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ANTIAPPROXIMATIVELY COMPACT PLANES AND CAVERNS IN NORMED SPACES

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Abstract. A subset M of a normed linear space is approximatively compact if, for each point x, each minimizing sequence from M for x contains a subsequence convergent to a point from M. It is well known that a Banach space is a Efimov–Stechkin space (or a reflexive Kadec–Klee space) if and only if each hyperplane of this space is approximatively compact. For a partially inverse problem, we study spaces with antiapproximatively compact balls with respect to strong, weak, and Deutsch convergences. It turns out that in such spaces each hyperplane is antiapproximatively compact in the corresponding sense. We also consider the problem of existence of antiproximinal caverns and obtain, in this way, extensions of available results from the classical CLUR-spaces to more general spaces corresponding to other kinds of approximative compactness.

1. Introduction

A subset M of a normed linear space is approximatively compact if, for each point x, each minimizing sequence from M for x contains a subsequence convergent to a point from M. This definition, which was introduced by N. V. Efimov and S. B. Stechkin in the 1960s, proved very important in approximation theory. There are many various generalizations of this concept: approximative weak compactness, approximative compactness in the sense of F. Deutsch, approximative compactness in metric and semimetric spaces, etc. It is well known that a Banach space is a Efimov–Stechkin space (or a reflexive Kadec–Klee space) if and only if each hyperplane of this space is approximatively compact (or, equivalently, if each nonempty closed convex set is approximatively compact). E. V. Oshman and L. P. Vlasov (see, e.g., [2], [3]) characterized the normed linear spaces where each proximinal hyperplane is approximatively (strongly/weakly) compact. Their results have been recently advanced and generalized in [2], [3], [1].

In the present paper, we consider a partially inverse problem and introduce the antiapproximatively compact sets — these are the sets for which no points of approximative compactness lie outside the set (see Definition 2.3). In § 2, we study

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spaces with antiapproximatively compact balls with respect to strong, weak, and Deutsch convergences. It turns out that the space has antiapproximatively compact ball if and only if each hyperplane in this space is antiapproximatively compact. Generalizations to weak and Deutsch antiapproximative compactness are also obtained. In § 3, we consider the problem of existence of antiproximinal caverns. According to Pyatyshev [16], if $X \in$ (CLUR), then X contains an approximatively compact cavern (the complement of a bounded open convex set). We extend this result to approximative Deutsch compactness (and, in particular, to approximative weak compactness). The corresponding results (Theorem 3.3 and Corollary 3.2) are obtained in D-SVD- and SVD-spaces.

In what follows:
$X = (X, \|\cdot\|)$ is a normed linear space over $\mathbb{R}$,
$\mathring{B}(x,r) = \{y \in X \mid \|y-x\| < r\}$ is the open ball centered at x of radius r;
$B(x,r) = \{y \in X \mid \|y-x\| \le r\}$ is the closed ball;
$S(x,r) = \{y \in X \mid \|y-x\| = r\}$ is the sphere.

For brevity, $B := B(0,1)$ is the unit ball, $\mathring{B} := \mathring{B}(0,1)$ is the open unit ball, $S = S(0,1)$ is the unit sphere; S^* is the unit sphere of the dual space.

The *distance* from a point $x \in X$ to a nonempty set $M \subset X$ is defined by $\rho(x,M) := \inf_{y\in M} \|y-x\|$. The set of *nearest points* in M for a point $x \in X$ is defined by

$$P_M x := \bigl\{y \in M \mid \rho(x,M) = \|y-x\|\bigr\}.$$

A set M is an existence set (a proximinal set) if $P_M x \ne \varnothing$ for each $x \in X$. A set M is antiproximinal if $P_M x = \varnothing$ for each $x \in X$. The term *antiproximinal set* was proposed by Holmes [15]. A simplest example of a closed antiproximinal set is the hyperplane generated by a continuous functional which does not attain its norm.

2. Spaces with antiapproximatively compact balls with respect to strong, weak, and Deutsch convergences

Following Deutsch [12] and de Boor [10] (see also [4, Sec. 4.4]), we recall the definition of one fairly general type of convergence of nets (sequences) in normed linear spaces. The machinery of this convergence proved useful in obtaining existence of best approximation for a number of classical and abstract objects of approximation theory (see, e.g., [6], [12], and [17]).

