Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 00, Number 0, XXXX, Pages 000–000

COMPLETENESS OF THE SYSTEM OF EIGEN AND ASSOCIATED VECTORS OF OPERATORS GENERATED BY PARTIAL OPERATOR-DIFFERENTIAL EXPRESSIONS IN HILBERT SPACE

GAMIDULLA I. ASLANOV, KAMILLA A. ALIMARDANOVA, AND GUNEL M. EYVAZLI

Abstract. In the paper, we prove the completeness of the system of eigen and associated functions of the operator generated by higher order partial operator-differential expression with unbounded operator coefficients in Hilbert space.

1. Introduction

Let $H_0, H_1, ..., H_{2m}$ be Hilbert spaces, $H_{i+1} \subset H_i, i = 0, 1, ..., 2m-1$ and all embeddings be compact.

Let us consider the following differential expression

$$L(x, D) = \sum_{|\alpha| \le 2m} A_\alpha(x) D^\alpha u, \quad x \in R^n \tag{1.1}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n), |\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}$. The function $u(x) \in H_{2m}$ is such that $D^\alpha u \in H_{2m-|\alpha|}$. It is assumed that for each $x \in R^n$ the operator $A_\alpha(x),\ \alpha \ne 0$ is a bounded operator: $H_{2m-|\alpha|} \to H_0$.

$A_0(x) = A_0 + q(x)$, where A_0 is a positive-definite self-adjoint operator $H_{2m} \to H_0$, such that A_0^{-1} is completely continuous. The complex-valued function $q(x)$ is assumed to be measurable and locally bounded.

In this paper, we establish the completeness of the system of eigen and associated functions of the operator L generated by the expression $L(x, D)$.

The definition of m-fold completeness of the system of eigen and associated functions belongs to M.V. Keldysh [11]. He has established m-fold completeness of the system of eigen and associated functions of some polynomial operator pencil. And this was followed by a large series of works on theory of non selfadjoint operators.

The works of J.E. Allahverdiyev [7], V.M. Lidskii [15], A.S. Markus [16], V.I.Matsayev [17], M.G. Gasymov [9], A.G. Kostyuchenko [12], S.S.Mirzoev [18],

---

2020 *Mathematics Subject Classification.* 35J25, 47A10, 58J40.

Key words and phrases. Hilbert space, operator-differential expressions, operator, eigen and associated vectors, completeness, resolvent, scalar product, inverse operator, Fourier transform, eigenvalues.

A.A. Shkalikov [20], S. Agmon [1], F.Browder [8] and others made a significant contribution to this theory.

2. Main results

Before deriving the main estimate for our purpose of the resolvent of the operator $L(x,D)$, we formulate conditions that the coefficients of the operator $L(x,D)$ must satisfy.

We will denote

$$R_0(x,\xi)=\left[\sum_{|\alpha|=2m}A_\alpha(x)(i\xi)^\alpha\right]^{-1},\quad \xi=(\xi_1,\xi_2,...,\xi_n),\quad \xi^\alpha=\xi_1^{\alpha_1},\xi_2^{\alpha_2}...\xi_n^{\alpha_n}.$$

Assume that

1) $R_0(x,\xi)$ for all $\xi\in R^n\setminus\{0\}$ is a bounded operator $H_0\to H_{2m}$, and

$$\sum_{i=0}^{2m}|\xi|^i\,\|R_0\|_{H_0\to H_{2m-i}}\le\delta_1$$

where $\delta_1=const$ does not depend on x and ξ.

2) There exists a ray $l=\{\lambda:\arg\lambda=\beta\}$ of the complex plane λ such that the operator

$$R(x,\xi,\lambda)=\left[\sum_{|\alpha|\le 2m}A_\alpha(x)(i\xi)^\alpha-\lambda E\right]^{-1}$$

is a bounded operator $H_0\to H_{2m}$ for $\lambda\in l$, $\xi\in R^n$, $|x|>c$ and

$$(|q(x)|+|\lambda|)\,\|R_0\|_{H_0\to H_0}+\sum_{i<2m}|\xi|^{2m-i}\,\|R_0\|_{H_0\to H_{2m-i}}\le\delta_2.$$

$\delta_2=const$ does not depend on x and ξ; c is a positive number.

3) The quantities $\sup\limits_{|x-x_0|\le h}\|A_\alpha(x)-A_\alpha(x_0)\|,\sup\left|\frac{q(x)-q(x_0)}{q(x_0)}\right|$ tend to zero as $h\to 0$ uniformly with respect to x_0, $|\alpha|=2m$.

