Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

REGULAR VARIATION AND ALMOST A-STATISTICAL CONVERGENCE

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Abstract. This paper introduces the theory of almost $A-$statistical regular variation for sequences of positive real numbers. The central classes $ASRV_s$, and $ASTr(RV_s)$ are defined and their fundamental properties are established. A key decomposition theorem characterizes sequences that are almost $A-$statistically regularly varying as sums of sequences that converge classically and are asymptotically small. We also prove the equivalence between strong almost $A-$summable regular variation and statistical regular variation for RV-bounded sequences and provide characterizations via exponential representations and limit superior/inferior conditions. Our results extend classical regular variation theory systematically to the setting of almost $A-$statistical convergence.

1. Introduction and preliminaries

Originally formulated by Fast [16] and Steinhaus [23] as a natural generalization of ordinary convergence, statistical convergence has evolved into a central tool in analysis. Following its introduction, the fundamental properties of statistical convergence were thoroughly investigated. This type of convergence was later integrated with matrix summability theory, leading to numerous generalizations that have been explored across various fields. Specifically, these generalizations have found applications in approximation theory [7, 10, 25], number theory, the study of trigonometric series [26], locally convex spaces [22], and strong integral summability. In [15], the concept of statistical convergence has also been successfully applied to Karamata's theory [18, 19] of regular variation. This theory focuses on the asymptotic analysis of divergent processes [3, 6, 11, 13].

Recently, in [10] a new generalization of statistical convergence, called almost $A-$statistical convergence, has been introduced. Some of its fundamental properties have already been examined and applied to approximation theory. In this article, we will utilize this novel convergence concept in the theory of variation. Particular attention is paid to characterizing regular and translationally regular variation in the context of almost $A-$statistical convergence.

We begin by examining several fundamental properties associated with the almost $A-$statistical convergence of a sequence $\mathbf{x}$. Recall the following:

2020 *Mathematics Subject Classification.* 26A12, 40A05, 40A35.

Key words and phrases. statistical convergence, almost convergence, almost regular matrix, regularly varying.

Let $A = (a_{jm})$ be a summability matrix and $\mathbf{x} = (x_m)$ be a real-valued sequence. If the sequence

$$(Ax)_j = \sum_{m=0}^{\infty} a_{jm} x_m$$

exists, i.e., the series $\sum\limits_{m=0}^{\infty} a_{jm} x_m$ is convergent for each $j \in \mathbb{N}_0$, where $\mathbb{N}_0$ is the set of all nonnegative integers, then the sequence $A\mathbf{x}$ is called the A-transformation of $\mathbf{x}$. If the sequence $A\mathbf{x}$ is convergent to a number s, then the sequence $\mathbf{x} = (x_m)$ is said to be *$A-$summable to s*, and we write $A-\lim x_m = s$.

King [20] introduced the concept of almost $A-$summability as a natural extension of classical summability when the convergence of the A-transform is relaxed to that of almost convergence. A sequence $\mathbf{x}$ is called *almost $A-$summable to L* if the $A-$transform of $\mathbf{x}$ is almost convergent to L. The matrix A is said to be *almost regular* if it maps the entire space c, (the space of convergent sequences) into the space of almost convergent sequences, and it acts as a limit-preserving operator, ensuring that the almost limit of the transform equals the ordinary limit of the original sequence. Recall that a matrix $A = (a_{jm})$ is almost regular if and only if it meets the following conditions:

(i) $\sup \sum\limits_{m=0}^{\infty} |a_{jm}| < \infty$, $j = 0, 1, 2, ...,$

(ii) $\lim\limits_{p\to\infty} \frac{1}{p} \sum\limits_{j=k}^{k+p-1} a_{jm} = 0$ uniformly in k, $m = 0, 1, 2, ...,$

(iii) $\lim\limits_{p\to\infty} \frac{1}{p} \sum\limits_{j=k}^{k+p-1} \sum\limits_{m=0}^{\infty} a_{jm} = 1$ uniformly in k.

It is a well-known fact that the class of regular matrices is strictly contained within the class of almost regular matrices. The following example demonstrates that this inclusion is indeed proper (i.e., the converse is not true).

Example 1.1. The matrix $A = (a_{jm})$ whose general term is defined by

$$a_{jm} = \begin{cases} \frac{1}{j+1}\left[1 + (-1)^j\right], & 0 \leq m \leq j, \\ 0, & j < m. \end{cases}$$

is non-negative, almost regular but non-regular.

Definition 1.1. ([10]) Let $A = (a_{jm})$ be a non-negative almost regular matrix. Then *almost $A-$density* of $E \subseteq \mathbb{N}_0$, denoted $\delta_A^a(E)$, is given by

$$\delta_A^a(E) = \lim_{p\to\infty} \frac{1}{p} \sum_{j=k}^{k+p-1} \sum_{m\in E} a_{jm} = \lim_{p\to\infty} \frac{1}{p} \sum_{j=k}^{k+p-1} (A\chi_E)_j, \text{uniformly in } k,$$

whenever this limit exists.

Let A be a non-negative and almost regular matrix. Then, the concept of almost $A-$density has the following properties.

Lemma 1.1. *(*[10]*) (Fundamental Properties of Almost A-Density) i)* $\delta_A^a(\mathbb{N}_0) = 1$,

ii) For any subset E of $\mathbb{N}_0$, $0 \leq \delta_A^a(E) \leq 1$,

iii) If E is a subset of $\mathbb{N}_0$, and $\delta_A^a(E)$ exists, then $\delta_A^a(E^c)$ exists and $\delta_A^a(E)+\delta_A^a(E^c)=1$,

Assume that E and F are subsets of $\mathbb{N}_0$ and $\delta_A^a(E)$, $\delta_A^a(F)$ exists.

iv) Monotonicity: If $E\subseteq F$, $\delta_A^a(E)\leq\delta_A^a(F)$,

v) Finite Additivity: $\delta_A^a(E\cup F)\leq\delta_A^a(E)+\delta_A^a(F)$,

vi) If E is a finite subset of $\mathbb{N}_0$, then $\delta_A^a(E)=0$.

From this point forward, every matrix A considered is non-negative and almost regular.

A sequence $\mathbf{x}=(x_m)$ is said to be *almost $A-$statistically convergent* to $\mathcal{L}$ if, for every $\varepsilon>0$,

$$\delta_A^a\left(\{m\in\mathbb{N}_0:|x_m-\mathcal{L}|\geq\varepsilon\}\right)=0.$$

When this holds, we write $st_A^a\text{-}\lim\mathbf{x}=\mathcal{L}$ ([10]).

Based on the definition provided above, we derive the following theorem.

Theorem 1.1. *If $st_A^a\text{-}\lim x_m=\mathcal{L}_1$, $st_A^a\text{-}\lim y_m=\mathcal{L}_2$, then*

(a) $st_A^a\text{-}\lim(x_m+y_m)=\mathcal{L}_1+\mathcal{L}_2$.

(b) $st_A^a\text{-}\lim(x_my_m)=\mathcal{L}_1\mathcal{L}_2$.

Now let us give the following characterization for almost $A-$statistical convergence.