Definition 2.1. Let X be a normed linear space. By a *Deutsch convergence* (or. briefly, *D-convergence*) $x_\delta \overset{D}{\to} x$ of nets (sequences) in X we mean a convergence D with the following properties:

(i) D is norm dominated, i.e., $x_\delta \overset{D}{\to} x$ implies that $\|x\| \le \limsup \|x_\delta\|$;

(ii) D is translation invariant, i.e., $x_\delta \overset{D}{\to} x$ implies that $x_\delta + y \overset{D}{\to} x+y$ for each $y \in X$;

(iii) D is homogeneous, i.e., $x_\delta \overset{D}{\to} x$ implies that $\alpha x_\delta \overset{D}{\to} \alpha x$ for each $\alpha \in \mathbb{R}$.

If nets in Definition 2.1 are replaced by sequences, we get the definition of a sequential Deutsch convergence.

In what follows, by Deutsch convergence we will always mean a sequential Deutsch convergence (sequential D-convergence).

Particular cases of Deutsch convergence include the strong convergence ($D = n$) and weak convergence ($D = w$). We also note the following fairly general (almost weak) convergence: $x_n \overset{\text{aw}}{\to} x$ (if a w^*-dense subset Λ of the set of all extreme points of the dual space X^* such that $x^*(x_\delta) \to x^*(x)$ for each $x^* \in \Lambda$ (here $D = \text{aw}$). This convergence is also considered for nets and for sequences.

Definition 2.2. Let D be a Deutsch convergence. A set M is *approximatively D-compact* (written: $M \in ({}^D\text{AC})$) if, for each point $w \notin M$ and each sequence $(z_n) \subset M$ with $\|z_n - w\| \to \rho(w, M)$, there exists a subsequence (z_{n_k}) D-convergent to some point in M.

For $D = n$ (strong convergence), Definition 2.2 is the classical definition of approximative compactness, and for $D = w$ (weak convergence), this is the definition of approximative weak compactness (for the last case, see, e.g., [3]).

Correspondingly, for a given nonempty set M,

– $\text{AC}(M)$ is the set of points of approximative compactness of a set M,

– $\text{wAC}(M)$ is the set of points of approximative weak compactness of M;

– ${}^D\text{AC}(M)$ is the set of points of approximative D-compactness of M.

Definition 2.3. A nonempty set M is:

– *approximatively compact* if $\text{AC}(M) = X$;

– *antiapproximatively compact* if $\text{AC}(M) = M$;

– *approximatively weakly compact* if $\text{wAC}(M) = X$;

– *antiapproximatively weakly compact* if $\text{wAC}(M) = M$.

It is well known that a Banach space is a Efimov–Stechkin space (or a reflexive Kadec–Klee space) if and only if each (closed) subspace of this space is approximatively compact (or, equivalently, if each nonempty closed convex set is approximatively compact). In the next theorem, we obtain a characterization in the inverse problem.

Theorem 2.1. *In a normed linear space, the following conditions are equivalent*:

a) *each hyperplane in X is antiapproximatively compact*;

b) $\text{AC}(B) = B$ (*i.e., each point outside the unit ball is not a point of approximative compactness of the unit ball*).

We note in passing that the spaces where each proximinal hyperplane is approximatively (strongly/weakly) compact were characterized by R. E. Megginson (see, e.g., [3]).

Remark 2.1. In the space $c_0(\Gamma)$, the unit ball B is antiapproximatively compact (i.e., $\text{AC}(B) = B$) (see Hu Fang [14, Corollary 4.1]; for c_0 this was proved earlier by Pyatyshev [16]). The same holds for the space $L^1(\mu)$ with atomless σ-finite σ-additive measure.

Corollary 2.1. *In the following spaces, each hyperplane is antiapproximatively compact:*

a) $c_0(\Gamma)$ (Γ *is an infinite set*);

b) $L^1(\mu)$ *with atomless σ-finite σ-additive measure.*

Remark 2.2. For c and c_0, the following partial analogue of Corollary 2.1 is known: the spaces c and c_0 have no proper infinite-dimensional approximatively compact subspaces [9].