4) $\sum\limits_{0<|\alpha|\le 2m}\|A_\alpha(x)\|\le\delta_3,\ \delta_3=const$ does not depend in x and ξ.

5) The operator A_0 is a self-adjoint positive definite operator $H_0\to H_0$, bounded $H_{2m}\to H_0$ and such that A_0^{-1} is completely continuous, $\lambda_k(A_0^{-1})\ge ck^{-\frac{n}{2m}}$ and here $0<\lambda_1<\lambda_2<...$ is a spectrum of the operator A_0^{-1}.

We denote $\mathcal{H}_i$ space with a scalar product

$$(f,g)_{\mathcal{H}_i}=\int_{R^n}(f(x),g(x)_{H_i}dx,\quad f,g\in H_i, i=\overline{1,2m}.$$

Now let's formulate a theorem on the completeness of eigen and associated functions of the operator L.

The operator L_0 is defined as $L_0u=L(x,D)u$ on the set of functions from $C_0^\infty(R^n)$. If conditions 1)-4) are fulfilled, then the operator L_0 admits closure in $\mathcal{H}_0$. This closure is $\overline{L}_0$ and is considered as an operator L.

Theorem 2.1. *Let conditions 1)-5) be fulfilled, and the condition 2) be fulfilled for all β such that $\beta_1 < \beta < \beta_2$.*

And besides $\beta_0 = \max\left[2\pi - (\beta_2 - \beta_1), (\beta_2 - \beta_1)\right] < \frac{2m\pi}{n}$.

Let also

$$|q(x)| \geq c\,|x|^{\gamma}\,, \quad \textit{where}\ \gamma > \frac{2m\beta_0}{\frac{2m\pi}{n} - \beta_0}.$$

Then the system of eigen and associated functions of the operator L is complete in the space $\mathcal{H}_0$.

Proof. The first stage of the proof is to show that the resolvent of the operator L belongs to a certain σ_p.

Let us consider the auxiliary equation

$$L_1 u \equiv (-1)^m \sum_{i=1}^{2m} \frac{\partial^{2m} u}{\partial x_i^{2m}} + A_0 u + |q(x) - \lambda|\, u = f, \quad x \in R^n. \tag{2.1}$$

We determine the operator L_1 as a closure on $C_0^{\infty}(R^n)$ of the operator $u \to L_1 u$ by the norm $\mathcal{H}_0$.

From the results of the work [6] it follows that the operator L_1 has a completely continous inverse operator L_1^{-1}. Indeed, from the condition $\lim\limits_{|x|\to\infty} |q(x)| = \infty$ it follows that for any $\varepsilon > 0$ there exists such R that

$$\int\limits_{|x|>R} \|u\|_{H_0}^2 \, dx < \varepsilon.$$

and if $\|f\|_{\mathcal{H}_0} \leq 1$, then the estimation

$$\left\| (-1)^m \sum_{i=0}^{2m} \frac{\partial^{2m} u}{\partial x_i^{2m}} + A_0 u \right\|_{\mathcal{H}_0} \leq C.$$

Applying the Fourier transform, we obtain:

$$\int\limits_{R^n} \left\| |\xi|^{2m} \, |\widetilde{u}|^2 + A_0 \widetilde{u} \right\|_{H_0}^2 d\xi \leq C.$$

Since the operator A_0 is a positive definite operator, we have

$$\int\limits_{R^n} |\xi|^{2m} \, |\widetilde{u}|^2 \, d\xi \leq C_1, \quad \int\limits_{R^n} \|A_0 \widetilde{u}\|_{H_0}^2 \, d\xi \leq C_2$$

i.e.

$$\int\limits_{R^n} \left(\|u\|_{H_0}^2 + \sum_{|\alpha|=2m} \|D^{\alpha} u\|_{H_0}^2 \right) dx \leq C_2$$

Moreover

$$\sum_{|\alpha|=2m} \int\limits_{|x|>R} \|D^{\alpha} u\|_{H_0}^2 \, dx \leq C_2.$$

It follows from the Arzela-Ascoli theorem that the totality of contractions of functions $u(x)$ such that $\|L_1 u\| \leq 1$ on the ball $S_R = \{x : |x| < R\}$ is compact in