Theorem 1.2. *A sequence $\mathbf{x}=(x_m)$ is almost $A-$statistically convergent to $\mathcal{L}$ if and only if there exists a subset E of $\mathbb{N}_0$, such that $\delta_A^a(E)=1$ and*

$$\lim_{\substack{m\to\infty\\ m\in E}}x_m=\mathcal{L}.$$

Proof. Let $\mathcal{L}$ be the almost A-statistical limit of the sequence $\mathbf{x}$. Define

$$E_r=\{m\in\mathbb{N}_0:|x_m-\mathcal{L}|\geq\tfrac{1}{r+1}\},F_r=\{m\in\mathbb{N}_0:|x_m-\mathcal{L}|<\tfrac{1}{r+1}\}.$$

Then $\delta_A^a(E_r)=0$ and

$$F_0\supset F_1\supset F_2\supset\cdots\supset F_r\supset F_{r+1}\supset\cdots \tag{1.1}$$

Moreover,

$$\delta_A^a(F_r)=1,\ r=0,1,2,\ldots \tag{1.2}$$

We now show that (x_m) converges to L for $m\in F_r$. Suppose, on the contrary, that (x_m) does not converge to $\mathcal{L}$. Then there exists $\varepsilon>0$ such that

$$|x_m-\mathcal{L}|\geq\varepsilon$$

for infinitely many terms. Let

$$F_\varepsilon=\{m\in\mathbb{N}_0:|x_m-\mathcal{L}|<\varepsilon\},\quad \varepsilon>\tfrac{1}{r+1},\ r=0,1,2,\ldots$$

Then

$$\delta_A^a(F_\varepsilon)=0.$$

By (1.1), $F_r\subset F_\varepsilon$, hence $\delta_A^a(F_r)=0$, which contradicts (1.2). Therefore, (x_m) converges to $\mathcal{L}$.

For the reverse implication, assume the existence of a subset $F\subseteq\mathbb{N}_0$ such that $\delta_A^a(F)=1$ and

$$\lim_{\substack{m\to\infty\\ m\in F}}x_m=\mathcal{L}.$$

Equivalently, one can find an index $N \in \mathbb{N}_0$ ensuring that for an arbitrary $\varepsilon > 0$,

$$|x_m - \mathcal{L}| < \varepsilon, \quad \forall m \geq N \text{ and } m \in F.$$

Define

$$F_\varepsilon = \{m \in \mathbb{N}_0 : |x_m - \mathcal{L}| \geq \varepsilon\}.$$

Then

$$F_\varepsilon \subseteq \mathbb{N}_0 \setminus \{m \in F : m \geq N\}.$$

Hence,

$$\delta_A^a(F_\varepsilon) = 0.$$

This leads to the conclusion that $\mathbf{x}$ is almost A-statistically convergent to $\mathcal{L}$. The theorem has been proved.

The set of almost $A-$statistically convergent sequences will be denoted by $\mathsf{ST}_\mathsf{A}^\mathsf{a}$.

Now, we will give the new notions of almost $A-$statistically Cauchy sequence, almost $A-$statistical boundedness, almost $A-$statistical limit superior, almost $A-$statistical limit inferior, and almost $A-$statistical limit at infinity.

Definition 1.2. A sequence of positive real numbers, denoted by $\mathbf{x} = (x_m)$, is said to be *almost $A-$statistically Cauchy* if there exists an index $M \in \mathbb{N}_0$ corresponding to every $\varepsilon > 0$, such that the relation

$$\delta_A^a(\{m \in \mathbb{N}_0 : |x_m - x_M| \geq \varepsilon\}) = 0$$

holds.

Definition 1.3. A real sequence $\mathbf{x} = (x_m)$ is called *almost $A-$statistically bounded* if there exists a constant H such that

$$\delta_A^a(\{m \in \mathbb{N}_0 : \ |x_m| > H\}) = 0.$$

The theoretical framework concerning statistical limit superior and limit inferior was primarily established in the seminal work of Fridy and Orhan [17], and further investigated in [9]. We give now their almost $A-$statistical analogs.

For a number sequence $\mathbf{x} = (x_m)$ we write

$$U_\mathbf{x} := \{\alpha \in \mathbb{R} : \delta_A^a \{m \in \mathbb{N}_0 : \ x_m > \alpha\} \neq 0\},$$

and

$$V_\mathbf{x} = \{\beta \in \mathbb{R} : \delta_A^a \{m \in \mathbb{N}_0 : \ x_m < \beta\} \neq 0\}.$$

Throughout this paper, note that the notation $\delta_A^a(E) \neq 0$ indicates that either $\delta_A^a(E) > 0$ or the set E does not possess an $A-$density.

Definition 1.4. The *almost $A-$statistical limit superior* of $\mathbf{x}$ is given by

$$st_A^a\text{−}\limsup x_m = \begin{cases} \sup U_\mathbf{x}, & \text{if } U_\mathbf{x} \neq \emptyset, \\ -\infty, & \text{if } U_\mathbf{x} = \emptyset. \end{cases}$$

Also, the *almost $A-$statistical limit inferior* of $\mathbf{x}$ is defined by

$$st_A^a - \liminf x_m = \begin{cases} \inf V_\mathbf{x} & \text{if } V_\mathbf{x} \neq \emptyset, \\ +\infty, & \text{if } V_\mathbf{x} = \emptyset. \end{cases}$$

Theorem 1.3. st_A^a-$\limsup x_m = \alpha_1$ *is finite if and only if for every positive number* ε,

$$\delta_A^a\left(\{m \in \mathbb{N}_0 : \ x_m > \alpha_1 - \varepsilon\}\right) \neq 0, \ \delta_A^a\left(\{m \in \mathbb{N}_0 : \ x_m > \alpha_1 + \varepsilon\}\right) = 0.$$

Similarly st_A^a-$\liminf x_m = \beta_1$ *is finite if and only if for every positive number* ε,

$$\delta_A^a\left(\{m \in \mathbb{N}_0 : \ x_m < \beta_1 + \varepsilon\}\right) \neq 0, \ \delta_A^a\left(\{m \in \mathbb{N}_0 : \ x_m < \beta_1 - \varepsilon\}\right) = 0.$$

Proof. Let st_A^a-$\limsup x_m = \alpha_1$. Since

$$U_{\mathbf{x}} := \{\alpha \in \mathbb{R} : \delta_A^a \{m \in \mathbb{N}_0 : \ x_m > \alpha\} \neq 0\} \neq \emptyset,$$

we have $\alpha_1 = \sup\left\{\alpha \in \mathbb{R} : \delta_A^a\left(\{m \in \mathbb{N}_0 : \ x_m > \alpha\}\right) \neq 0\right\}$. By the properties of the supremum:

∘ For all $\alpha \in U_{\mathbf{x}}$, $\alpha \leq \alpha_1$,

∘ For every $\varepsilon > 0$, there exists a $\alpha \in U_{\mathbf{x}}$ such that $\alpha > \alpha_1 - \varepsilon$.