Proof of Theorem 2.1. b) $\Rightarrow$ a). Let H be a proximinal hyperplane. We can assume that $H = \{z \in X \mid f(z) = 1\}$, where $f \in S^*$. Let $x \in P_H 0$ and let (y_n) be a minimizing sequence from the hyperplane H for the point $2x$. It is clear that $\rho(2x, H) = \rho(2x, B) = 1$. We set $v_n := y_n/\|y_n\|$. We have $\|2x - v_n\| \to 1$, and hence (v_n) is a minimizing sequence from the unit ball B for $2x$. By the assumption, $2x \notin \mathrm{AC}(B)$. Hence $2x \notin \mathrm{AC}(H)$. Now to prove the equality $\mathrm{AC}(H) = H$ it remains to employ the theorem on linearity of the metric projection along a subspace (see, e.g., Proposition 1.6 in [4]). If H is not proximinal, then H is antiproximinal (by the same theorem on linearity along a subspace). Hence, as above, $\mathrm{AC}(H) = H$.

a) $\Rightarrow$ b). Let $x \in S$ and let $\alpha > 1$. We claim that $\alpha x \notin \mathrm{AC}(B)$. Let H be a support proximinal hyperplane to the unit ball B at the point x. By the assumption, $\mathrm{AC}(H) = H$, and hence $\alpha x \notin \mathrm{AC}(H)$. Normalizing and proceeding as above, we find that $\alpha x \notin \mathrm{AC}(B)$. It is well known that if $u \notin M$, $u \in \mathrm{AC}(M)$, $v \in P_M u$, then $[u, v] \subset \mathrm{AC}(M)$. (A similar result also holds for approximatively Deutsch compact sets, and, in particular, for approximatively weakly compact set.) Hence $\lambda x \notin \mathrm{AC}(B)$ for each $\lambda > \alpha$. Since α is arbitrary, we have $\alpha x \notin \mathrm{AC}(B)$ for each $\alpha > 1$. Now the required equality $\mathrm{AC}(B) = B$ follows since $x \in S$ is arbitrary. $\square$

The proof of the next result is similar.

Theorem 2.2. *In a normed linear space, the following conditions are equivalent*:

a) *each hyperplane in X is antiapproximatively weakly compact*;

b) $\mathrm{wAC}(B) = B$ (*i.e., each point outside the unit ball is not a point of approximative weak compactness of the ball*).

Remark 2.3. Hu Fang showed that the unit ball B of the space c_0 is antiapproximatively weakly compact (i.e., $\mathrm{wAC}(B) = B$). As a result, in the space c_0 each hyperplane is antiapproximatively weakly compact and each subspace of finite codimension is antiapproximatively weakly compact.

3. Antiproximinal caverns. The classes of spaces (CLUR), (SVD), (D-SVD)

In 1966, Klee conjectured that if a Hilbert space contains a nonconvex Chebyshev set, then this space also contains a Chebyshev set with convex bounded complement. This conjecture was solved affirmatively by E. Asplund, who proposed the name “Klee cavern” for such sets. In general, by a *cavern* we mean the complement of a bounded convex open set; a *body* is a set with nonempty interior. Many interesting results on approximative properties of caverns were obtained by S. Cobzaş, V. S. Balaganskii, and I. A. Pyatyshev (see, e.g., [11] and [8]). We start with one result of Pyatyshev.

Theorem 3.A. *Let the unit ball B of a normed linear space X be antiapproximatively compact, i.e.,*

$$\mathrm{AC}(B) = B$$

and let $M \subset X$ be approximatively compact convex body. Then the cavern $\overline{X \setminus M}$ is antiproximinal.

Remark 3.1. Theorem 3.A was established by Pyatyshev [16] in the Banach space setting, however, the completeness of the space is of course superfluous.

Remark 3.2. Pyatyshev [16] constructed a bounded approximatively compact convex body in c_0 (and in c). Hence by Theorem 3.A and Remark 2.1 the space c_0 contains an antiproximinal set (of course, this result was known earlier; see, e.g., [11]).

Below, we obtain analogues of Theorem 3.A for the weak and Deutsch convergences.

Theorem 3.1. *Let the unit ball B of a normed linear space X be antiapproximatively weakly compact, i.e.,*

$$\mathrm{wAC}(B) = B$$

and let a convex body $M \subset X$ be approximatively weakly compact. Then the cavern $\overline{X \setminus M}$ is antiproximinal.