$H_0(S_R)$. This means that from the set of such $u(x)$ we can choose $u_\varepsilon(x), \varepsilon$ is a set in $H_0(S_R)$. The totality of functions

$$u'_\varepsilon(x) = \begin{cases} u_\varepsilon(x) & |x| < R \\ 0 & |x| > R \end{cases}$$

forms 2ε- set in $H_0(R^n)$. Hence it follows that L_1^{-1} is a compact operator. Since the operator L_1 is symmetric and invertible it is a selfadjoint and unbounded operator $\mathcal{H}_0 \to \mathcal{H}_0$. The operator L_1^{-1}, inverse to the self-adjoint one, is also a selfadjoint operator. Let $\varphi_1, \varphi_2, ...$ be a complete in $\mathcal{H}_0$ orthonormal system of eigen elements of the operator L_1^{-1}, while $\mu_1 \geq \mu_2 \geq ...$ are appropriate eigenvalues. From the variational principle for self-adjoint operators it follows that

$$\mu_1 = \max_{f\in\mathcal{H}_0} \frac{(L_1^{-1}f, f)}{(f, f)}, \quad \mu_{j+1} = \max_{\substack{f\in\mathcal{H}_0 \\ f\perp P_j}} \frac{(L_1^{-1}f, f)}{(f, f)}, \tag{2.2}$$

where P_j is a linear span of the elements $\varphi_1, \varphi_2, ..., \varphi_j$.

Let $\lambda^k = \mu_k^{-1}$.

In this case

$$\lambda_1 = \min_{u\in D(L_1)} \frac{(L_1, u, u)}{(u, u)_{\mathcal{H}_0}} =$$

$$= \min_{u\in D(L_1)} \frac{\int\limits_{R^n} \left\{ \sum\limits_{i=1}^{n} \left\| \frac{\partial^m u}{\partial x_i^m} \right\|_{H_0}^2 + (A_0, u, u) + |q(x) - \lambda| \, \|u\|_{H_0}^2 \right\} dx}{\|u\|_{\mathcal{H}_0}^2}$$

where $D(L_1)$ is a domain of definition of the operator L_1.

By the Courant principle (see [10])

$$\lambda_1 = \max_{P\in N_j} \min_{\substack{u\in D(L_1) \\ u\perp P}} \frac{\int\limits_{R^n} \left\{ \sum\limits_{i=1}^{n} \left\| \frac{\partial^m u}{\partial x_i^m} \right\|_{H_0}^2 + (A_0, u, u) + |q(x) - \lambda| \, \|u\|_{H_0}^2 \right\} dx}{\|u\|_{\mathcal{H}_0}^2},$$

where N_j the set of all j-dimensional lineals from $D(L_1)$.

Let Q_y be a cube of edge length 1 centered at y with the edges parallel to the coordinate axes. Denote by D' the set of functions $u(x) \in \mathcal{H}_0$ such that

$$\int\limits_{Q_y} \|u\|_{H_{2m}}^2 \, dx < \infty$$

It follows from the theorem conditions that

$$|q(x) - \lambda_0| \geq cr(x) \quad x \in R^n, \quad c > 0,$$

where

$$r(x) = \begin{cases} 1, & for \quad |x| \leq N \\ |x|^\gamma & for \quad |x| > N \end{cases}$$

Let us consider the function $\rho(x), x \in R^n$ determined as follows:

$$\rho(x) = \rho_l = \min r(x), \quad l = 0, 1, 2, ...$$

$$I_l = \{x : |x_i| \leq l\}$$

Note that $\rho_l(x)$ is a piecewise constant function. Then

$$\widetilde{\lambda}_1 = \inf_{u\in D(L)} \frac{\int\limits_{R^n} \left[\sum\limits_{i=1}^{n} \left\| \frac{\partial^m u}{\partial x_i^m} \right\|_{H_0}^2 + \rho(x) \left\| u \right\|_{H_0}^2 + (A_0 u, u) \right] dx}{\int\limits_{R^n} \left\| u \right\|_{H_0}^2 dx} \leq \lambda_1$$

$$\widetilde{\lambda}_{j+1} = \sup_{P\in M_j} \inf_{\substack{u\in D(L)\\ u\in P}} \frac{\int\limits_{R^n} \left[\sum\limits_{i=1}^{n} \left\| \frac{\partial^m u}{\partial x_i^m} \right\|_{H_0}^2 + \rho(x) \left\| u \right\|_{H_0}^2 + (A_0 u, u) \right] dx}{\left\| u \right\|_{\mathcal{H}_0}^2} \leq \lambda_{j+1}$$