Furthermore, if $\alpha \in U_{\mathbf{x}}$, $\delta_A^a\left(\{m \in \mathbb{N}_0 : \ x_m > \alpha\}\right) \neq 0$. For every $\varepsilon > 0$ and for the specific $\alpha \in U_{\mathbf{x}}$ satisfying $\alpha > \alpha_1 - \varepsilon$, we have $x_m > \alpha > \alpha_1 - \varepsilon$, which implies:

$$\{m \in \mathbb{N}_0 : \ x_m > \alpha\} \subset \{m \in \mathbb{N}_0 : \ x_m > \alpha_1 - \varepsilon\}.$$

Thus, since $\delta_A^a\left(\{m \in \mathbb{N}_0 : \ x_m > \alpha\}\right) \neq 0$, $\delta_A^a\left(\{m \in \mathbb{N}_0 : \ x_m > \alpha_1 - \varepsilon\}\right) \neq 0$ hold. On the other hand, for all $\alpha \in U_{\mathbf{x}}$, $\alpha \leq \alpha_1$, so for every $\varepsilon > 0$, $\alpha + \varepsilon \leq \alpha_1 + \varepsilon$. This gives the inclusion:

$$\{m \in \mathbb{N}_0 : \ x_m > \alpha_1 + \varepsilon\} \subset \{m \in \mathbb{N}_0 : \ x_m > \alpha + \varepsilon\}. \tag{1.3}$$

If $\alpha + \varepsilon$ were in $U_{\mathbf{x}}$, then we would have $\delta_A^a\left(\{m\imath\mathbb{N}_0 : \ x_m > \alpha + \varepsilon\}\right) = 0$. From inclusion (1.3), we conclude that:

$$\delta_A^a\left(\{m \in \mathbb{N}_0 : \ x_m > \alpha_1 + \varepsilon\}\right) = 0.$$

The proof of the converse is clear. The proof of $st_A^a - \liminf x_m = \beta_1$ can be carried out in a similar manner. The theorem has been proved.

As in the statistical superior or inferior limit, one obtains

$$\liminf x_m \leq st_A^a - \liminf x_m \leq st_A^a - \limsup x_m \leq \limsup x_m \tag{1.4}$$

and also that, $\mathbf{x} = (x_m)$ is almost $A-$statistically bounded and st_A^a-$\lim x_m = \mathcal{L}$ if and only if st_A^a-$\liminf x_m = st_A^a$-$\limsup x_m = \mathcal{L}$.

Now, let us define st_A^a-lim at infinity for functions by using a similar idea to that in [4].

Definition 1.5. A function $f : [a, \infty) \to \mathbb{R}$ has an st_A^a-lim *at infinity* equal to $\mathcal{L}$ if st_A^a-$\lim f(x_m) = \mathcal{L}$ with every $\mathbf{x} = (x_m) \subset [a, \infty) : st_A^a$-$\lim x_m = \infty$, and it will be denoted as st_A^a-$\lim\limits_{t\to\infty} f(t) = \mathcal{L}$.

2. Almost $A-$statistical variations

In this section, we extend the classical theory of regular variation by introducing its almost A-statistical analogues, covering both standard and translational regular variation. Additionally, the concept of a strong almost A-summable regularly varying sequence is formulated.

2.1. Regular variation.

Definition 2.1. ([5]) Suppose that F is a positive real function defined on an interval $[a,\infty)$, $(a>0)$. Then F is said to be *regularly varying* if it is measurable and satisfies

$$\lim_{t\to\infty}\frac{F(\zeta t)}{F(t)}=h(\zeta)<\infty,\ \zeta>0.$$

If $h(\zeta)=1$, for each $\zeta>0$, then F is called *slowly varying.*

The function $h(\zeta)$ is assumed to be representable in the form $h(\zeta)=\zeta^{\rho}$ for some $\rho\in\mathbb{R}$, where the parameter ρ is referred to as *the index of variability* of F.

The classes of regularly varying and slowly varying sequences are abbreviated as RV_f and SV_f, respectively. Moreover, $RV_{f,\rho}$ refers to the set of regularly varying sequences characterized by the variability index ρ.

Definition 2.2. ([5]) A sequence $\mathbf{x}=(x_m)$ of positive real numbers is called *regularly varying* if

$$\lim_{m\to\infty}\frac{x_{[\zeta m]}}{x_m}=k(\zeta)<\infty,\ \text{for all } \zeta>0.$$

If $k(\zeta)=1$, for every $\zeta>0$, then $\mathbf{x}$ is said to be *slowly varying.*

We denote by RV_s (resp. SV_s) the class of regularly varying (resp. slowly varying) sequences. In addition, $RV_{s,\rho}$ represents the class of regularly varying sequences with variability index ρ.

Theorem 2.1. *([5]) If $\mathbf{x}=(x_m)\in RV_s$, then $k(\zeta)=\zeta^{\rho}$ for some $\rho\in\mathbb{R}$. ρ is called the index of variability of $\mathbf{x}$.*

Definition 2.3. Suppose that F is a positive function, defined on an interval $[a,\infty)$, $(a>0)$. Then F is said to be *almost A-statistically regularly varying* if it is measurable and satisfies

$$st_A^a\text{–}\lim_{t\to\infty}\frac{F(\zeta t)}{F(t)}=h(\zeta)<\infty,\ \zeta>0.$$

If $h(\zeta)=1$, holds for all $\zeta>0$, then F is said to be *almost $A-$statistically slowly varying.* The collection of all functions satisfying this property is denoted by $ASRV_f$.

Definition 2.4. A sequence of positive real numbers $\mathbf{x}=(x_m)$ belongs to the class $ASRV_s$ of *almost $A-$statistically regularly varying sequences* if

$$st_A^a-\lim\frac{x_{[\zeta m]}}{x_m}=k(\zeta)<\infty,\ \text{for all } \zeta>0.$$

Consequently, the class of regularly varying sequences is contained within the class of almost A-statistically regularly varying sequences. This inclusion is strict, parallel to the relationship between ordinary and almost A-statistical convergence; that is, the converse assertion does not hold.

Example 2.1. Consider the sequence (x_m) defined by

$$x_m=\begin{cases} m\ln m, & m=2k \text{ and } m\neq 0,\\ m!, & \text{otherwise,}\end{cases}\quad k=0,1,2,....$$

Also, let us take the matrix $A=(a_{jm})$ whose general term is given by

$$a_{jm}=\begin{cases}1, & \text{if } j \text{ is even and } m\in\{j,j+2\},\\ 0, & \text{otherwise.}\end{cases}$$

This matrix is non-negative, almost regular but non-regular. Also, it is clear that

$$\delta_A^a\left(\{m=2k+1:\ k=0,1,2,...\}\right)=0$$

and

$$\delta_A^a\left(\{m=2k:\ k=0,1,2,...\}\right)=1.$$

Then, for $\zeta\geq 1$ and for every $m=2k$, $k=1,2,...$,

$$\frac{x_{[\zeta m]}}{x_m}=\frac{[\zeta m]\ln[\zeta m]}{m\ln m}\leq\zeta\frac{\ln\zeta m}{\ln m}\to\zeta,$$

hence, by Theorem 1.2, (x_m) is almost A-statistically regularly varying sequence, but since $\lim\frac{x_{[\zeta m]}}{x_m}$ does not exist, (x_m) is not regularly varying sequence.

Definition 2.5. A sequence of positive real numbers $\mathbf{x}=(x_m)$ is said to be an *almost $A-$statistically regularly varying Cauchy sequence* if, for every $\varepsilon>0$ and $\zeta>0$, one can find an index $M=M(\varepsilon)$ with the property that

$$\delta_A^a\left(\left\{m\in\mathbb{N}_0:\left|\frac{x_{[\zeta m]}}{x_m}-\frac{x_{[\zeta M]}}{x_m}\right|\geq\varepsilon\right\}\right)=0.$$

Definition 2.6. A sequence $\mathbf{x}=(x_m)$ of positive real numbers is called *almost $A-$statistically RV-bounded* if, for each $\zeta>0$, there exists a finite number B such that $\delta_A^a\left(\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}>B\}\right)=0$.

