, also merit deeper analysis and may provide additional insight into approximation-theoretic and computational features of these polynomials. Overall, the present work lays a solid foundation for both theoretical developments and practical applications of two-parameter q-Gould–Hopper–based Appell polynomials, and it is hoped that it will stimulate further research in this evolving area of q-special function theory.

Looking ahead, future research may focus on examining their orthogonality properties, establishing links with various special functions and combinatorial structures, and exploring their usefulness in areas such as fractional calculus, q-information theory, and signal processing.

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