$j = 1, 2, ...$ where M_j is the set of all j-dimensional lineals in $D(L)$. In the cube Q_y we consider the following problem

$$(-1)^m \sum_{i=1}^{n} \frac{\partial^{2m} u}{\partial x_i^{2m}} + A_0 u + \rho(x) u = \lambda u, \quad x \in Q_y \tag{2.3}$$

$$u(x) \in H_{2m}, \qquad \int\limits_{Q_y} \left\| u \right\|_{H_{2m}}^2 dx < \infty.$$

Let's estimate from above the number of eigenvalues of the equation (2.3) not exceeding the number Λ. Eigenvalues of the equation (2.3) have the form

$$\nu_{j_1} + \nu_{j_2} + ...\nu_{j_n} + \nu_{j_{n+1}} + q_0$$

where $\nu_k,\ k \leq n$ are the eigen-values of the following boundary value problem for an ordinary differential operator:

$$\begin{cases} (-1)^m y^{(2m)}(t) = \lambda y(t) \\ y^{(m)}(0) = y^{(m+1)}(0) = ... = y^{(2m-1)}(0) = 0, \\ y^{(m)}(1) = y^{(m+1)}(1) = ... = y^{(2m-1)}(1) = 0 \end{cases} \tag{2.4}$$

$\nu_{j_{n+1}}$ is a point of the spectrum of the operator A_0.

In theory of boundary value problems for ordinary differential operators it is proved that (see [17])

$$\nu_k \sim c k^{2m} \tag{2.5}$$

Let Λ be a rather large positive number. We denote by $r_j(\Lambda)$ the number of eigenvalues not exceeding Λ of equation (2.3) in the cube $Q_y \subset I_j \setminus I_{j-1}$. The number of such cubes with nonintersecting interiors equals $2^n[j^n - (j-1)^n]$.

Since the function $r(x)$ is an increasing function, there is such an integer $S > 0$ that $\rho(x) > \Lambda$ in $I_{s+1}|I_s$.

Therefore the number $r(\Lambda)$ of eigenvalues of the equation (2.4) not exceeding Λ is such that

$$r(\Lambda) \leq 2^n \left\{ r_1(\Lambda) + (2^n - 1) r_2(\Lambda) + (3^n - 2^n) r_3(\Lambda) + ... + [S^n - (S-1)^n] r_s(\Lambda). \right.$$

Due to (2.5) $r_1(\Lambda)$ does not exceed the number $N(\Lambda)$ of whole non-negative solutions of the inequality

$$\rho_0 + c_1(j_1^{2m} + j_2^{2m} + ... + j_n^{2m}) + \nu_{jn+1} \leq \Lambda.$$

Hence it follows that $N(\Lambda)$ does not exceed the volume of n-dimensional cube with the rib $\left(\frac{\Lambda-q_0-\nu_{jn+1}}{c_1}\right)^{\frac{1}{2m}}+1$.

From the inequality $\rho(x)\le\Lambda$ fulfilled in $I_{s+1}\setminus I_s$ follows that

$$S\le\left(\frac{\Lambda}{c}\right)^{\frac{1}{\gamma}}+1\quad(c=const>0)$$

Thus, for Λ rather large

$$r(\Lambda)\le c\Lambda^{n\left(\frac{1}{2m}+\frac{1}{\gamma}\right)}$$

It follows that

$$\lambda_j\ge\left(\frac{j}{c}\right)^{\frac{2m\gamma}{n(2m+\gamma)}}$$

Since, $\mu_j^{-1}=\lambda_j$, then $\sum\limits_{j=1}^{\infty}\mu_j^p<\infty$ for $p>\frac{n}{\gamma}+\frac{n}{2m}$,i.e. $L_1^{-1}\in\sigma_p$ for $p>\frac{n}{\gamma}+\frac{n}{2m}$.

It is easy to see that

$$L-\lambda_0E=(L-\lambda_0E)L_1^{-1}L_1.$$

This follows from $D(L)=D(L_1)$ (i.e. the domains of definition of the operators L and L_1 coincide).

It is easily seen that

$$(L-\lambda_0E)^{-1}=L_1^{-1}\left[L_1(L-\lambda_0E)^{-1}\right]$$

It is not difficult to show that the operator $L_1(L-\lambda_0E)^{-1}$ is a bounded operator $\mathcal{H}_0\to\mathcal{H}_0$. Let $f\in\mathcal{H}_0$. Denote $u_0=(L-\lambda_0E)^{-1}f$. Then

$$\left\|L_1(L-\lambda_0E)^{-1}f\right\|_{\mathcal{H}_0}\le\|L_1u_0\|_{\mathcal{H}_0}\le c\,\|f\|_{\mathcal{H}_0}.$$

Since the operator $L_1^{-1}\in\sigma_p$, $p>\frac{n}{\gamma}+\frac{n}{2m}$, while the operator $L_1(L-\lambda_0E)^{-1}$ is bounded, the operator $(L-\lambda_0E)^{-1}\in\sigma_p$, where p is any number such that $p>\frac{n}{\gamma}+\frac{n}{2m}$.

