

STRUCTURES AND BEHAVIORS OF GLOBAL CONTINUA OF NONTRIVIAL SOLUTIONS OF SOME LINEARIZABLE EIGENVALUE PROBLEMS

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Abstract. This paper considers a nonlinear Sturm-Liouville problem with an indefinite weight, which is linearizable at zero and at infinity. Note that problems of this type arise from selection-migration models in population genetics. Here we show the existence of four families of continua of nontrivial solutions of this problem that have the usual nodal properties, and connect bifurcation points with respect to the trivial solutions with the corresponding asymptotic bifurcation points.

1. Introduction

We consider the following nonlinear Sturm-Liouville eigenvalue problem

$$\ell(y) \equiv -(p(x)y')' + q(x)y = \lambda r(x)y + g(x, y, y')y, \quad x \in (0, 1), \quad (1.1)$$

$$\alpha_0 y(0) - \beta_0 y'(0) = 0, \quad (1.2)$$

$$\alpha_1 y(1) + \beta_1 y'(1) = 0, \quad (1.3)$$

where $\lambda \in \mathbb{R}$ is a spectral parameter, p is a positive continuously differentiable function on $[0, 1]$, q is a nonnegative continuous function on $[0, 1]$, r is a real-valued continuous function on $[0, 1]$ that changes sign, $\alpha_0, \beta_0, \alpha_1$ and β_1 are real constants such that

$$|\alpha_0| + |\beta_0| > 0, \quad |\alpha_1| + |\beta_1| > 0, \quad \alpha_0\beta_0 \geq 0 \quad \text{and} \quad \alpha_1\beta_1 \geq 0$$

(note that the case $q \equiv 0$ and $\alpha_0 = \alpha_1 = 0$ is excluded). Furthermore g is a non-positive continuous function on $[0, 1] \times \mathbb{R}^2$ and satisfies the following conditions:

$$g(x, y, s) \rightarrow 0 \quad \text{as} \quad |y| + |s| \rightarrow 0, \quad (1.4)$$

and

$$g(x, y, s) \rightarrow 0 \quad \text{as} \quad |y| + |s| \rightarrow +\infty, \quad (1.5)$$

uniformly with respect to $x \in [0, 1]$.

Problem (1.1)-(1.3) arises when separating variables in a boundary value problem obtained when modeling selection-migration in population genetics (see [6, 10, 11]).

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The global bifurcation from zero or infinity of nontrivial solutions of nonlinear eigenvalue problems for ordinary differential equations of second and fourth orders and elliptic partial differential equations of second order with indefinite weight was considered in papers [1-3, 5-7, 9, 12, 13, 15, 18, 19, 23]. The authors of these papers prove the existence of four families of global continua of nontrivial solutions to these nonlinear problems, branching from bifurcation points and intervals of the line of trivial solutions, or from bifurcation points and intervals of the line $\mathbb{R} \times \{\infty\}$. They also established that, in the case of bifurcation from zero, these global continua are contained in classes of functions possessing the oscillatory properties of the corresponding linear problems and are unbounded (in this case, these global continua either meet the line $\mathbb{R} \times \{\infty\}$, or their projection onto the line of trivial solutions is unbounded); however, in the case of bifurcation from infinity, these global continua are contained in classes of functions possessing oscillatory properties of the corresponding linear problems only in certain neighborhoods of bifurcation points and intervals, and either they meet other bifurcation points and intervals, or meet the line of trivial solutions, or their projection onto the line of trivial solutions is unbounded.

As can be seen from (1.1), unlike in [3, 5], the nonlinear term here does not depend on the eigenvalue parameter. Therefore, unlike the results presented above, we show that when conditions (1.4) and (1.5) are satisfied simultaneously, the global continua branching from zero have a bounded projection onto the line of trivial solutions and therefore intersect the line $\mathbb{R} \times \{\infty\}$. Moreover, we obtain a stronger result, namely, we show that the corresponding global continua branching from the line $\mathbb{R} \times \{\infty\}$ coincide with them.

