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NUMERICAL RADIUS, BEREZIN RADIUS AND NORM BEHAVIOR OF SOME OPERATORS

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Abstract. We prove some results for the numerical radius and Berezin radius of Toeplitz and truncated Toeplitz operators. We also study the behavior of the operator norm $\|T^nS\|$ for some operators T and S on a Hilbert space.

1. Introduction and Notation

In the present article, we establish estimates for the numerical radius and Berezin radius of Toeplitz and truncated Toeplitz operators. We also prove some results related to the behavior of the operator norm $\|T^nS\|$ for some operators on a Hilbert space.

Recall that for a bounded linear operator T on the reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(Q)$ of complex-valued functions on some set Q with reproducing kernel $k_{\mathcal{H},\lambda}(z)$, its Berezin symbol (or Berezin transform) (see Zhu [29] and Nordgren and Rosenthal [23]) $\widetilde{T}$ is defined by

$$\widetilde{T}(\lambda) := \left\langle T\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle_{\mathcal{H}}, \quad \lambda \in Q,$$

where $\widehat{k}_{\mathcal{H},\lambda} := \frac{k_{\mathcal{H},\lambda}}{\|k_{\mathcal{H},\lambda}\|}$ is the normalized reproducing kernel of the space $\mathcal{H}$. The numerical range of $T \in \mathcal{B}(\mathcal{H})$ is defined by $W(T) := \{\langle Tx, x\rangle : x \in \mathcal{H} \text{ and } \|x\| = 1\}$. The numerical radius is the number $w(T) := \sup\{|\langle Tx, x\rangle| : \|x\| = 1\}$.

The Berezin symbol of an operator provides important information about it. For example, it is well known that on most familiar RKHS's, including the Hardy, Bergman and Fock spaces, the Berezin symbol uniquely determines the operator, i.e., $A_1 = A_2$ if and only if $\widetilde{A}_1 = \widetilde{A}_2$ (See, Zhu [29].) It is one of the most useful tools in the study of Toeplitz and Hankel operators on the Hardy space $H^2(\mathbb{D})$ and the Bergman space $L_a^2(\mathbb{D})$. The concept of the Berezin symbol of an operator arose in connection with quantum mechanics and non-commutative geometry (see, for instance, [23]). On the other hand, the method of reproducing kernels is actively developed by Saitoh and Castro and their collaborators, in order to solve various problems of applied mathematics (see, [4, 5, 6], and the references therein). For other applications of reproducing kernels and Berezin

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symbols, and also about new results for numerical radii see for instance the papers [1, 2, 3, 7, 9, 10, 11, 12, 13, 14, 15, 23, 26, 27, 28, 29].

For a bounded operator A on a RKHS, the corresponding Berezin set and Berezin number of A are defined, respectively, as follows (see [14]):

$$Ber(A) = Range(\widetilde{A}) \ = \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\}$$
$$ber(A) = \sup \left\{ |\mu| : \mu \in Ber(A) \right\}.$$

We define two Berezin norms as follows:

$$\|A\|_{B,1} := \sup_{\lambda \in Q} \left\| A\widehat{k}_{\mathcal{H},\lambda} \right\| \text{ and } \|A\|_{B,2} := \sup_{\lambda,\mu \in Q} \left| \left\langle A\widehat{k}_{\lambda}, \widehat{k}_{\mu} \right\rangle \right|,$$

clearly $ber(A) \leq \|A\|_{B,2} \leq \|A\|_{B,1}$. A function θ is an inner function if $|\theta(z)| \leq 1$ for all $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $|\theta(\xi)| = 1$ for almost all $\xi \in \mathbb{T} := \partial\mathbb{D} = \{\xi : |\xi| = 1\}$. The basic RKHS in which our objects live are the Hardy space $H^2 = H^2(\mathbb{D})$ and model space $K_\theta := H^2 \ominus \theta H^2$; H^2 is the Hilbert space of analytic functions $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ defined in the unit disk $\mathbb{D}$ such that $\sum\limits_{n=0}^{\infty} |a_n|^2 < \infty$. Alternately, it can be defined with a closed subspace of the Lebesgue space $L^2(\mathbb{T})$ on the unit circle $\mathbb{T}$, by associating to each analytic function its radial limit. The Banach algebra of bounded analytic functions on $\mathbb{D}$ is denoted by $H^\infty = H^\infty(\mathbb{D})$. Let $\varphi \in L^\infty$ and θ be inner. The Toeplitz operator $T_\varphi : H^2 \to H^2$, and the truncated Toeplitz operator $A_\varphi^\theta : K_\theta \to K_\theta$ are defined respectively by (see [24, 25])

$$T_\varphi = P_+ M_\varphi, \ A_\varphi^\theta = \mathcal{P}_\theta M_\varphi.$$

Here, $P_+ : L^2 \ (\mathbb{T}) \to H^2$ and $\mathcal{P}_\theta : L^2 \ (\mathbb{T}) \to K_\theta$ are orthogonal projections, and M_φ is a multiplication operator on L^2 $(\mathbb{T})$ which is defined by $M_\varphi f = \varphi f$. It is easy to see that

$$\mathcal{P}_\theta = P_+ - \theta P_+ \overline{\theta}, \quad P_\theta := \mathcal{P}_\theta|_{H^2} = I - T_\theta T_{\overline{\theta}}.$$