Definition 2.7. Let q be a positive real number. A sequence of positive real numbers $\mathbf{x}=(x_m)$ is said to be a *strongly almost $A-$summable regularly varying* sequence if it satisfies the condition

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty}a_{jm}\left|\frac{x_{[\zeta m]}}{x_m}-k(\zeta)\right|^q=0,\ \text{uniformly in } k,\ \text{for all } \zeta>0.$$

We denote the space of all such sequences by $wARV_{s,q}$.

The following initial result offers a convergence criterion that does not necessitate prior knowledge of the limit value.

Theorem 2.2. *Let $\mathbf{x}=(x_m)$ be a sequence of positive real numbers. The following conditions must be met in order for the equivalence to hold:*

(a) $\mathbf{x}$ *belongs to the class of almost A-statistically regularly varying sequences,*

(b) $\mathbf{x}$ *is classified as an almost A-statistically regularly varying Cauchy sequence,*

(c) *One can find a regularly varying sequence* $\mathbf{y}$ *such that*

$$\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\neq\frac{y_{[\zeta m]}}{y_n}\right\}\right)=0.$$

Proof. As an auxiliary result, we first prove that (a) implies (b). Assume that condition (a) holds, so $st_A^a\text{-}\lim\frac{x_{[\zeta m]}}{x_m}=k(\zeta)$. Let $\varepsilon>0$ be arbitrary. By the definition of almost $A-$statistical limit, the set of indices $m\in\mathbb{N}_0$ for which $\left|\frac{x_{[\zeta m]}}{x_m}-k(\zeta)\right|<\frac{\varepsilon}{2}$ has almost $A-$statistical density 1. Then $M=M(\varepsilon)$ is chosen so that $\left|\frac{x_{[\zeta M]}}{x_m}-k\left(\zeta\right)\right|<\frac{\varepsilon}{2}$. Applying the triangle inequality on the set where the density is 1, we obtain

$$\delta_A^a\left(\left\{m\in\mathbb{N}_0:\left|\frac{x_{[\zeta m]}}{x_m}-\frac{x_{[\zeta M]}}{x_m}\right|<\varepsilon\right\}\right)=1.$$

Since ε was arbitrary, $\mathbf{x}$ is an almost $A-$statistically regularly varying Cauchy sequence.

Next, we assume that (b) holds and choose M_1 so that closed interval $J=\left[\frac{x_{[\zeta M_1]}}{x_{M_1}}-1,\frac{x_{[\zeta M_1]}}{x_{M_1}}+1\right]$ contains $\frac{x_{[\zeta m]}}{x_m}$ such that $\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\in J\right\}\right)=1$. Also choose M_2 so that the interval $J^{'}=\left[\frac{x_{[\zeta M_2]}}{x_{M_2}}-\frac{1}{2},\frac{x_{[\zeta M_2]}}{x_{M_2}}+\frac{1}{2}\right]$ contains $\frac{x_{[\zeta m]}}{x_m}$ such that $\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\in J^{'}\right\}\right)=1$. Let $J_1=J\cap J^{'}$. Since the almost A-statistical density is finitely subadditive, we have

$$\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\in J_1\right\}\right)=1.$$

We have

$$\begin{aligned}&\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\notin J\cap J^{'}\right\}\\ =&\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\notin J\right\}\cup\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\notin J^{'}\right\}.\end{aligned}$$

Hence

$$\begin{aligned}\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\notin J\cap J^{'}\right\}\right)&\leq\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\notin J\right\}\right)\\&\quad+\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\notin J^{'}\right\}\right)\\&=0.\end{aligned}$$

Consequently, J_1 is a closed interval of length not exceeding 1 which contains $\frac{x_{[\zeta m]}}{x_m}$ such that $\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\in J_1\right\}\right)=1$. Now, we proceed by choosing M_3 so that $J''=\left[\frac{x_{[\zeta M_3]}}{x_m}-\frac{1}{4},\frac{x_{[\zeta M_3]}}{x_m}+\frac{1}{4}\right]$ contains $\frac{x_{[\zeta m]}}{x_m}$, $\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\in J''\right\}\right)=1$ and similarly, $J_2=J_1\cap J''$ contains $\frac{x_{[\zeta m]}}{x_m}$, $\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\in J_2\right\}\right)=1$, and J_2 has length not greater than $\frac{1}{2}$. This inductive argument yields a sequence of closed intervals $(J_n)_{n=1}^{\infty}$ exhibiting the specific features that for each n, $J_{n+1}\subseteq J_n$, J_n contains $\frac{x_{[\zeta m]}}{x_m}$, $\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\in J_n\right\}\right)=1$ and the length of J_n is less than or equal to 2^{1-n}. As indicated earlier, by Nested Intervals Theorem there exists a number, say, μ such that $\mu=\bigcap\limits_{n=1}^{\infty}J_n$. By virtue of J_n contains $\frac{x_{[\zeta m]}}{x_m}$, $\delta_A^a\left(\left\{m\in\mathbb{N}_0:\frac{x_{[\zeta m]}}{x_m}\in J_n\right\}\right)=1$, we can choose an increasing positive integer sequence $(P_n)_{n=1}^{\infty}$ such that

$$\delta_A^a\left(\left\{m\in\mathbb{N}_0:m>P_n,\frac{x_{[\zeta m]}}{x_m}\notin J_n\right\}\right)<\frac{1}{n}.\tag{2.1}$$

Consider the subsequence $\mathbf{z}$ derived from $\mathbf{x}$ whose terms x_m are determined by the condition that

$$m > P_0 \text{ and } P_n < m \leq P_{n+1}, \text{ then } \frac{x_{[\zeta m]}}{x_m} \notin J_n.$$

Now we consider the sequence $\mathbf{y} = \{y_m\}$ with

$$\frac{y_{[\zeta m]}}{y_m} = \begin{cases} \mu, & \text{if } x_m \text{ is included in } z, \\ \frac{x_{[\zeta m]}}{x_m}, & \text{otherwise.} \end{cases}$$

Then $\lim \frac{y_{[\zeta m]}}{y_m} = \mu$: for if $P_n < m$ and $0 < \frac{1}{n} < \varepsilon$, then either x_m is included in $\mathbf{z}$, i.e., $\frac{y_{[\zeta m]}}{y_m} = \mu$ or $\frac{y_{[\zeta m]}}{y_m} = \frac{x_{[\zeta m]}}{x_m} \in J_n$ and $\left|\frac{y_{[\zeta m]}}{y_m} - \mu\right|$ is not greater than the length of I_n. Next, we assert that $\frac{x_{[\zeta m]}}{x_m} = \frac{y_{[\zeta m]}}{y_n}$, $\delta_A^a\left(\left\{m \in \mathbb{N}_0 : \frac{x_{[\zeta m]}}{x_n} = \frac{y_{[\zeta m]}}{y_m}\right\}\right) = 1$. Note that for $P_n < m \leq P_{n+1}$, we have

$$\left\{\mathbb{N}_0 : \frac{x_{[\zeta m]}}{x_m} \neq \frac{y_{[\zeta m]}}{y_m}\right\} \subseteq \left\{m \in \mathbb{N}_0 : \frac{x_{[\zeta m]}}{x_m} \notin J_n\right\}.$$