2. Preliminaries

By (1.4) for any $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for any $x \in [0, 1]$ and any $(y, s) \in \mathbb{R}^2$ satisfying the condition $|y| + |s| < \delta_\varepsilon$ the inequality

$$|g(x, y, s)| < \varepsilon \quad (2.1)$$

holds. Then, for any $x \in [0, 1]$ and any $(y, s) \in \mathbb{R}^2$ satisfying the condition $|y| + |s| < \delta_\varepsilon$, we have

$$\frac{|g(x, y, s)|}{|y| + |s|} < \frac{\varepsilon}{|y| + |s|} \leq \varepsilon, \quad (2.2)$$

i.e., the relation

$$g(x, y, s) = o(|y| + |s|) \text{ as } |y| + |s| \rightarrow 0 \quad (2.3)$$

holds uniformly with respect to $x \in [0, 1]$. Then, by [16, Ch. IV, § 2, Theorem 2.1], problem (1.1)-(1.3) is linearizable in a neighborhood of $y = 0$ and the corresponding linear problem is the following eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda r(x)y(x), & x \in (0, 1), \\ \alpha_0 y(0) - \beta_0 y'(0) = 0, \\ \alpha_1 y(1) + \beta_1 y'(1) = 0. \end{cases} \quad (2.4)$$

It is known (see [14, §10.61, p. 237]) that there exists an infinite set of real eigenvalues of problem (2.4) which have limit points $+\infty$ and $-\infty$; if the positive

and negative eigenvalues are arranged in order of increasing numerical value, and are denoted by

$$\lambda_1^+, \lambda_2^+, \dots, \lambda_k^+, \dots, \\ \lambda_1^-, \lambda_2^-, \dots, \lambda_k^-, \dots,$$

and the corresponding eigenfunctions by

$$y_1^+, y_2^+, \dots, y_k^+, \dots, \\ y_1^-, y_2^-, \dots, y_k^-, \dots,$$

then the functions $y_k^+(x)$ and $y_k^-(x)$ have exactly $k-1$ zeros in the interval $(0, 1)$. Moreover (see [14, § 10.71, p. 239]), for every $k \in \mathbb{N}$ the following inequalities hold:

$$\int_0^1 r(x)(y_k^+(x))^2 dx > 0 \text{ and } \int_0^1 r(x)(y_k^-(x))^2 dx < 0. \quad (2.5)$$

Let

$$E = C^1[0, 1] \cap \{y : \alpha_0 y(0) - \beta_0 y'(0) = 0, \alpha_1 y(1) + \beta_1 y'(1) = 0\}$$

be the Banach space with the norm $\|y\|_1 = \|y\|_\infty + \|y'\|_\infty$, where

$$\|y\|_\infty = \max_{x \in [0, 1]} |y(x)|.$$

From now on, σ (respectively, ν) will denote either $+$ or $-$, and on $-\sigma$ (respectively, $-\nu$) will denote the opposite sign to σ (respectively, to ν).

Using the definition of classes S_k^ν , $k \in \mathbb{N}$, from [20] (see also [21, 22]) and inequality (2.5), the author of [18] constructed classes $S_{k, \sigma}^\nu$, $k \in \mathbb{N}$, of functions $y \in E$ that satisfy the following conditions:

(i) the zeros of the function y contained in $[0, 1]$ are simple and this function has exactly $k-1$ such zeros in $(0, 1)$;

(ii) $\sigma \int_0^1 r(x)y^2(x)dx > 0$;

(iii) the function $\nu y(x)$ is positive in a deleted neighborhood of the point $x = 0$.