In particular, when $\varphi \in H^\infty$, the truncated Toeplitz operator A_φ^θ is called the model operator, denoted by $\varphi(M_\theta)$, where

$$M_\theta := A_z^\theta = \mathcal{P}_\theta M_z$$

is the model operator. Evaluations at points $\lambda \in \mathbb{D}$ are bounded functionals on H^2 and the corresponding reproducing kernel is the function

$$k_{H^2,\lambda}(z) = \frac{1}{1 - \overline{\lambda} z}, \ \lambda, z \in \mathbb{D};$$

thus,

$$f(\lambda) = \left\langle f, k_{H^2,\lambda} \right\rangle, \quad \forall f \in H^2.$$

If $\varphi \in H^\infty$, then $k_{H^2,\lambda}$ is an eigenvector for $T_\varphi^* = T_{\overline{\varphi}}$ i.e.,

$$T_\varphi^* k_{H^2,\lambda} = \overline{\varphi(\lambda)} k_{H^2,\lambda}.$$

The normalized reproducing kernel is denoted, as before, by

$$\widehat{k}_{H^2,\lambda} = \frac{k_{H^2,\lambda}}{\left\| k_{H^2,\lambda} \right\|} = \sqrt{1 - |\lambda|^2} k_{H^2,\lambda} = \frac{\sqrt{1 - |\lambda|^2}}{1 - \overline{\lambda} z}.$$

In what follows, we will use short notation $\widehat{k}_{\lambda}$ instead of $\widehat{k}_{H^2,\lambda}$. In K_θ the reproducing kernel for a point $\lambda \in \mathbb{D}$ is the function

$$k_{\theta,\lambda}(z) := \frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\overline{\lambda}z},$$

and the normalized reproducing kernel is

$$\widehat{k}_{\theta,\lambda}(z) = \left(\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\right)^{1/2} k_{\theta,\lambda}(z).$$

By the spectrum $\sigma(\theta)$ of θ we mean the complement (with respect to the whole closed disk $\overline{\mathbb{D}}$) of the set of all points $\lambda \in \overline{\mathbb{D}}$ such that the function $1/\theta$ can be continued analytically into a (full) neighborhood of λ. Also

$$\mathbb{Z}(\theta) = \left\{\lambda \in \overline{\mathbb{D}} : \theta(\lambda) = 0\right\}$$

is the null set of θ and $supp\mu$ denotes the support of any measure μ. If

$$\theta = B\exp\left(-\int \frac{\xi+z}{\xi-z} d\mu^s(\xi)\right)$$

is the canonical factorization of θ, then

$$\sigma(\theta) = \overline{\mathbb{Z}(\theta)} \cup supp\mu^s = \left\{\lambda \in \overline{\mathbb{D}} : \underline{\lim}_{z\to\lambda, z\in\mathbb{D}} |\theta(z)| = 0\right\}$$

and

$$\sigma(M_\theta) = \sigma(\theta), \quad \sigma_p(M_\theta) = \sigma(\theta) \cap \mathbb{D} = \mathbb{Z}(\theta).$$

For any Toeplitz operator T_φ ($\varphi \in L^\infty$) on H^2 it is known that $\widetilde{T}_\varphi = \widetilde{\varphi}$ where $\widetilde{\varphi}$ denotes the harmonic extension of φ into $\mathbb{D}$ (see, for example, [11, 20, 24, 33, 37]).

This paper is organized as follows. In Section 2, we prove some results concerning Berezin and numerical radii of operators. We demonstrate some new relations among these numerical characteristics of operators on RKHS's (Section 3). In Section 4, we prove some new conclusions for $\lim\limits_{n} \|T^n S\|$.

2. Berezin radius of truncated Toeplitz operators

It is clear that always $Ber(T) \subset W(T)$ (numerical range) and $ber(T) \leq w(T)$ (numerical radius). Therefore, the study of these new characteristics of operators is important. In the present section, we investigate the Berezin radius of some truncated Toeplitz operators.

Theorem 2.1. *Let θ be an inner function such that $\sigma(\theta) \cap \mathbb{T} \neq \emptyset$ and $\varphi \in L^\infty(\mathbb{T})$ be a function such that $|\varphi(\eta)| = \underline{\lim}_{\lambda\to\eta} \left|\widetilde{\varphi(\lambda)}\right|$ exists and is finite for all $\eta \in \sigma(\theta) \cap \mathbb{T}$. If A_φ^θ is a truncated Toeplitz operator on K_θ then*

$$\sup_{\eta\in\sigma(\theta)\cap\mathbb{T}} |\varphi(\eta)| \leq ber(A_\varphi^\theta) \leq \|\varphi\|_\infty . \tag{2.1}$$

Proof. First, let us calculate the Berezin symbol of the truncated Toeplitz operator $A_{\varphi}^{\theta}=\mathcal{P}_{\theta}T_{\varphi}|K_{\theta}$:

$$\begin{aligned}
\widetilde{A_{\varphi}^{\theta}}(\lambda) &= \left\langle A_{\varphi}^{\theta}\widehat{k}_{\theta,\lambda},\widehat{k}_{\theta,\lambda}\right\rangle \\
&= \left\langle A_{\varphi}^{\theta}\left(\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\right)^{1/2}\frac{1-\overline{\theta(\lambda)}\theta}{1-\overline{\lambda}z},\left(\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\right)^{1/2}\frac{1-\overline{\theta(\lambda)}\theta}{1-\overline{\lambda}z}\right\rangle \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left\langle \mathcal{P}_{\theta}T_{\varphi}(1-\overline{\theta(\lambda)}\theta)\widehat{k}_{\lambda},(1-\overline{\theta(\lambda)}\theta)\widehat{k}_{\lambda}\right\rangle \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left\langle T_{\varphi}(1-\overline{\theta(\lambda)}\theta)\widehat{k}_{\lambda},(1-\overline{\theta(\lambda)}\theta)\widehat{k}_{\lambda},\right\rangle \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left\langle T_{\varphi(1-\overline{\theta(\lambda)}\theta)}\widehat{k}_{\lambda},T_{(1-\overline{\theta(\lambda)}\theta)}\widehat{k}_{\lambda}\right\rangle \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left\langle T_{1-\theta(\lambda)}T_{\varphi(1-\overline{\theta(\lambda)}\theta)}\widehat{k}_{\lambda},\widehat{k}_{\lambda}\right\rangle \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left\langle T_{\varphi|(1-\overline{\theta(\lambda)}\theta)|^2}\widehat{k}_{\lambda},\widehat{k}_{\lambda}\right\rangle = \frac{1}{1-|\theta(\lambda)|^2}\left(\left(\varphi\left|1-\overline{\theta(\lambda)}\theta\right|^2\right)^{\sim}\right)(\lambda) \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left(\varphi\left(1-2Re\left(\overline{\theta(\lambda)}\theta\right)+|\theta(\lambda)|^2\right)\right)^{\sim}(\lambda) \\
&= \frac{1}{1-|\theta(\lambda)|^2}\left(\left(1+|\theta(\lambda)|^2\right)\widetilde{\varphi}(\lambda)-2\left(\varphi Re\left(\overline{\theta(\lambda)}\theta\right)\right)^{\sim}(\lambda)\right),
\end{aligned}$$

and thus,

$$\left(1+|\theta(\lambda)|^2\right)\widetilde{\varphi}(\lambda)=\left(1-|\theta(\lambda)|^2\right)\widetilde{A_{\varphi}^{\theta}}(\lambda)+2\left(\varphi Re\left(\overline{\theta(\lambda)}\theta\right)\right)^{\sim}(\lambda),\ \forall\lambda\in\mathbb{D}; \tag{2.2}$$

here $\widetilde{\varphi}$ is the harmonic extension of φ into $\mathbb{D}$ and

$$\widehat{k}_{\lambda}(z)=\frac{\left(1-|\lambda|^2\right)^{1/2}}{1-\overline{\lambda}z}$$

is the normalized reproducing kernel of the Hardy space $H^2(\mathbb{D})$. Since $1-|\theta(\lambda)|^2\leq 1(\forall\lambda\in\mathbb{D})$, it follows from formula (2.2) that

$$(1+|\theta(\lambda)|^2)\,|\widetilde{\varphi}(\lambda)|\leq\left|\widetilde{A_{\varphi}^{\theta}}(\lambda)\right|+2\left|\left(\varphi Re\left(\overline{\theta(\lambda)}\theta\right)\right)^{\sim}(\lambda)\right|,\ \forall\lambda\in\mathbb{D}.$$

Then

$$(1+|\theta(\lambda)|^2)\,|\widetilde{\varphi}(\lambda)|\leq ber\left(A_{\varphi}^{\theta}\right)+2\left|\left(\varphi Re\left(\overline{\theta(\lambda)}\theta\right)\right)^{\sim}(\lambda)\right|,\ \forall\lambda\in\mathbb{D} \tag{2.3}$$

On the other hand, for any $\lambda \in \mathbb{D}$ we have

$$\begin{aligned}
\left|\left(\varphi Re\left(\overline{\theta(\lambda)}\theta\right)\right)^{\sim}(\lambda)\right| &= \left|\int_{\mathbb{T}} \frac{1-|\lambda|^2}{|\xi-\lambda|^2} Re\overline{(\theta(\lambda)}\theta(\xi))\varphi(\xi)dm(\xi)\right| \\
&\leq \int_{\mathbb{T}} \frac{1-|\lambda|^2}{|\xi-\lambda|^2} \left|Re\overline{(\theta(\lambda)}\theta(\xi))\right| |\varphi(\xi)| \, dm(\xi) \\
&\leq \int_{\mathbb{T}} \frac{1-|\lambda|^2}{|\xi-\lambda|^2} \left|\overline{(\theta(\lambda)}\theta(\xi)\right| |\varphi(\xi)| \, dm(\xi) \\
&= |\theta(\lambda)| \int_{\mathbb{T}} \frac{1-|\lambda|^2}{|\xi-\lambda|^2} |\varphi(\xi)| \, dm(\xi) \\
&= |\theta(\lambda)| \, |\varphi|^{\tilde{}}(\lambda) \leq |\theta(\lambda)| \, \|\varphi\|_{\infty};
\end{aligned}$$

here, dm is the usual normalized Lebesgue measure on the unit circle $\mathbb{T}$. By considering this inequality in (2.3), we have

$$\left(1+|\theta(\lambda)|^2\right) |\widetilde{\varphi}(\lambda)| \leq ber(A_{\varphi}^{\theta}) + 2\,|\theta(\lambda)| \, \|\varphi\|_{\infty}, \quad \forall \lambda \in \mathbb{D}. \tag{2.4}$$

Let $\eta \in \sigma(\theta) \cap \mathbb{T}$ be any point. Since $\underline{\lim}_{\lambda \to \eta} |\theta(\lambda)| = 0$ and by condition of the theorem, $\varphi(\eta)$ exists and is finite, we obtain by (2.4) that

$$|\varphi(\eta)| = \lim_{\lambda \to \eta} |\widetilde{\varphi}(\lambda)| \leq ber(A_{\varphi}^{\theta}), \quad \forall \eta \in \sigma(\theta) \cap \mathbb{T}.$$

Hence,

$$\sup_{\eta \in \sigma(\theta) \cap \mathbb{T}} |\varphi(\eta)| \leq ber(A_{\varphi}^{\theta}). \tag{2.5}$$

On the other hand, it is obvious that

$$ber(A_{\varphi}^{\theta}) \leq \left\|A_{\varphi}^{\theta}\right\| \leq \|\varphi\|_{\infty},$$

which together with (2.5) gives (2.1). This proves the theorem. $\square$

The following corollary is immediate from Theorem 2.1.