Notice that if λ belongs to the regular ray and is rather large, then

$$\|R(\lambda,L)\|_{\mathcal{H}_0\to\mathcal{H}_0}=O\left(\frac{1}{|\lambda|}\right)\quad\text{as}\quad|\lambda|\to\infty,\arg\lambda=\beta\tag{2.6}$$

Show that if $f^*\in\mathcal{H}_0$ is orthogonal to $Sp(L)$,then $f^*=0$.

By $Sp(L)$ we denote the closure of the space of all eigen and associated elements of the operator L. This means that $Sp(L)=\mathcal{H}_0$. We can assume that (this does not violate generality) the zero is not a point of the spectrum of the operator L.

In this case $R(\lambda,L)$ is a meromorphic function with poles at points of the spectrum of the operator L. Let $L^{-1}=T$. Let us consider the function

$$F(\lambda)=\left(f^*,R\left(\frac{1}{\lambda},T\right)f\right)_{\mathcal{H}_0}$$

where $f\in\mathcal{H}_0$.

$F(\lambda)$ is analytic at regular points of the operator L and has poles at its points of spectrum. Since $(f^{j*},Sp(L))_{\mathcal{H}_0}$, then singular part of Laurent expansion of the function $F(\lambda)$ equals zero, and consequently $F(\lambda)$ is an entire function. It follows from [8] that

$$F(\lambda) = O(|\lambda|) \text{ as } \lambda \to \infty, \arg\lambda = \beta.$$

It follows from Agmon's theorem [1] that there exists a sequence of positive numbers $r_i \to \infty$ such that

$$|F(\lambda)| \le e^{|\lambda|^{p+\varepsilon}} \tag{2.7}$$

for $|\lambda| = r_i, i = 1, 2, ...$ and for any $\varepsilon > 0$.

Let us consider the function $F(\lambda)$ in the closure of the angle $\beta_1 < \beta < \beta_2$, formed by the rays arg $\arg\lambda = \beta_1$ and $\arg\lambda = \beta_2$. By the theorem conditions the size of such an angle is less than P. Choosing ε in (2.7) rather small and applying the Phragmen-Lindelof theorem, we obtain $F(\lambda) = O(|\lambda|)$ as $\lambda \to \infty$. According to the Liouville theorem this means that $F(\lambda)$ is a linear function, $F(\lambda) = c_1 + c_2\lambda$.

On the other hand, $\left(R\left(\frac{1}{\lambda}\right), T\right) = \lambda E + \lambda^2 T + ...$, consequently,

$$F(\lambda) = \lambda(f^*, f)_{\mathcal{H}_0} + \lambda^2(f^*, Tf)_{\mathcal{H}_0} + ...$$

This and property of linearity of $F(\lambda)$ means that $(f^*, Tf)_{\mathcal{H}_0} = 0$. If f is an arbitrary element from the domain of definition of the operator T and since the range of values of the operator T is closed, then we obtain $f^* = 0$.

Theorem is proved. □

Compared to ordinary operator-differential equations, partial operator-differential equations have been investigated much less. In this direction, we refer to the works of S. Agmon, A. Douglis, L. Nirenberg [2], G.I. Aslanov [3-6], A.A. Shkalikov [20], V.B. Shakhmurov [21], V.B. Shakhmurov and A.A. Babaev [22] and others. In general for the detailed study of operator-differential equations we refer to fundamental monographs of S.G. Krein [13], J.L. Lions and E. Magenes [14], S.Ya. Yakubov [23].

Acknowledgments. We express our deep gratitude to professors M. Bayramoglu and S.S. Mirzoyev for useful discussions of the results obtained.

References

[1] S. Agmon, On the eigenfunctions and on the eigen-values of general elliptic boundary value problems. *Comm.Pure Appl. Math.,* **16** (1963), 121-139.

[2] S. Agmon, A.Douglis, L.Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. *Comm.Pure.Appl Math.* I. 1959, 12(4), pp.623-727; II, 17, (1964), no.1, pp.35-92.