For each $k \in \mathbb{N}$, each σ and each ν the set $S_{k, \sigma}^\nu$ is an open subset of E , and $S_{k', \sigma'}^{\nu'} \cap S_{k, \sigma}^\nu = \emptyset$ for any different (k, σ, ν) and (k', σ', ν') . Moreover, if $y \in \partial S_{k, \sigma}^\nu$,

then either there exists $\xi \in [0, 1]$ such that $y(\xi) = y'(\xi) = 0$, or $\int_0^1 r(x)y^2(x)dx = 0$ (see [19, p. 348]).

Remark 2.1. By the above arguments for each $k \in \mathbb{N}$ and each σ we have $y_k^\sigma \in S_{k, \sigma} = S_{k, \sigma}^+ \cup S_{k, \sigma}^-$.

Multiplying both sides of equation (1.1) by $y(x)$ and integrating the obtained result with respect to x from 0 to 1, using integration by parts, and taking into account boundary conditions (1.2) and (1.3) we get

$$\int_0^1 \{p(x)y'^2(x) + q(x)y^2(x)\} dx + N[y] = \lambda \int_0^1 r(x)y^2(x)dx + \\ \int_0^1 g(x, y(x), y'(x))y^2(x)dx \quad (2.6)$$

where

$$N[y] = -p(1)y'(1)y(1) + p(0)y'(0)y(0).$$

By conditions $\alpha_0\beta_0 \geq 0$ and $\alpha_1\beta_1 \geq 0$ we have

$$N[y] \geq 0. \quad (2.7)$$

Since the case $q \equiv 0$ and $\alpha_0 = \alpha_1 = 0$ is excluded, by (2.7), we get the following relation

$$\int_0^1 \{p(x)y'^2(x) + q(x)y^2(x)\} dx + N[y] > 0 \quad (2.8)$$

Then, in view of inequality (2.7), we have the following properties of solutions of problem (1.1)-(1.3).

Lemma 2.1. *Let $(\lambda, y) \in \mathbb{R} \times E$ be a nontrivial solution of problem (1.1)-(1.3). Then $\lambda \neq 0$, and consequently, either $\lambda \in \mathbb{R}^+ = (0, +\infty)$, or $\lambda \in \mathbb{R}^- = (-\infty, 0)$.*

Proof. If $(\lambda, y) \in \mathbb{R} \times E$ is a nontrivial solution of problem (1.1)-(1.3) such that $\lambda = 0$, then it follows from (2.6) that

$$\int_0^1 \{p(x)y'^2(x) + q(x)y^2(x)\} dx + N[y] = \int_0^1 g(x, y(x), y'(x))y^2(x) dx \quad (2.9)$$

Since $g(x, y(x), y'(x)) \leq 0$ for any $x \in [0, 1]$, the right-hand side of relation (2.9) is non-positive, and the left-hand side of this relation, according to (2.8), is positive. The resulting contradiction completes the proof of Lemma 2.1.

Lemma 2.2. *Let $(\lambda, y) \in \mathbb{R} \times E$ be the solution of problem (1.1)-(1.3) such that $y \in \partial S_{k,\sigma}^\nu$ for some $k \in \mathbb{N}$. Then $y \equiv 0$.*

Proof. If $(\lambda, y) \in \mathbb{R} \times E$ be the solution of problem (1.1)-(1.3) such that $y \in \partial S_{k,\sigma}^\nu$,

then either $y(\xi) = y'(\xi) = 0$ for some $\xi \in [0, 1]$, or $\int_0^1 r(x)y^2(x) dx = 0$.

If $y(\xi) = y'(\xi) = 0$ for some $\xi \in [0, 1]$, then the proof of this lemma is similar to that of [17, Lemma 2.2].

Now let $\int_0^1 r(x)y^2(x) dx = 0$. Then from relation (2.6) we obtain relation (2.9).

Next, the proof in this case proceeds in the same fashion as Lemma 2.1. The proof of Lemma 2.2 is complete.

3. Structures and behaviors of global continua of nontrivial solutions of problem (1.1)-(1.3)

By (2.3) and Lemmas 2.1, 2.2 it follows from [5, Theorem 3.1] (see also [20, Theorem 2.3]) that the following global bifurcation result holds for the problem (1.1)-(1.3).