Corollary 2.1. *If φ is a function in $L^{\infty}(\mathbb{T})$ such that*

$$\sup_{\eta \in \sigma(\theta) \cap \mathbb{T}} |\varphi(\eta)| = \|\varphi\|_{\infty},$$

then

$$ber(A_{\varphi}^{\theta}) = \|\varphi\|_{\infty}.$$

3. Estimates for Berezin radii and numerical radii

In this section, we prove some new inequalities for Berezin radius and numerical radii of operators on the Hardy space H^2.

Let $(\sum)$ denote the set of all inner functions in H^2 and $\left(H^2\right)_1 := \left\{f \in H^2 : \|f\|_2 = 1\right\}$, the unit sphere of H^2.

Proposition 3.1. *Let $A : H^2 \to H^2$ be an arbitrary bounded linear operator. Then*

$$\sup_{\theta \in (\sum) \cup \{1\}} ber\left(T_{\overline{\theta}} A T_\theta\right) \leq w(A) \leq \sup_{f \in H^\infty \cap (H^2)_1} ber\left(T_{\overline{f}} A T_f\right). \tag{3.1}$$

Proof. Since H^∞ is dense in H^2 it is easy to show that

$$\sup\left\{|\langle Af, f\rangle| : f \in (H^2)_1\right\} = \sup\left\{|\langle Af, f\rangle| : f \in H^\infty \cap (H^2)_1\right\}$$

Then we have:

$$\begin{aligned}
w(A) &= \sup\left\{|\langle Af, f\rangle| : f \in H^\infty \cap (H^2)_1\right\} \\
&= \sup\left\{\left|\langle T_f^* A T_f 1, 1\rangle\right| : f \in H^\infty \cap (H^2)_1\right\} \\
&= \sup\left\{\left|\left\langle T_{\overline{f}} A T_f \widehat{k}_0, \widehat{k}_0\right\rangle\right| : f \in H^\infty \cap (H^2)_1\right\} \\
&= \sup\left\{\left|\left(T_{\overline{f}} A T_f\right)^{\sim}(0)\right| : f \in H^\infty \cap (H^2)_1\right\} \\
&\leq \sup_{f \in H^\infty \cap (H^2)_1} \sup\left\{\left|\left(T_{\overline{f}} A T_f\right)^{\sim}(\lambda)\right| : \lambda \in \mathbb{D}\right\} \\
&= \sup_{f \in H^\infty \cap (H^2)_1} ber\left(T_{\overline{f}} A T_f\right).
\end{aligned}$$

i.e.,

$$w(A) \leq \sup_{f \in H^\infty \cap (H^2)_1} ber\left(T_{\overline{f}} A T_f\right). \tag{3.2}$$

On the other hand, if $\theta \in (\sum)$ is an arbitrary inner function, then by considering that $\theta g \in (H^2)_1$ for every $g \in (H^2)_1$ we have:

$$\begin{aligned}
ber\left(T_{\overline{\theta}} A T_\theta\right) &= \sup_{\lambda \in \mathbb{D}} \left|\widetilde{T_{\overline{\theta}} A T_\theta}\right| = \sup_{\lambda \in \mathbb{D}} \left|T_{\overline{\theta}} A T_\theta \widehat{k}_\lambda, \widehat{k}_\lambda\right| \\
&= \sup_{\lambda \in \mathbb{D}} \left|\left\langle A T_\theta \widehat{k}_\lambda, A T_\theta \widehat{k}_\lambda\right\rangle\right| = \sup_{\lambda \in \mathbb{D}} \left|\left\langle A \theta \widehat{k}_\lambda, \theta \widehat{k}_\lambda\right\rangle\right| \\
&\leq \sup_{\lambda \in \mathbb{D}} \sup_{h \in (H^2)_1} |\langle Ah, h\rangle| = \sup_{h \in (H^2)_1} |\langle Ah, h\rangle| = w(A),
\end{aligned}$$

hence

$$ber\left(T_{\overline{\theta}} A T_\theta\right) \leq w(A)$$

for any $\theta \in (\sum)$. Thus

$$ber\left(T_{\overline{\eta}} A T_\eta\right) \leq w(A)$$

for any $\eta \in (\sum) \cup \{1\}$. Therefore, we obtain

$$\sup_{\eta \in (\sum) \cup \{1\}} ber\left(T_{\overline{\eta}} A T_\eta\right) \leq w(A). \tag{3.3}$$

Now, the desired inequality (3.1) follows from (3.2) and (3.3), which completes the proof. $\square$

Corollary 3.1. *If* $\sup_{f\in H^{\infty}\cap(H^2)_1} ber\left(T_{\overline{f}}AT_f\right) = berT_{\overline{\theta}}AT_\theta$ *for some* $\theta \in (\sum)$ *then* $w(A) = ber(T_{\overline{\theta}}AT_\theta)$.

Corollary 3.2. *If* $\sup_{\eta\in(\sum)\cup\{1\}} ber\left(T_{\overline{\eta}}AT_\eta\right) = \sup_{f\in H^{\infty}\cap(H^2)_1} ber\left(T_{\overline{f}}AT_f\right) = ber(A)$, *then* $w(A) = ber(A)$.