This implies, by (2.1),

$$\frac{1}{p}\sum_{j=k}^{k+p-1} \sum_{m\in\mathbb{N}_0 : \frac{x_{[\zeta m]}}{x_m} \neq \frac{y_{[\zeta m]}}{y_m}} a_{jm} \leq \frac{1}{p}\sum_{j=k}^{k+p-1} \sum_{m\in\mathbb{N}_0 : \frac{x_{[\zeta m]}}{x_m} \notin I_n} a_{jm} < \frac{1}{n}.$$

Taking supremum over k and limit as $p \to \infty$, we get

$$\delta_A^a\left(\left\{n\imath\mathbb{N}_0 : \frac{x_{[\zeta n]}}{x_m} \neq \frac{y_{[\zeta n]}}{y_n}\right\}\right) = 0,$$

i.e., $\frac{x_{[\zeta m]}}{x_m} = \frac{y_{[\zeta m]}}{y_m}$, $\delta_A^a\left(\left\{m \in \mathbb{N}_0 : \frac{x_{[\zeta m]}}{x_m} = \frac{y_{[\zeta m]}}{y_m}\right\}\right) = 1$. Therefore, (c) holds.

Lastly, we suppose that (c) is satisfied, i.e.,

$\frac{x_{[\zeta m]}}{x_m} = \frac{y_{[\zeta m]}}{y_m}$, $\delta_A^a\left(\left\{m \in \mathbb{N}_0 : \frac{x_{[\zeta m]}}{x_m} = \frac{y_{[\zeta m]}}{y_n}\right\}\right) = 1$ and $\lim \frac{y_{[\zeta m]}}{y_m} = k(\zeta)$.

Let $\varepsilon > 0$. Then

$$\begin{aligned} &\left\{m \in \mathbb{N}_0 : \left|\frac{x_{[\zeta m]}}{x_m} - k(\zeta)\right| \geq \varepsilon\right\} \\ \subseteq &\left\{m \in \mathbb{N}_0 : \frac{x_{[\zeta m]}}{x_m} \neq \frac{y_{[\zeta m]}}{y_m}\right\} \cup \left\{m \in \mathbb{N}_0 : \left|\frac{y_{[\zeta m]}}{y_m} - k(\zeta)\right| > \varepsilon\right\}. \end{aligned}$$

Therefore, $\delta_A^a\left(\left\{m \in \mathbb{N}_0 : \left|\frac{x_{[\zeta m]}}{x_m} - k(\zeta)\right| \geq \varepsilon\right\}\right) = 0$, i.e., $\mathbf{x}$ is an almost A-statistically regularly varying sequence and hence the theorem. The theorem has been proved.

Based on the established equivalences in Theorem 2.2 and the basic results related to regular variation (Theorem 2.1), we are now in a position to state the following conclusion regarding the limit function $k(\zeta)$.

Corollary 2.1. *If a positive real sequence* $\mathbf{x}$ *satisfies the almost A-statistical limit condition* $st_A^a\text{-}\lim \frac{x_{[\zeta m]}}{x_m} = k(\zeta)$, $\zeta > 0$, *then it possesses the following property:*

$\lim \frac{y_{[\zeta m]}}{y_m} = k(\zeta)$ *and* $k(\zeta) = \zeta^\rho$, *for some* $\rho \in \mathbb{R}$.

The class of all almost $A-$statistically regularly varying sequences possessing the index of variability ρ is denoted by $ASRV_{s,\rho}$.

Now, we proceed to present a decomposition theorem specifically for almost A-statistically regularly varying sequences.

Theorem 2.3. *Let $\mathbf{x} = (x_m)$ be a sequence of positive real numbers. The following conditions are then equivalent:*

(a) st_A^a-$\lim \frac{x_{[\zeta m]}}{x_m} = k(\zeta)$,

(b) There exist sequences $\mathbf{y} = \{y_m\}$ and $\mathbf{z} = \{z_m\}$ such that $\frac{x_{[\zeta m]}}{x_m} = \frac{y_{[\zeta m]}}{y_m} + \frac{z_{[\zeta m]}}{z_m}$ with $\lim \frac{y_{[\zeta m]}}{y_m} = k(\zeta)$, $\delta_A^a\left(supp\frac{z_{[\zeta m]}}{z_m}\right) = 0$, where $suppz = \left\{m \in \mathbb{N}_0 : \frac{z_{[\zeta m]}}{z_m} \neq 0\right\}$.

Proof. Assume (a) holds. There is a set $K = \{m_0 < m_1 < ... < m_k < ...\}$ with $\delta_A^a(K) = 1$ such that $\lim \frac{x_{[\zeta m]}}{x_m} = k(\zeta)$. Define the sequence $\mathbf{y} = \{y_m\}$ by

$$\frac{y_{[\zeta m]}}{y_m} = \begin{cases} \frac{x_{[\zeta m]}}{x_m}, & m \in K, \\ k(\zeta), & m \in \mathbb{N} \setminus K. \end{cases} \tag{2.2}$$

It is clear that $\lim \frac{y_{[\zeta m]}}{y_m} = k(\zeta)$. Now, set $\frac{z_{[\zeta m]}}{z_m} = \frac{x_{[\zeta m]}}{x_m} - \frac{y_{[\zeta m]}}{y_m}$, $m \in \mathbb{N}_0$. Since $\left\{m \in \mathbb{N}_0 : \frac{x_{[\zeta m]}}{x_m} \neq \frac{y_{[\zeta m]}}{y_m}\right\} \subset \mathbb{N} \setminus K$, $\delta_A^a(\mathbb{N} \setminus K) = 0$, $\delta_A^a\left(\left\{m \in \mathbb{N}_0 : \frac{z_{[\zeta m]}}{z_m} \neq 0\right\}\right) = 0$. Thus $\delta_A^a\left(supp\frac{z_{[\zeta m]}}{z_m}\right) = 0$ and $\frac{x_{[\zeta m]}}{x_m} = \frac{y_{[\zeta m]}}{y_m} + \frac{z_{[\zeta m]}}{z_m}$.

Conversely, suppose there exist sequences $\mathbf{y} = \{y_m\}$ and $\mathbf{z} = \{z_m\}$ satisfying $\frac{x_{[\zeta m]}}{x_m} = \frac{y_{[\zeta m]}}{y_m} + \frac{z_{[\zeta m]}}{z_m}$ with $\lim \frac{y_{[\zeta m]}}{y_m} = k(\zeta)$ and $\delta_A^a\left(supp\frac{z_{[\zeta m]}}{z_m}\right) = 0$, where $supp\mathbf{z} = \left\{m \in \mathbb{N}_0 : \frac{z_{[\zeta m]}}{z_m} \neq 0\right\}$. We show that st_P-$\lim \frac{x_{[\zeta m]}}{x_m} = k(\zeta)$. Define $K = \left\{m \in \mathbb{N}_0 : \frac{z_{[\zeta m]}}{z_m} = 0\right\} \subset \mathbb{N}_0$. Since $\delta_A^a\left(supp\frac{z_{[\zeta m]}}{z_m}\right) = 0$, it follows that $\delta_A^a(K) = 1$. For $m \in K$, we have $\frac{x_{[\zeta m]}}{x_m} = \frac{y_{[\zeta m]}}{y_m}$. Consequently, there exists a set $K = \{m_0 < m_1 < m_2 < ... < m_k < ...\}$ with $\delta_A^a(K) = 1$ such that $\lim \frac{x_{[\zeta m_k]}}{x_{m_k}} = k(\zeta)$. Hence, completing the proof. The theorem has been proved.