Theorem 3.1. *For each $k \in \mathbb{N}$, each σ and each ν there exists a component $C_{k,\sigma}^\nu$ of nontrivial solutions of problem (1.1)-(1.3) which meets $(\lambda_k^\sigma, 0)$, is contained in $\mathbb{R}^\sigma \times S_{k,\sigma}^\nu$ and is unbounded in $\mathbb{R} \times E$ (in this case either $C_{k,\sigma}^\nu$ meets (λ, ∞) for some $\lambda \in \mathbb{R}^\sigma$, or its projection onto $\mathbb{R} \times \{0\}$ is unbounded).*

Note that using the condition (1.5) we can study in more detail the behavior of the components $\mathcal{C}_{k,\sigma}^\nu$, $k \in \mathbb{N}$, of the set of nontrivial solutions of problem (1.1)-(1.3).

Lemma 3.1. *For each $k \in \mathbb{N}$, each σ and each ν the natural projection of the set $\mathcal{C}_{k,\sigma}^\nu$ onto $\mathbb{R} \times \{0\}$ is bounded.*

Proof. Let $(\eta, v) \in \mathcal{C}_{k,\sigma}^\nu$. Then, by Theorem 3.1, $\eta \in \mathbb{R}^\sigma$ and $v \in S_{k,\sigma}^\nu$.

We introduce the notation:

$$\varphi(x) = -g(x, v(x), v'(x)), \quad x \in [0, 1]. \quad (3.1)$$

Since $g \in C([0, 1] \times \mathbb{R}^2)$, $g(x, y, s) \leq 0$ for any $[0, 1] \times \mathbb{R}^2$, and $v \in E$, it follows that

$$\varphi(x) \in C[0, 1] \text{ and } \varphi(x) \geq 0, \quad x \in [0, 1]. \quad (3.2)$$

By (1.5) for any $\varepsilon > 0$ there exists $\Delta_\varepsilon > 0$ such that for any $x \in [0, 1]$ and any $(y, s) \in \mathbb{R}^2$ satisfying $|y| + |s| > \Delta_\varepsilon$ we have

$$|g(x, y, s)| < \varepsilon. \quad (3.3)$$

Since $g \in C([0, 1] \times \mathbb{R}^2)$ there exists a positive constant M_ε such that for any $x \in [0, 1]$ and any $(y, s) \in \mathbb{R}^2$ satisfying $\delta_\varepsilon \leq |y| + |s| \leq \Delta_\varepsilon$ the inequality

$$|g(x, y, s)| < M_\varepsilon. \quad (3.4)$$

holds.

Let

$$\mathcal{M}_1 = \max \{1, M_1\}.$$

Then by (2.1), (3.3) and (3.4) it follows from (3.1) that

$$0 \leq \varphi(x) \leq \mathcal{M}_1, \quad x \in [0, 1]. \quad (3.5)$$

By (3.1) the number η is an eigenvalue of the linear problem

$$\begin{cases} \ell(y)(x) + \varphi(x)y(x) = \lambda r(x)y(x), & x \in (0, 1), \\ \alpha_0 y(0) - \beta_0 y'(0) = 0, \\ \alpha_1 y(1) + \beta_1 y'(1) = 0, \end{cases} \quad (3.6)$$

and v is the corresponding eigenfunction. Since $v \in S_{k,\sigma}^\nu$, by (3.2), it follows from [14, § 10-61, p. 237] that $\eta = \lambda_{k,\varphi}^\sigma$, where $\lambda_{k,\varphi}^\sigma$ is k th eigenvalue of problem (3.6) with the sign σ .