Corollary 3.3. *Let* $\varphi \in L^{\infty}(\mathbb{T})$ *and* T_φ *be a Toeplitz operator on the Hardy space* H^2. *Then*

$$ber(T_\varphi) = w(T_\varphi) = \|T_\varphi\| = \|\varphi\|_\infty .$$

The proof of the latter corollary uses the well-known result that $\widetilde{T_\varphi} = \widetilde{\varphi}$ for any Toeplitz operator $T_\varphi, \varphi \in L^{\infty}(\mathbb{T})$ on the Hardy space H^2, where $\widetilde{\varphi}$ denotes the harmonic extension of φ.

Similar arguments allow us to state the following result in the Bergman space $L^2_0(\mathbb{D})$.

Proposition 3.2. *Let* $\varphi \in L^{\infty}(\mathbb{D})$ *and* T_φ *be a Toeplitz operator on the Bergman space* $L^2_0(\mathbb{D})$. *Then*

$$w(T_\varphi) \leq \sup_{f\in H^{\infty}\cap(L^2_a)_1} ber\left(T_{|f|^2\varphi}\right).$$

Recall that for any inner function θ *and any symbol* $\varphi \in L^{\infty}(\mathbb{T})$, *the truncated Toeplitz operator* $A^\theta_\varphi : K_\theta \to K_\theta$ *is defined by* $A^\theta_\varphi f = P_\theta(\varphi f)$. *Our next result estimates the numerical radius of* A^θ_φ.

Proposition 3.3. *Let* θ *be an inner function,* $\varphi \in L^{\infty}(\mathbb{T})$ *be a function and* $A^\theta_\varphi : K_\theta \to K_\theta$ *be a truncated Toeplitz operator. Then the numerical radius of* A^θ_φ *satisfies the following inequality:*

$$w(A^\theta_\varphi) \geq \sup_{\lambda\in\mathbb{D}} \frac{\left|\left(\left|1-\overline{\theta(\lambda)}\theta\right|^2\varphi\right)^{\sim}(\lambda)\right|}{1-|\theta(\lambda)|^2}$$

Proof. Since $\varphi \in L^{\infty}(\mathbb{T})$, the truncated Toeplitz operator A^θ_φ is bounded . For any $f \in K_\theta, \|f\|_2 = 1$, we have

$$\left\langle A^\theta_\varphi f, f\right\rangle = \langle P_\theta T_\varphi, f, f\rangle = \langle T_\varphi, f, f\rangle .$$

By considering this and the formula $\widehat{k}_{\theta,\lambda}(z)=\left(\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\right)^{\frac{1}{2}}\frac{1-\overline{\theta(\lambda)}\theta(z)}{1-\overline{\lambda}z}$ we have

$$\begin{aligned}
\sup_{\|f\|_2=1}\left|\left\langle A_{\varphi}^{\theta}f,f\right\rangle\right| &= \sup_{\|f\|_2=1}\left\langle T_{\varphi},f,f\right\rangle \geq \sup_{\lambda\in\mathbb{D}}\left|\left\langle T_{\varphi}\widehat{k}_{\theta,\lambda},\widehat{k}_{\theta,\lambda}\right\rangle\right| \\
&= \sup_{\lambda\in\mathbb{D}}\frac{1-|\lambda|^2}{1-|\theta(\lambda)|^2}\left|\left\langle T_{\varphi}\frac{1-\overline{\theta(\lambda)}\theta}{1-\overline{\lambda}z},\frac{1-\overline{\theta(\lambda)}\theta}{1-\overline{\lambda}z}\right\rangle\right| \\
&= \sup_{\lambda\in\mathbb{D}}\frac{1}{1-|\theta(\lambda)|^2}\left|\left\langle T_{1-\theta(\lambda)\overline{\theta}}T_{\varphi}T_{1-\overline{\theta(\lambda)}\theta}\right.\right. \\
&\qquad \left.\left.\frac{\left(1-|\lambda|^2\right)^{\frac{1}{2}}}{1-\overline{\lambda}z},\frac{\left(1-|\lambda|^2\right)^{\frac{1}{2}}}{1-\overline{\lambda}z}\right\rangle\right| \\
&= \sup_{\lambda\in\mathbb{D}}\frac{1}{1-|\theta(\lambda)|^2}\left|\left\langle T_{\varphi\left|1-\overline{\theta(\lambda)}\theta\right|^2}\widehat{k}_{\theta,\lambda},\widehat{k}_{\theta,\lambda}\right\rangle\right| \\
&= \sup_{\lambda\in\mathbb{D}}\frac{1}{1-|\theta(\lambda)|^2}\left|\widetilde{T}_{\varphi\left|1-\overline{\theta(\lambda)}\theta\right|^2}(\lambda)\right| \\
&= \sup_{\lambda\in\mathbb{D}}\frac{\left(\left|1-\overline{\theta(\lambda)}\theta\right|^2\varphi\right)^{\sim}(\lambda)}{1-|\theta(\lambda)|^2}.
\end{aligned}$$

In the last equality we again used the well known results that $\widetilde{T_h}=\widetilde{h}$ for every $h\in L^{\infty}(\mathbb{T})$ (see, for example, Zhu [29]). Thus we have

$$w(A_{\varphi}^{\theta})\geq \sup_{\lambda\in\mathbb{D}}\frac{\left(\left|1-\overline{\theta(\lambda)}\theta\right|^2\varphi\right)^{\sim}(\lambda)}{1-|\theta(\lambda)|^2},$$

which proves the proposition. □

Our next result establishes an inequality for the Berezin radius of an abstract operator.