Remark 3.3. Above, we noted that the first condition of Theorem 3.1 (i.e., $\mathrm{wAC}(B) = B$) is met in the space c_0.

Proof of Theorem 3.1. Assume on the contrary that the cavern $X \setminus M$ is not antiproximinal. Then there exists a ball with center at an interior point $a \in M$ which supports the boundary of the set M at some point x. We can assume without loss of generality that $a = 0$, $\|x - a\| = \|x\| = 1$. Let H be a support hyperplane to the ball B at the point x. It can be assumed that $H = \{z \in X \mid f(z) = 1\}$, where $f \in S^*$. It is clear that $x \in P_M(2x)$. Indeed, if $y \in M$, then

$$\|2x - y\| \geq \frac{f(2x - y)}{\|f\|} = f(2x - y) = 2 - f(y) \geq 1,$$

but $\|2x - x\| = 1$, and hence $x \in P_M(2x)$. Let (y_n) be a minimizing sequence from $B \subset M$ for $2x$, i.e., $\|2x - y_n\| \to \rho(2x, B) = 1$. By the above $\rho(2x, M) = 1$, and hence (y_n) is also a minimizing sequence from M for $2x$. By the assumption, $2x \notin \mathrm{wAC}(B)$, and hence (y_n) does not contain a weakly convergent subsequence. However, by the assumption $2x \in \mathrm{wAC}(M)$. This contradiction proves Theorem 3.1. $\square$

Remark 3.4. An example of a convex approximatively compact body in $X = c_0$ (and also in $X = c$) was constructed by Pyatyshev [16]. Balaganskii [7] constructed an example of a convex Chebyshev approximatively compact body with 2-Lipschitz metric projection in $L^1[0, 1]$.

The next result (see [7]) follows from Theorem 3.A and Remarks 2.1 and 3.4.

Corollary 3.1. *The space $L^1(\mu)$ with atomless σ-finite σ-additive measure contains an antiproximinal cavern.*

Replacing the weak convergence by Deutsch convergence (for nets or sequences) in Theorem 3.1, we get the following result.

Theorem 3.2. *Let D be a Deutsch convergence in a normed linear space X, let the unit ball B of X be antiapproximatively Deutsch compact, i.e.,*

$$^{D}\mathrm{AC}(B) = B,$$

and let $M \subset X$ be a D-approximatively (Deutsch) compact convex body. Then the cavern $\overline{X \setminus M}$ is antiproximinal.

Remark 3.5. The authors are unaware of a space with the property $^{D}\mathrm{AC}(B) = B$, except the strong $D = n$ and weak $D = w$ convergences (see Remarks 2.1 and 2.3 above).

Definition 3.1. Let X be a normed linear space. A point x of the unit sphere S of X is a CLUR-*point* if the conditions $y_n \in S$ and $\|x + y_n\|/2 \to 1$ imply that (y_n) has a converging subsequence. The class[1] (CLUR) is defined as follows: $X \in$ (CLUR) if each point of the unit sphere S is a CLUR-point.

The class (CLUR) appears as a criterion condition for many classical problems of min- and max-approximation (see [5], [3], [1]).

Definition 3.2. A normed linear space X is called a *Vlasov–Dutta–Shunmugaraj space* [3] (written: $X \in$ (SVD)) if each point of the unit sphere S is an SVD-*point*, i.e., the conditions $x, y_n \in S$ and $\|x + y_n\|/2 \to 1$ imply that (y_n) has a weakly convergent subsequence to a point in S.

Replacing the weak convergence by the strong convergence in Definition 3.2, we get the classical class (CLUR) of normed linear spaces (see, e.g., [5]).

Remark 3.6. The class (SVD) (which is one of several generalizations of the class (CLUR)) was introduced by L. P. Vlasov (with a different name; see, e.g., [19] and [8]). Vlasov obtained several characterizations of SVD-spaces in terms of generalized best approximants (from the second dual space). The name and definitions of these classes have changed over time; in [8], the classes (wCLUR) and (VDS) were denoted, respectively, by (wCLUR°) and (wCLUR). Dutta and Shunmugaraj [13], who used the name "weakly CLUR"=spaces" (weakly CLUR) for what we now call (VDS), obtained some characterizations of these spaces.