Corollary 2.2. *Let $\mathbf{x} = (x_m)$ be a sequence of positive real numbers. Then, st_A^a-$\lim \frac{x_{[\zeta n]}}{x_n} = k(\zeta)$ if and only if there exist sequences $\mathbf{y} = \{y_m\}$ and $\mathbf{z} = \{z_m\}$ such that $\frac{x_{[\zeta m]}}{x_m} = \frac{y_{[\zeta m]}}{y_m} + \frac{z_{[\zeta m]}}{z_m}$ with $\lim \frac{y_{[\zeta m]}}{y_m} = k(\zeta)$ and st_A^a-$\lim \frac{z_{[\zeta m]}}{z_m} = 0$.*

Proof. Let $\frac{z_{[\zeta m]}}{z_m} = \frac{x_{[\zeta n]}}{x_n} - \frac{y_{[\zeta n]}}{y_n}$, where $\{y_m\}$ is the sequence defined by (2.2) of the previous proof. Then $\lim \frac{y_{[\zeta m]}}{y_m} = k(\zeta)$, and by Theorem 1.1 we conclude that st_A^a-$\lim \frac{z_{[\zeta m]}}{z_m} = 0$.

For the converse, assume $\frac{x_{[\zeta m]}}{x_m} = \frac{y_{[\zeta m]}}{y_m} + \frac{z_{[\zeta m]}}{z_m}$, where $\lim \frac{y_{[\zeta n]}}{y_n} = k(\zeta)$ and st_A^a-$\lim \frac{z_{[\zeta m]}}{z_m} = 0$. Since st_A^a-$\lim \frac{y_{[\zeta m]}}{y_m} = k(\zeta)$, applying Theorem 1.1 we get st_A^a-$\lim \frac{y_{[\zeta m]}}{y_m} = k(\zeta)$. The corollary has been proved.

The following result is derived from Theorem 1.3.

Theorem 2.4. *If $\alpha_1 = st_A^a$-$\limsup \frac{x_{[\zeta m]}}{x_m}$ is finite, then for all $\varepsilon > 0$*

$$\delta_A^a\left(\left\{m : \frac{x_{[\zeta m]}}{x_m} > \alpha_1 - \varepsilon\right\}\right) \neq 0 \text{ and } \delta_A^a\left(\left\{m : \frac{x_{[\zeta m]}}{x_m} > \alpha_1 + \varepsilon\right\}\right) = 0. \tag{2.3}$$

Conversely, assuming the validity of the condition (2.3) for an arbitrary ε*, then* $\alpha_1 = st_A^a\text{-}\limsup \frac{x_{[\zeta m]}}{x_m}$.

The corresponding characterization for the almost A-statistical limit inferior is presented next.

Theorem 2.5. *If* $\beta_1 = st_A^a\text{-}\liminf \frac{x_{[\zeta m]}}{x_m}$ *is finite, then for every* $\varepsilon > 0$

$$\delta_A^a\left(\left\{m : \frac{x_{[\zeta m]}}{x_m} < \beta_1 + \varepsilon\right\}\right) \neq 0 \text{ and } \delta_A^a\left(\left\{m : \frac{x_{[\zeta m]}}{x_m} < \beta_1 - \varepsilon\right\}\right) = 0. \quad (2.4)$$

Conversely, if (2.4) hold for every positive ε*, then* $\beta_1 = st_A^a\text{-}\liminf \frac{x_{[\zeta n]}}{x_n}$.

Theorem 2.6. *An almost* $A-$*statistically* $RV-$*bounded sequence* $\mathbf{x}$ *is almost* $A-$*statistically regularly varying if and only if*

$$st_A^a-\liminf \frac{x_{[\zeta m]}}{x_m} = st_A^a-\limsup \frac{x_{[\zeta m]}}{x_m}.$$

Proof. Let $\alpha = st_A^a\text{-}\limsup \frac{x_{[\zeta m]}}{x_m}$ and $\beta = st_A^a\text{-}\liminf \frac{x_{[\zeta m]}}{x_m}$. First, assume that $st_A^a\text{-}\lim \frac{x_{[\zeta m]}}{x_m} = k(\zeta)$ and let $\varepsilon > 0$. Then $\delta_A^a\left(\left\{m \in \mathbb{N}_0 : \left|\frac{x_{[\zeta m]}}{x_m} - k(\zeta)\right| \geq \varepsilon\right\}\right) = 0$. Consequently, $\delta_A^a\left(\left\{m \in \mathbb{N}_0 : \frac{x_{[\zeta m]}}{x_m} > k(\zeta) + \varepsilon\right\}\right) = 0$, which implies that $\alpha \leq k(\zeta)$. Similarly, $\delta_A^a\left(\left\{m : \frac{x_{[\zeta m]}}{x_m} < k(\zeta) - \varepsilon\right\}\right) = 0$, implying $k(\zeta) \leq \beta$. Therefore $\alpha \leq \beta$. Using the general inequality $\alpha \geq \beta$ (which holds by definition), we conclude that $\alpha = \beta$.

For the converse, suppose $\alpha = \beta$ and define $k(\zeta) = \alpha$. Given $\varepsilon > 0$, conditions (2.3) and (2.4) from Theorem 2.4 and Theorem 2.5 respectively, yield $\delta_A^a\left(\left\{m : \frac{x_{[\zeta m]}}{x_m} > k(\zeta) + \frac{\varepsilon}{2}\right\}\right) = 0$ and $\delta_A^a\left(\left\{m : \frac{x_{[\zeta m]}}{x_m} < k(\zeta) - \frac{\varepsilon}{2}\right\}\right) = 0$. which is equivalent to $st_A^a\text{-}\lim \frac{x_{[\zeta m]}}{x_m} = k(\zeta)$. This completes the proof. The theorem has been proved.

Theorem 2.7. *If a positive real sequence* $\mathbf{x}$ *is* RV*-bounded above and strongly almost* $A-$*summable regularly varying to the number* $st_A^a\text{-}\limsup \frac{x_{[\zeta m]}}{x_m} = \alpha$*, then* $\mathbf{x}$ *is almost* $A-$*statistically regularly varying to* α.