Along with eigenvalue problems (2.4) and (3.6), we consider the following eigenvalue problems

$$\begin{cases} \ell(y)(x) - \lambda r(x)y(x) = \mu y(x), & x \in (0, 1), \\ \alpha_0 y(0) - \beta_0 y'(0) = 0, \\ \alpha_1 y(1) + \beta_1 y'(1) = 0, \end{cases} \quad (3.7)$$

$$\begin{cases} \ell(y)(x) + \varphi(x)y(x) - \lambda r(x)y(x) = \mu y(x), & x \in (0, 1), \\ \alpha_0 y(0) - \beta_0 y'(0) = 0, \\ \alpha_1 y(1) + \beta_1 y'(1) = 0. \end{cases} \quad (3.8)$$

It is well known (see, for example, [14, § 10-6, p. 233]) that for each fixed $\lambda \in \mathbb{R}$ the eigenvalues of problems (3.7) and (3.8) are real and simple and form infinitely increasing sequences $\{\mu_k(\lambda)\}_{k=1}^\infty$ and $\{\mu_{k,\varphi}(\lambda)\}_{k=1}^\infty$, respectively. Moreover, for each $k \in \mathbb{N}$, the eigenfunctions $v_k(x, \lambda)$ and $v_{k,\varphi}(x, \lambda)$, corresponding to the

eigenvalues $\mu_k(\lambda)$ and $\mu_{k,\varphi}(\lambda)$, respectively, have exactly $k - 1$ simple zeros in the interval $(0, 1)$. It should be noted that the eigenvalues λ_k^σ and $\lambda_{k,\varphi}^\sigma$, $k \in \mathbb{N}$, of problems (3.7) and (3.8) are the zeros of the functions $\mu_k(\lambda)$ and $\mu_{k,\varphi}(\lambda)$, respectively.

In view of (3.5), by formula (3.3) of [2] for each $k \in \mathbb{N}$ and each $\lambda \in \mathbb{R}$ we have the following relation:

$$\mu_k(\lambda) \leq \mu_{k,\varphi}(\lambda) \leq \mu_k(\lambda) + \mathcal{M}_1. \quad (3.9)$$

By (2.8) we have

$$\mu_k(0) > 0 \text{ and } \mu_{k,\varphi}(0) > 0.$$

Then it follows from (3.9) that

$$0 < \mu_k(0) \leq \mu_{k,\varphi}(0) \leq \mu_k(0) + \mathcal{M}_1. \quad (3.10)$$

By [2, Lemma 1] for each $k \in \mathbb{N}$ the continuous on $\mathbb{R}$ functions $\mu_k(\lambda)$ and $\mu_{k,\varphi}(\lambda)$ are strictly decreasing on $\mathbb{R}^+$ and strictly increasing on $\mathbb{R}^-$. Moreover, by [4, Theorem 2.2] we have

$$\mu_k(\lambda) \rightarrow -\infty \text{ and } \mu_{k,\varphi}(\lambda) \rightarrow -\infty \text{ as } |\lambda| \rightarrow +\infty. \quad (3.11)$$

Since $\mu_{k,\varphi}(\lambda_{k,\varphi}^\sigma) = 0$, by (3.9), we get $\mu_k(\lambda_{k,\varphi}^\sigma) \geq -\mathcal{M}_1$. Then by (3.10) there exists a positive integer m_1 such that

$$\mu_{k+m_1}(\lambda_{k,\varphi}^\sigma) \geq \mu_k(\lambda_{k,\varphi}^\sigma) + 2\mathcal{M}_1 \geq \mathcal{M}_1 > 0.$$

Hence by (3.11) we get the following relation:

$$\sigma \lambda_k^\sigma \leq \sigma \lambda_{k,\varphi}^\sigma < \sigma \lambda_{k+m_1}^\sigma. \quad (3.12)$$