[3] G.I. Aslanov, On the solvability and asymptotic behavior of solutions of differential equations in Hilbert space international Conference on differential equation and Related Questions (15 th joint of the Petrovskii seminar and the Moscow Mathematical Society, 19-22, January 1993. Russ. *Math. Sirv.***48** (1993), no.4, 190-191.

[4] G.I. Aslanov, Differential equations with operator coefficients in Hilbert space. *Math. Notes,* **53**, (1993), no.3, 351-353.

[5] G.I. Aslanov, Differential equations with unbounded operator coefficients in Hilbert space. *Dokl. Math.* **50** (1955). no.1, 5-9.

[6] G.I. Aslanov, On the boundedness of the resolvent of the operator generated by partial operator-differential expressions of Higher order in Hilbert space, *Baku Math. J.* **3**, (2024), no.2, 129-135.

[7] J.E. Allahverdiyev, On the completeness of the system of eigen and associated vectors of a class of parameter dependent non self-adjoint operators, *DAN SSSR*, **196** (1965), no.6 1231-1234.

[8] F. Browder, On the eigenfunctions and eigenvalues of the general linear elliptic differential operator, *Proc. Nat. Acad. Sci.,* **39** (1953) 433-439.

[9] M.G. Gasymov, On multiple completeness of a part of eigen and associated vectors of polynomial operator pencils, *Izv. AN Arm. SSR ser Math.* **155** (1964) no.2, 273-276.

[10] R. Courant., D.Hilbert, *Mathematical physics methods*, 1989.

[11] M.V. Keldysh, On the completeness of eigen functions of some classes of nonselfadjoint operators, UMN, **26** (1971) no.4, 15-4

[12] A.G. Kostyuchenko, *On some spectral properties of differential operators*, Doctoral dissertation MSU 1966.

[13] S.G. Krein, *Linear Differential Equations in Banach space.* Amer. Math. Soc. Providence, R.I. 1971.

[14] J.I. Lions, E. Magenes, *Non-Homogeneous boundary value problems and applications*, Dinod, Paris, 1968; Mir, Moscow, 1971, Springer, Berlin, 1972.

[15] V.B. Lidskii , On the completeness of the system of eigen and associated functions of non self-adjoint differential operators, *DAN SSR* **110** (1956),no.2, 172-175.

[16] A.S. Markus, Some criteria for the completeness of a system of root vectors of a linear operator in a Banach space, *Mat. Sb.,* **70** (112):4 (1966), 526–561.

[17] V.I. Matsaev, Several theorems on completeness of root vectors of completely continuous operators, *Dok. AN SSSR*, **155** (1964) no.2, 273-276.

[18] S.S. Mirzoev, Multiple completeness of root vectors of polynomial operator pencils corresponding to boundary-value problems on the semiaxis, *Funct. Anal. Appl.*, **17** (1983), no. 2, 151-153; (translated from *Funktsional. Anal. Prilozhen.*, **17** (1983), no. 2, 84-85).

[19] M.A. Naimark, *Linear differential operators*, Nauka, Moscow, 1969.

[20] A.A. Shkalikov, Elliptic equations in Hilbert space and associated spectral problems, *J. Soviet Math.*, **51**, 1990, no.4, 2399-2467 (translated from *Trudy Sem. Petrovsk.*, 14, 1989, 140-224).

[21] V.B. Shakhmurov, Coercive boundary value problems for strongly degenerated differential operator equations, *DAN SSSR,* **290**, (1986), no.3, 553-556.

[22] V.B. Shakhmurov, A.A. Babayev, Coercive problems for an equation with parameters, *DAN SSSR,* **315**, (1992), no.1, 37-40.

[23] S.Ya. Yakubov, *Linear differential operator equations and their applications*, Baku, Elm, 1985, 220 pp.

Gamidulla I. Aslanov

Ministry of Science of Education of the Republic of Azerbaijan, Institute of Mathematics, Baku, Azerbaijan

E-mail address: `aslanovhamidulla50@gmail.com`

Kamilla A. Alimardanova

Ministry of Science of Education of the Republic of Azerbaijan, Institute of Mathematics, Baku, Azerbaijan

E-mail address: `kamillaalimardanova@gmail.com`

Gunel M. Eyvazli

Azerbaijan Technical University, Baku Azerbaijan
E-mail address: gunel.eyvazli@aztu.edu.az

Received: December 23, 2025; Revised: May 18, 2026; Accepted: June 12, 2026