Proposition 3.4. *Let $\mathcal{H}=\mathcal{H}(Q)$ be a reproducing kernel Hilbert space of complex-valued functions on some set Ω with reproducing kernel*

$$k_{\mathcal{H},\lambda}(z)=\sum_{n=0}^{\infty}\overline{e_n(\lambda)}e_n(z),$$

where $\{e_n(z)\}_{n\geq 0}$ is any orthonormal basis of the space $\mathcal{H}$. Let $A:\mathcal{H}\to\mathcal{H}$ be a bounded linear operator with matrix entries

$$a_{n,m}:=\left\langle Ae_n(z),e_m(z)\right\rangle,n,m=0,1,2,...,$$

satisfying the inequality

$$|a_{n,m}|\leq C\left|a_n\right|\left|b_m\right|\leq\|A\|\quad(\forall n,m\geq 0), \tag{3.4}$$

for some sequences $\{a_n\}_{n\geq 0}\in l^2$, $\{b_n\}_{n\geq 0}\in l^2$ and $C>0$. Then

$$ber(A)\leq C\left\|\{a_n\}\right\|_{l^2}\left\|\{b_n\}\right\|_{l^2}.$$

Proof. First, let us determine the Berezin symbol of the operator A:

$$\begin{aligned}
\widetilde{A}(\lambda) &= \left\langle A\widehat{k}_{\mathcal{H},\lambda}, \widehat{k}_{\mathcal{H},\lambda} \right\rangle = \frac{1}{\left\|\widehat{k}_{\mathcal{H},\lambda}\right\|^2} \left\langle Ak_\lambda, k_\lambda \right\rangle \\
&= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \left\langle A\sum_{n=0}^{\infty}\overline{e_n(\lambda)}e_n(z), \sum_{n=0}^{\infty}\overline{e_n(\lambda)}e_n(z) \right\rangle \\
&= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \left\langle \sum_{n=0}^{\infty}\overline{e_n(\lambda)}Ae_n(z), \sum_{n=0}^{\infty}\overline{e_n(\lambda)}e_n(z) \right\rangle \\
&= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \sum_{n=0}^{\infty}\overline{e_n(\lambda)}e_m(z) \left\langle Ae_n(z), e_m(z) \right\rangle \\
&= \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \sum_{n,m=0}^{\infty} a_{n,m}\overline{e_n(\lambda)}e_m(\lambda),
\end{aligned}$$

or

$$\widetilde{A}(\lambda) = \frac{1}{\|k_{\mathcal{H},\lambda}\|^2} \sum_{n,m=0}^{\infty} a_{n,m}\overline{e_n(\lambda)}e_m(\lambda), \tag{3.5}$$

for all $\lambda \in \Omega$.

Now using condition (3.4) and formula (3.5), we have

$$\begin{aligned}
\left|\widetilde{A}(\lambda)\right| &\leq C\frac{1}{\|k_{\mathcal{H},\lambda}\|^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |a_n|\,|b_m|\,|e_n(\lambda)|\,|e_m(\lambda)| \\
&= C\frac{1}{\|k_{\mathcal{H},\lambda}\|^2}\sum_{n=0}^{\infty} |a_n|\,|e_n(\lambda)| \sum_{m=0}^{\infty} |b_m|\,|e_m(\lambda)| \\
&\leq C\frac{1}{\|k_{\mathcal{H},\lambda}\|^2}\left(\sum_{n=0}^{\infty}|a_n|^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}|e_n(\lambda)|^2\right)^{\frac{1}{2}}\left(\sum_{m=0}^{\infty}|b_m|^2\right)^{\frac{1}{2}} \\
&\times \left(\sum_{m=0}^{\infty}|e_m(\lambda)|^2\right)^{\frac{1}{2}} = C\,\|\{a_n\}\|_{l^2}\,\|\{b_n\}\|_{l^2}
\end{aligned}$$

for all $\lambda \in \Omega$, which implies that

$$ber(A) = \sup_{\lambda\in\Omega}\left|\widetilde{A}(\lambda)\right| \leq C\,\|\{a_n\}\|_{l^2}\,\|\{b_n\}\|_{l^2}\,.$$

The proposition is proved. □

4. Some operator norm inequalities

In this section, we prove some new results about operator norm inequalities. Namely, we will estimate $\lim\limits_{n\to\infty}\|T^nS\|$ for some appropriate operators T and S on a Hilbert space. These types of limits are important, in particular, for the study of the compactness property of the operator S. For, it is enough to remember, for instance, the well known Hartman-Sarason theorem for compact model operators $\varphi(M_\theta)(\varphi \in H^\infty)$ defined on the model space $K_\theta = H^2 \ominus \theta H^2$ by

$\varphi(M_\theta)f = P_\theta\varphi, f \in K_\theta$. There are also many other recent papers where the $\lim_{n\to\infty} \|T^n S\|$ is investigated for different goals, see, for instance, [8, 16, 17, 18, 19, 20, 21].

Trivially, $\lim_{n\to\infty} \|T^n\| = 0$ when $\|T\| < 1$. In the present section we prove the same assertion under weaker conditions. Our observation is based on the following simple assertion: $\lim_{n\to\infty} \|T^n\| = 0$ if and only if $\lim_{n\to\infty} \|T^n S\| = 0$ for every $S \in \mathcal{B}(H)$. In fact, if $\lim_{n\to\infty} \|T^n S\| = 0$, then in particular, for $S = I$ (identity operator), we have that $\lim_{n\to\infty} \|T^n\| = 0$. Conversely, if $\lim_{n\to\infty} \|T^n\| = 0$, then by using the submultiplicative inequality

$$\|T^n S\| \leq \|T^n\| \|S\|,$$

we deduce that $\lim_{n\to\infty} \|T^n S\| = 0$, as desired.