Thus, it follows from (3.12) that

$$\lambda_{k,\varphi}^+ \in [\lambda_k^+, \lambda_{k+m_1}^+) \text{ and } \lambda_{k,\varphi}^- \in (\lambda_{k+m_1}^-, \lambda_k^-], \quad (3.13)$$

whence implies that

$$\eta \in [\lambda_k^+, \lambda_{k+m_1}^+) \text{ if } \eta = \lambda_{k,\varphi}^+ \text{ and } \eta \in (\lambda_{k+m_1}^-, \lambda_k^-] \text{ if } \eta = \lambda_{k,\varphi}^-. \quad (3.14)$$

Thus (3.14) shows that the natural projection of $\mathcal{C}_{k,\sigma}^\nu$ onto $\mathbb{R} \times \{0\}$ is bounded. The proof of Lemma 3.1 is complete.

Remark 3.1. By Lemma 3.1 it follows from Theorem 3.1 that for each $k \in \mathbb{N}$, each σ and each ν the component $\mathcal{C}_{k,\sigma}^\nu$ of solutions of problem (1.1)-(1.3) meets (λ, ∞) for some $\lambda \in \mathbb{R}^\sigma$.

Lemma 3.2. For each $k \in \mathbb{N}$, each σ and each ν the set $\mathcal{C}_{k,\sigma}^\nu$ meets $(\lambda_k^\sigma, \infty)$.

Proof. Let $\{(\lambda_{k,n,\sigma}^\nu, y_{k,n,\sigma}^\nu)\}_{n=1}^\infty \subset \mathcal{C}_{k,\sigma}^\nu$ be a sequence such that

$$\lambda_{k,n,\sigma}^\nu \rightarrow \hat{\lambda} \text{ and } \|y_{k,n,\sigma}^\nu\|_1 \rightarrow +\infty \text{ as } n \rightarrow \infty. \quad (3.15)$$

Note that for each $n \in \mathbb{N}$ the following relations hold:

$$\begin{cases} \ell(y_{k,n,\sigma}^\nu) = \lambda_{k,n,\sigma}^\nu r(x) y_{k,n,\sigma}^\nu + g(x, y_{k,n,\sigma}^\nu, (y_{k,n,\sigma}^\nu)') y_{k,n,\sigma}^\nu, & x \in (0, 1), \\ \alpha_0 y_{k,n,\sigma}^\nu(0) - \beta_0 (y_{k,n,\sigma}^\nu)'(0) = 0, \\ \alpha_1 y_{k,n,\sigma}^\nu(1) + \beta_1 (y_{k,n,\sigma}^\nu)'(1) = 0, \end{cases} \quad (3.16)$$

Let

$$w_{k,n,\sigma}^\nu(x) = \frac{y_{k,n,\sigma}^\nu(x)}{\|y_{k,n,\sigma}^\nu\|_1} \quad (3.17)$$

and

$$g_{k,n,\sigma}^\nu(x) = \frac{g(x, y_{k,n,\sigma}^\nu(x), (y_{k,n,\sigma}^\nu)'(x)) y_{k,n,\sigma}^\nu(x)}{\|y_{k,n,\sigma}^\nu\|_1}, \quad x \in [0, 1], \quad n \in \mathbb{N}. \quad (3.18)$$

Let $\Delta_{1,\varepsilon} > \Delta_\varepsilon$ be chosen so large that the following inequality holds:

$$\frac{M_\varepsilon \Delta_\varepsilon}{\Delta_{1,\varepsilon}} < \varepsilon. \quad (3.19)$$

Let $y \in E$ such that $\|y\|_1 > \Delta_{1,\varepsilon}$ and let

$$\hat{g}(x) = \frac{g(x, y(x), y'(x)) y(x)}{\|y\|_1}, \quad x \in [0, 1].$$

If $|y(x)| + |y'(x)| \leq \Delta_\varepsilon$, then by (2.1), (3.4) and (3.19) for any $x \in [0, 1]$ we get

$$|\hat{g}(x)| = \frac{|g(x, y(x), y'(x))| |y(x)|}{\|y\|_1} < \frac{M_\varepsilon \Delta_\varepsilon}{\Delta_{1,\varepsilon}} < \varepsilon.$$