Proof. Assume that $\mathbf{x}$ is not almost $A-$statistically regularly varying to α. Then by Theorem 1.3, $st_A^a\text{-}\liminf x_m < \alpha$, so there is a number $M < \alpha$ such that $\delta_A^a(\{m \in \mathbb{N}_0 : x_m < \alpha\}) \neq 0$. Let $E' = \{m \in \mathbb{N}_0 : x_m < M\}$. By the definition of α, $\delta_A^a(\{m \in \mathbb{N}_0 : x_m > \alpha + \varepsilon\}) = 0$ for every $\varepsilon > 0$. Define $E'' = \{m \in \mathbb{N}_0 : M \leq x_m \leq \alpha + \varepsilon\}$, $E''' = \{m \in \mathbb{N}_0 : x_m > \alpha + \varepsilon\}$ and let $C := \sup_m x_m$. Since $\delta_A^a\left(E'\right) \neq 0$, there are infinitely many j such that $\limsup \sum_{m \in E'} a_{jm} \geq$

$d > 0$. Hence we have

$$
\begin{aligned}
& \frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty} a_{jm}\left|\frac{x_{[\zeta m]}}{x_m}-\alpha\right|^q \\
= & \frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in E'} a_{jm}\left|\frac{x_{[\zeta m]}}{x_m}-\alpha\right|^q + \frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in E''} a_{jm}\left|\frac{x_{[\zeta m]}}{x_m}-\alpha\right|^q \\
+ & \frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in E'''} a_{jm}\left|\frac{x_{[\zeta m]}}{x_m}-\alpha\right|^q \\
\leq & M\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in E'} a_{jm} + (\alpha+\varepsilon)\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in E''} a_{jm} \\
+ & B\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in E'''} a_{jm} \\
= & M\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in E'} a_{jm} + (\alpha+\varepsilon)\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty} a_{jm} \\
- & (\alpha+\varepsilon)\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in E''} a_{jm} + O(1) \\
= & -\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in E'} a_{jm}\left[-M+(\alpha+\varepsilon)\right] + (\alpha+\varepsilon)\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty} a_{jm} + O(1) \\
\leq & \alpha\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty} a_{jm} - d(\alpha-M) + \varepsilon\left(\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty} a_{jm} - d\right) + O(1).
\end{aligned}
$$

Since $\varepsilon > 0$ is arbitrary it follows that

$$\limsup \frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty} a_{jm}x_m \leq \alpha - d(\alpha-M) < \alpha.$$

Hence, $\mathbf{x}$ is not strongly almost $A-$summable regularly varying to α, whence the result. The theorem has been proved.

Theorem 2.8. *Suppose that* $\mathbf{x}$ *is a sequence of positive real numbers which is bounded below in the RV-sense. If* $\mathbf{x}$ *is further assumed to be strongly almost A-summable regularly varying corresponding to the value* $st_A^a\text{-}\limsup \frac{x_{[\zeta m]}}{x_m} = \beta$, *then* $\mathbf{x}$ *is almost* $A-$*statistically regularly varying to* β.

Proof. Since the verification of this assertion follows lines similar to those of Theorem 2.7, the detailed proof is omitted to avoid redundancy.

The main motivation for this section is the following.

Theorem 2.9. *Let* q *be a positive real number. Then:*

(a) a sequence of positive real numbers $\mathbf{x} = (x_m)$ *is almost* $A-$*statistically regularly varying sequence if it is strongly almost* $A-$*summable regularly varying.*

(b) $wARV_{s,q} \cap RVl_\infty = ASRV_s \cap RVl_\infty$.

Proof. (a) Assume $\mathbf{x} \in wARV_{s,q}$. For a fixed $\varepsilon > 0$, define the set $F_\varepsilon(q) = \left\{ m \in \mathbb{N}_0 : \left| \frac{x_{[\zeta m]}}{x_m} - k(\zeta) \right|^q \geq \varepsilon \right\}$. We have:

$$\begin{aligned}
&\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty} a_{jm}\left|\frac{x_{[\zeta m]}}{x_m} - k(\zeta)\right|^q \\
= &\frac{1}{p}\sum_{j=k}^{k+p-1}\left\{\sum_{m\in F_\varepsilon(q)} a_{jm}\left|\frac{x_{[\zeta m]}}{x_m} - k(\zeta)\right|^q + \sum_{m\notin F_\varepsilon(q)} a_{jm}\left|\frac{x_{[\zeta m]}}{x_m} - k(\zeta)\right|^q\right\} \\
\geq &\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in F_\varepsilon(q)} a_{jm}\varepsilon
\end{aligned}$$

As $\mathbf{x}$ is strongly almost A-summable regularly varying, $\lim_{p\to\infty} \frac{1}{p} \sum_{j=k}^{k+p-1} \sum_{m\in F_\varepsilon(q)} a_{jm} = 0$, uniformly in k. Hence $x \in ASRV_s$.

(b) Let $\mathbf{x} \in ASRV_s \cap RVl_\infty$. Since $\mathbf{x}$ is a RV-bounded sequence, there exists a finite constant $M > 0$ such that $\left|\frac{x_{[\zeta m]}}{x_m}\right| + |k(\zeta)| \leq M$ for all m. Define the set $E_\varepsilon(q) = \left\{ m \in \mathbb{N}_0 : \left| \frac{x_{[\zeta m]}}{x_m} - k(\zeta) \right| \geq \left(\frac{\varepsilon}{2}\right)^{\frac{1}{q}} \right\}$. Since $\mathbf{x} \in ASRV_s$, we get

$$\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m\in E_\varepsilon(q)} a_{jm} < \frac{\varepsilon}{2M^q}.$$

Now, consider the strong summability expression

$$\begin{aligned}
&\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty} a_{jm}\left|\frac{x_{[\zeta m]}}{x_m} - k(\zeta)\right|^q \\
= &\frac{1}{p}\sum_{j=k}^{k+p-1}\left\{\sum_{m\in F_\varepsilon(q)} a_{jm}\left|\frac{x_{[\zeta m]}}{x_m} - k(\zeta)\right|^q + \sum_{m\notin F_\varepsilon(q)} a_{jm}\left|\frac{x_{[\zeta m]}}{x_m} - k(\zeta)\right|^q\right\} \\
< &\frac{\varepsilon}{2M^q}M^q + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

Hence, taking supremum over k and putting $p \to \infty$, we have

$$\lim_{p\to\infty}\frac{1}{p}\sum_{j=k}^{k+p-1}\sum_{m=0}^{\infty} a_{jm}\left|\frac{x_{[\zeta m]}}{x_m} - k(\zeta)\right|^q = 0, \text{ uniformly in } k, \text{ for all } \zeta > 0.$$

Therefore, $\mathbf{x} \in wARV_{s,q} \cap RVl_\infty$, completing the proof of the equality. The theorem has been proved.

In what follows, we characterize the relationship between almost A-statistical regular variation and strong almost A-summability in the context of regularly varying sequences.

Remark 2.1. If $0 < q_1 \leq q_2 < \infty$, then the following inclusion holds
$$wARV_{s,q_2} \subseteq wARV_{s,q_1}$$
and
$$wARV_{s,q} \cap RVl_\infty = wARV_{s,1} \cap RVl_\infty,$$
where RVl_∞ denotes the space of all RV-bounded sequences of positive real numbers.

2.2. Translational regular variation.

Definition 2.8. ([21, 24]) A sequence of positive real numbers $\mathbf{x} = (x_m)$ is in the class $Tr\left(RV_s\right)$ of *translationally regularly varying sequences* if for every real number $\zeta \in \mathbb{R}$,
$$\lim_{m\to\infty} \frac{x_{[m+\zeta]}}{x_m} = r\left(\zeta\right) < \infty.$$

The symbol $Tr\left(RV_{s,\rho}\right)$ stands for the collection of all sequences in $Tr\left(RV_s\right)$ of index of variability ρ.