Remark 3.7. An example of a VDS-space which is not a CLUR-space and is not a Kadec–Klee space was constructed in [13] (this is the space ℓ^2 with some special norm equivalent to the original norm on ℓ^2). We also note that

$$c_0.\ell^1, L^1[0,1] \notin (\mathrm{SVD}), \qquad (\mathrm{CLUR}) \subset (\mathrm{SVD}),$$
$$X \in (\mathrm{LUR}) \iff X \in (\mathrm{SVD}) \cap (\mathrm{R}),$$
$$X \in (\mathrm{CLUR}) \iff X \in (\mathrm{SVD}) \cap (\mathrm{KK}),$$

where (R) is the class of strictly convex (rotund) spaces, (LUR) is the class of locally uniformly convex spaces, (KK) is the class of Kadec–Klee spaces (see [3]).

Tsar'kov [18, Lemma 7] established the following characterization of CLUR-spaces in terms of points of approximative compactness of the set $M := X \setminus \mathring{B}$ (here, as above, $\mathring{B}$ is the open unit ball). This result proved very useful in revealing a particular importance of CLUR-spaces in problems of points of

[1]CLUR comes from "compactly locally uniformly rotund."

approximative compactness for max- and min-approximation in symmetric and asymmetric spaces see [5, Sec. 4], [2, Sec. 4], [1], [13], [14].

Theorem 3.B. *Let X be a normed linear space. Then*

$$X \in (\mathrm{CLUR}) \iff \mathrm{AC}(X \setminus \mathring{B}) \supset X \setminus \{0\}.$$

This result is extended to SVD-spaces as follows (see [13, Theorem 3.12] and [3, Sec. 3]):

$$X \in (\mathrm{SVD}) \iff \mathrm{wAC}(X \setminus \mathring{B}) \supset X \setminus \{0\}. \tag{3.1}$$

Definition 3.3. Let D be a Deutsch convergence. A normed linear space X lies in the class (D-SVD) if each point of its unit sphere S is a D-SVD-*point*, i.e., the conditions $x \in S$, $\|y_n\| \to 1$, $\|x + y_n\|/2 \to 1$ imply that (y_n) has a D-convergent subsequence to a point of the sphere S.

A space X lies in the class (D-SVD) if each point of its unit sphere is a D-SVD-point.

Remark 3.8. (SVD) = (w-SVD), (CLUR) = (n-SVD).

An analogue of Theorem 3.B and equality (3.1) for D-SVD-spaces is as follows:

$$X \in (D\text{-}\mathrm{SVD}) \Leftrightarrow {}^{D}\mathrm{AC}(X \setminus \mathring{B}) \supset X \setminus \{0\}. \tag{3.2}$$

Theorem 3.3. *Each D-SVD-space contains an approximatively D-compact cavern.*

Corollary 3.2. *Each* SVD-*space contains an approximatively weakly compact cavern.*

Corollary 3.3. *Each* CLUR-*space contains an approximatively compact cavern.*

Remark 3.9. Corollary 3.3 was established by Pyatyshev [16] in the Banach space setting, however, as above, the completeness of the space is of course superfluous.

Proof. Let $v \in X$, $\|v\| = 1/2$. Consider the set $M = X \setminus (\mathring{B} \cap (\mathring{B} + v))$. It is clear that M is a cavern. Let $x \notin M$ and let (w_n) be a minimizing sequence from M for x. If $x \neq 0$, $x \neq v$, then the sequence (w_n) lies in the union $(x_n) \cup (y_n)$, where (x_n) is a minimizing sequence from $X \setminus B$ for x and (y_n) is a minimizing sequence from $X \setminus (\mathring{B} + v)$ for x. The point x is not a center of the balls B and $B + v$, and hence by (3.2) the sequences (x_n) and (y_n) contain D-convergent subsequences. Hence (w_n) also has a D-convergent subsequence. For $x = 0$ and $x = v$, we have, respectively, $(w_n) \subset (y_n)$ and $(w_n) \subset (x_n)$, starting with some number. The origin 0 is not a center of the ball $B + v$ and v is not a center of the ball B, and so by (3.2) the sequences (x_n) and (y_n) have D-convergent subsequences. Therefore, (w_n) also contains a D-convergent subsequence. ☐

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