Before giving the results of this section, first let us introduce some additional notation. Let H be a complex Hilbert space. If $\{x_n\}_{n\geq 1} \subset H$, we denote by *span* $\{x_n : n = 1, 2, ...\}$ the closure of the linear hull generated by $\{x_n\}_{n\geq 1}$. The sequence $\mathbb{X} = \{x_n\}_{n\geq 1}$ is called: (i) complete if $\{x_n : n = 1, 2, ...\} = H$; (ii) Riesz basis if there exists an isomorphism $U : H \to H$ such that $\{Ux_n\}_{n\geq 1}$ is an orthonormal basis in H; (iii) the operator U will be called the orthogonalizer of $\mathbb{X}$.

It is well known that (see, for example, Nikolski [22]) $\{x_n\}$ is a Riesz basis in its closed linear span if there are constants $C_1 > 0, C_2 > 0$ such that

$$C_1 \left(\sum_{n\geq 1} |a_n|^2 \right)^{\frac{1}{2}} \leq \left\| \sum_{n\geq 1} a_n x_n \right\| \leq C_2 \left(\sum_{n\geq 1} |a_n|^2 \right)^{\frac{1}{2}}, \tag{4.1}$$

for all finite complex sequences $\{a_n\}_{n\geq 1}$. Note that $\left\|U^{-1}\right\|$ and $\|U\|^{-1}$ are the best constants in inequality (4.1).

Now we state our main results of this section:

Theorem 4.1. *Let H be an infinite dimensional separable Hilbert space with Riesz basis $\{R_i\}_{i=1}^{\infty}$ and let $T : H \to H$ be a bounded linear operator. Then:*

(i) $\|T^n S\| = O\left(\sup_{m\geq 1} \left(\sum_{i=1}^{m} \|T^{*n} R_i\|^2 \right)^{1/2} \right)$ *as $n \to \infty$, for every $S \in \mathcal{B}(H)$;*

(ii) if $\lim_{n\to\infty} \sup_{m\geq 1} \left(\sum_{i=1}^{m} \|T^{*n} R_i\|^2 \right)^{1/2} = 0$, *then* $\lim_{n\to\infty} \|T^n S\| = 0$ *for every $S \in \mathcal{B}(H)$, and hence* $\lim_{n\to\infty} \|T^n\| = 0$.

Proof. (i) Since $\{R_i\}_{i=1}^{\infty}$ is a Riesz basis in H (and hence is complete in H), for every $x \in H$ with $\|x\| = 1$ and $\varepsilon > 0$, there exists an integer $N := N(x, \varepsilon) > 0$ and scalars $c_i = c_i(x, \varepsilon) \in \mathbb{C}$ ($i = 1, 2, ..., N$) such that

$$\left\| x - \sum_{i=1}^{m} c_i R_i \right\| < \varepsilon,$$

which implies that

$$\left\| \sum_{i=1}^{m} c_i R_i \right\| < 1 + \varepsilon. \tag{4.2}$$

Taking into account the fact $\{R_i\}_{i=1}^{\infty}$ is a Riesz basis, and using (4.1) and (4.2) we obtain for any $S \in \mathcal{B}(H)$ and $n \geq 1$ that

$$\begin{aligned}
\|T^n S\| &= \|(T^n S)^*\| = \|S^* T^{*n}\| = \sup_{\|x\|=1} \|S^* T^{*n} x\| \\
&= \sup_{\|x\|=1} \left[\left\| S^* T^{*n} \left(x - \sum_{i=1}^{N} c_i R_i \right) + S^* T^{*n} \sum_{i=1}^{m} c_i R_i \right\| \right] \\
&\leq \sup_{\|x\|=1} \left[\|T^n S\| \, \varepsilon + \|S\| \sum_{i=1}^{N} |c_i| \, \|T^{*n} R_i\| \right] \\
&\leq \sup_{\|x\|=1} \left[\|T^n S\| \, \varepsilon + \|S\| \left(\sum_{i=1}^{N} |c_i|^2 \right)^{1/2} \left(\sum_{i=1}^{N} \|T^{*n} R\|^2 \right)^{1/2} \right]
\end{aligned}$$

$$\begin{aligned}
&\leq \sup_{\|x\|=1} \left[\|T^n S\| \, \varepsilon + \|S\| \, \|U\| \left\| \sum_{i=1}^{N} c_i R_i \right\| \left(\sum_{i=1}^{N} \|T^{*n} R_i\|^2 \right)^{1/2} \right] \\
&\leq \sup_{\|x\|=1} \left[\|T^n S\| \, \varepsilon + \|S\| \, \|U\| \, (\varepsilon + 1) \left(\sum_{i=1}^{N} \|T^{*n} R_i\|^2 \right)^{1/2} \right] \\
&\leq \|T^n S\| \, \varepsilon + \|S\| \, \|U\| \, (\varepsilon + 1) \sup_{\|x\|=1} \left(\sum_{i=1}^{N} \|T^{*n} R_i\|^2 \right)^{1/2} \\
&\leq \|T^n S\| \, \varepsilon + \|S\| \, \|U\| \, (\varepsilon + 1) \sup_{m \geq 1} \left(\sum_{i=1}^{m} \|T^{*n} R\|^2 \right)^{1/2},
\end{aligned}$$

where U is the orthogonalizer of $\{R_i\}_{i=1}^{\infty}$.