Theorem 2.10. ([21]) *If a sequence* $\mathbf{x} = (x_m)$ *belongs to the class* $Tr\left(RV_s\right)$, *then the limit function can be expressed as* $r\left(\zeta\right) = e^{\rho[\zeta]}$, *for some real number* $\rho \in \mathbb{R}$; *which is called the variability index of* x.

Definition 2.9. A positive real sequence $\mathbf{x} = (x_m)$ is said to belong to the class $ASTr\left(RV_s\right)$ i.e., $\mathbf{x}$ is an almost $A-$statistically translationally regularly varying sequence, if the almost A-statistical limit of $\frac{x_{[n+\zeta]}}{x_m}$ converges to a finite value $r\left(\zeta\right)$ for every $\zeta \in \mathbb{R}$, i.e.,
$$st_A^a - \lim \frac{x_{[m+\zeta]}}{x_m} = r\left(\zeta\right) < \infty.$$

It follows directly from the definitions that if a sequence belongs to the class $Tr\left(RV_s\right)$, then it is also a member of the class $ASTr\left(RV_s\right)$.

Example 2.2. Consider the sequence (x_m) defined by
$$x_m = \begin{cases} m^4, & m = 2k \text{ and } m \neq 0 \\ e^{\sqrt{m}}, & \text{otherwise,} \end{cases} \quad k = 0, 1, 2, ...$$

Also, let us take the matrix $A = (a_{jm})$ whose general term is given by
$$a_{jm} = \begin{cases} 2, & \text{if } j \text{ is even and } m = j^2, \\ 0, & \text{otherwise.} \end{cases}$$
We know that this matrix is almost regular but non-regular and
$$\delta_A^a\left(\{m = 2k + 1 : \ k = 0, 1, 2, ...\}\right) = 0$$
and
$$\delta_A^a\left(\{m = 2k : \ k = 0, 1, 2, ...\}\right) = 1.$$
Then, for every $m = 2k$, $k = 1, 2, ...$,
$$\frac{x_{[m+\zeta]}}{x_m} = \frac{[m+\zeta]^4}{m^4} \to 1,$$
hence, (x_m) is an almost $A-$statistically translationally regularly varying sequence, but since $\lim \frac{x_{[m+\zeta]}}{x_m}$ does not exist, (x_m) is not translationally regularly varying.

Definition 2.10. A sequence $\mathbf{x} = (x_m)$ consisting of positive real numbers is an *almost $A-$statistically translationally regularly varying Cauchy sequence* if, for every $\varepsilon > 0$ and every $\zeta \in \mathbb{R}$, one can find an index $M = M\ (\varepsilon)$ such that

$$\delta_A^a \left(\left\{ m \in \mathbb{N}_0 : \left| \frac{x_{[m+\zeta]}}{x_m} - \frac{x_{[M+\zeta]}}{x_m} \right| \geq \varepsilon \right\} \right) = 0.$$

Theorem 2.11. *Let $\mathbf{x} = (x_m)$ be a sequence of positive real numbers. The following conditions are then equivalent:*

(a) $\mathbf{x}$ *belongs to the class of almost $A-$statistically translationally regularly varying sequences,*

(b) $\mathbf{x}$ *is classified as an almost $A-$statistically translationally regularly varying Cauchy sequence,*

(c) *One can find a translationally regularly varying sequence* $\mathbf{y}$ *such that*

$$\delta_A^a \left(\left\{ n \in \mathbb{N}_0 : \tfrac{x_{[\zeta m]}}{x_m} \neq \tfrac{y_{[\zeta m]}}{y_m} \right\} \right) = 0.$$

As an immediate consequence of Theorem 2.11 (likely establishing equivalence or a key property) and Theorem 2.10 (likely the Cauchy functional equation solution for translational regular variation), we present the following result concerning the sequence limit $r\,(\zeta)$.

Corollary 2.3. *If a sequence $\mathbf{x} = (x_m)$ of positive real numbers is almost $A-$statistically translationally regularly varying with limit function $r\,(\zeta)$, then $\mathbf{x}$ possesses a subsequence $\mathbf{y} = \{y_m\}$ that is translationally regularly varying to $r\,(\zeta)$. Moreover, the limit function has the form $r\,(\zeta) = e^{\rho[\zeta]}$, for some real number $\rho \in \mathbb{R}$.*

The class comprising all almost $A-$statistically translationally regularly varying sequences that possess an index of variability ρ is denoted by $ASTr\,(RV_{s,\rho})$.

Theorem 2.12. *A sequence $\mathbf{x} = (x_m)$ of positive real numbers belongs to the class $ASTr\,(RV_{s,\rho})$, for $\rho \in \mathbb{R}$, if and only if it can be expressed as*

$$x_m = x_0 \cdot \exp\left(\sum_{i=0}^{m-1} a_i \right),\ m \geq 1,$$

where (a_m) is a real sequence satisfying $st_A^a - \lim \exp(a_m) = \exp(\rho)$ and $x_0 > 0$.

Proof. Let $\mathbf{x} = (x_m) \in ASTr\,(RV_{s,\rho})$, $\rho \in \mathbb{R}$. By Corollary 2.3, we have

$$st_A^a - \lim \frac{x_{m+1}}{x_m} = r\,(1) = \exp(\rho) < \infty.$$

This implies the existence of a sequence $\{b_m\}$ of positive numbers such that

$$st_A^a - \lim b_m = r\,(1) \text{ and } \frac{x_{m+1}}{x_m} = b_m,\ m \in \mathbb{N}_0.$$

For $m \geq 0$, it follows that

$$x_{m+1} = b_m x_m = b_m b_{m-1} ... b_0 x_0.$$

Setting $a_i = \ln b_i$ for each $i \in \mathbb{N}_0$, we obtain

$$st_A^a - \lim \exp(a_m) = st_A^a - \lim b_m = \exp(\rho),$$

and for every $m \in \mathbb{N}_0$,

$$x_{m+1} = x_0 \cdot \exp\left(\sum_{i=0}^{m} a_i\right).$$

Equivalently, for each $m \geq 1$

$$x_m = x_0 \cdot \exp\left(\sum_{i=0}^{m-1} a_i\right), \text{ where } st_A^a - \lim \exp(a_m) = \exp(\rho).$$

Conversely, assume

$$x_m = x_0 \cdot \exp\left(\sum_{i=0}^{m-1} a_i\right), \ m \geq 1,$$

with st_A^a-$\lim \exp(a_m) = \exp(\rho)$. Then, we have

$$r(1) = st_A^a - \lim \frac{x_{m+1}}{x_m} = st_A^a - \lim \exp(a_m) = \exp(\rho).$$

Consequently, for every $\zeta \in \mathbb{R}$

$$st_A^a - \lim \frac{x_{[m+\zeta]}}{x_m} = r(\zeta) = \exp(\rho[\zeta]).$$

Hence, $\mathbf{x} = (x_m) \in ASTr(RV_{s,\rho})$, completing the proof.

3. Final comments and open problems

In this paper we considered two variants of regular variation (in the sense of Karamata) in the context of almost $A-$statistical convergence. Among others, we gave a decomposition theorem for almost $A-$statistical regular variation and a representation theorem for almost $A-$statistically translationally regularly varying sequences. Our future work should be devoted to a similar study of $\mathcal{O}$-regular variation (see [1, 2, 12]) and rapid variation in the sense of de Haan (see [8, 14]).

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Received: February 26, 2026; Accepted: June 2, 2026