Since $n \geq 1$ is an arbitrary fixed number and ε is arbitrary, by letting ε tend to zero, from the latter we deduce for all $S \in \mathcal{B}(H)$ that

$$\|T^n S\| \leq \|S\| \, \|U\| \sup_{m \geq 1} \left(\sum_{i=1}^{m} \|T^{*n} R_i\|^2 \right)^{1/2}, \tag{4.3}$$

for all $n \geq 1$ which proves assertion (i) of the theorem.

Assertion (ii) is immediate from inequality (4.3). The theorem is proved. $\square$

Corollary 4.1. *If* $\sum_{i=1}^{\infty} \|T^{*n} R_i\|^2 < +\infty$ *for all* $n \geq 1$ *and* $\lim_{n \to \infty} \left(\sum_{i=1}^{\infty} \|T^{*n} R_i\|^2 \right)^{1/2} = 0$ *then* $\lim_{n \to \infty} \|T^n S\| = 0$ *for all* $S \in \mathcal{B}(H)$, *hence* $\lim_{n \to \infty} \|T^n\| = 0$.

A particular case of this corollary is the following.

Corollary 4.2. *If* $\lim_{n \to \infty} \|T^{*n} R_i\| = 0$ *for each* $i \geq 1$ *and* $\sum_{i=1}^{\infty} \|T^{*n} R_i\|^2 < +\infty$ *for each* $n \geq 1$ *then,* $\lim_{n \to \infty} \|T^n S\| = 0$ *for all* $S \in \mathcal{B}(H)$, *in particular,* $\lim_{n \to \infty} \|T^n\| = 0$.

Remark 4.1. Observe that since $R_i = U^{-1}e_i$ for some orthonormal basis $\{e_i\}_{n\geq 1} \subset H$, the condition

$$\sup_{m\geq 1}\left(\sum_{i=1}^{m}\|T^{*n}R_i\|^2\right)^{1/2} =: M_n < +\infty$$

is satisfied, for example, for Hilbert-Schmidt operator $T \in \mathcal{B}(H)$.

Proposition 4.1. *Let H be any infinite dimensional separable Hilbert space with Riesz basis $\{R_i\}_{i\geq 1}$ and let $T, S \in \mathcal{B}(H)$ be two operators such that: (i) there exists an isometry $V : H \to H$ such that $s-\lim\limits_{n} T^n = V$ and (ii) $\sum\limits_{i=1}^{\infty}\|SR_i\|^2 < +\infty$.*

Then

$$\lim_{n\to\infty}\|T^nS\| \leq \|U\|\left(\sum_{i=1}^{\infty}\|SR_i\|^2\right)^{1/2}. \tag{4.4}$$

Proof. The proof is similar to the proof of Theorem 4.1. Indeed, since $\lim\limits_{n\to\infty} T^n x = Vx$ for all $x \in H$, where V is an isometry, it is standard to show that T is power bounded, i.e. $\|T^n\| \leq C$ for all $n \geq 1$ and some constant $C > 0$. Then , as in the proof of Theorem 4.1, we have for any $\varepsilon \in (0,1)$ that

$$\begin{aligned}
\|T^nS\| &\leq \sup_{\|x\|=1}\left[C\|S\|\varepsilon + \|U\|(\varepsilon+1)\left(\sum_{i=1}^{N}\|T^nSR_i\|^2\right)^{1/2}\right] \\
&\leq \|S\|C\varepsilon + \|U\|(\varepsilon+1)\left(\sum_{i=1}^{\infty}\|T^nSR_i\|^2\right)^{1/2} \\
&\to \|U\|\left(\sum_{i=1}^{\infty}\|T^nSR_i\|^2\right)^{1/2}, \text{ as } \varepsilon \to 0.
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{n\to\infty}\|T^nS\| &\leq \|U\|\left(\sum_{i=1}^{\infty}\lim_{n\to\infty}\|T^nSR_i\|^2\right)^{1/2} \\
&= \|U\|\left(\sum_{i=1}^{\infty}\lim_{n\to\infty}\|VSR_i\|^2\right)^{1/2} = \|U\|\left(\sum_{i=1}^{\infty}\|SR_i\|^2\right)^{1/2},
\end{aligned}$$

which proves inequality (4.4) as desired. □

Remark 4.2. If $\Lambda := \{\lambda_n\}_{n\geq 1}$ is a sequence of distinct points in $\mathbb{D}$ and $B = B_\Lambda \prod\limits_{n\geq 1} b_{\lambda_n}$, where $b_{\lambda_n}(z) = \frac{|\lambda_n|}{\lambda_n}\,\frac{\lambda_n - z}{1-\overline{\lambda_n}z}$is the corresponding Blaschke product, then it is well known that (see, for instance, Nikolski [25]): (i) if $\Lambda := \{\lambda_n\}_{n\geq 1}$ is a Blaschke sequence (i.e. $\sum\limits_{i=1}^{\infty}\left(1-|\lambda_n|^2\right) < \infty$,) then the system $\left\{k_{H^2,\lambda_n}(z)\right\}_{n\geq 1} := \left\{\frac{1}{1-\overline{\lambda_n}z}\right\}_{n\geq 1}$ is complete in the model space $K_B := H^2 \ominus BH^2$,

(ii) the system $\left\{\widehat{k}_{H^2,\lambda_n}(z)\right\}_{n\geq 1} := \left\{\frac{\sqrt{1-|\lambda_n|^2}}{1-\overline{\lambda_n}z}\right\}_{n\geq 1}$ is a Riesz basis of K_B if only if $\{\lambda_n\}_{n\geq 1}$ satisfies Carleson's condition

$$\inf_{n\geq 1} |B_n(\lambda_n)| > 0,$$

where $B_n = \frac{B}{b_{\lambda_n}}$.

Thus, all results obtained above can be stated for the operators acting on the model space K_B.

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