If $|y(x)| + |y'(x)| > \Delta_\varepsilon$, then by (3.3) for any $x \in [0, 1]$ we obtain

$$|\hat{g}(x)| = \frac{|g(x, y(x), y'(x))| |y(x)|}{\|y\|_1} < |g(x, y(x), y'(x))| < \varepsilon.$$

Thus for any $\varepsilon > 0$ there exists $\Delta_{1,\varepsilon} > 0$ such that for any $x \in [0, 1]$ and any $y \in E$ satisfying $\|y\|_1 > \Delta_{1,\varepsilon}$ the relation

$$|\hat{g}(x)| = \frac{|g(x, y(x), y'(x))| |y(x)|}{\|y\|_1} < \varepsilon \quad (3.20)$$

holds.

Since $\|y_{k,n,\sigma}^\nu\|_1 \rightarrow +\infty$ as $n \rightarrow \infty$ it follows that there exists $n_\varepsilon \in \mathbb{N}$ such that for any $n > n_\varepsilon$ we get

$$\|y_{k,n,\sigma}^\nu\|_1 > \Delta_{1,\varepsilon}.$$

Then, by (3.20), from (3.18) we obtain

$$|g_{k,n,\sigma}^\nu(x)| < \varepsilon \quad \text{for any } x \in [0, 1] \text{ and } n > n_\varepsilon. \quad (3.21)$$

The last relation shows that

$$\|g_{k,n,\sigma}^\nu\|_\infty \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (3.22)$$

Let $n_{\varepsilon,1} \geq n_\varepsilon$ be chosen so that for any $n > n_{\varepsilon,1}$ the following inequality is satisfied:

$$|\lambda_{k,n,\sigma}^\nu - \hat{\lambda}| < \varepsilon. \quad (3.23)$$

By (3.17) and (3.18) it follows from (3.16) that

$$\begin{cases} \ell(w_{k,n,\sigma}^\nu)(x) = \lambda_{k,n,\sigma}^\nu r(x) w_{k,n,\sigma}^\nu(x) + g_{k,n,\sigma}^\nu(x), & x \in (0, 1), \\ \alpha_0 w_{k,n,\sigma}^\nu(0) - \beta_0 (w_{k,n,\sigma}^\nu)'(0) = 0, \\ \alpha_1 w_{k,n,\sigma}^\nu(1) + \beta_1 (w_{k,n,\sigma}^\nu)'(1) = 0, \end{cases} \quad (3.24)$$

and consequently, by (3.21) and (3.23), for any $x \in [0, 1]$ and any $n > n_{\varepsilon,1}$ we get

$$\begin{aligned} |(w_{k,n,\sigma}^\nu)''(x)| &\leq \frac{1}{p(x)} \left\{ |p'(x)| |(w_{k,n,\sigma}^\nu)'(x)| + (|\lambda_{k,n,\sigma}^\nu| |r(x)| + |q(x)|) |w_{k,n,\sigma}^\nu(x)| + |g_{k,n,\sigma}^\nu(x)| \right\} \\ &\leq \frac{1}{p_0} \left\{ \|p'\|_\infty + ((\hat{\lambda} + \varepsilon) \|r\|_\infty + \|q\|_\infty) + \varepsilon \right\} \leq \\ &\quad \frac{1}{p_0} \left\{ \|p'\|_\infty + ((\hat{\lambda} + 1) \|r\|_\infty + \|q\|_\infty) + 1 \right\}, \end{aligned}$$

where $p_0 = \min_{x \in [0,1]} p(x)$. Then by the Arzelà-Ascoli theorem there exists a subsequence $\{(\lambda_{k,n_m,\sigma}^\nu, w_{k,n_m,\sigma}^\nu)\}_{m=1}^\infty$ of a sequence $\{(\lambda_{k,n,\sigma}^\nu, w_{k,n,\sigma}^\nu)\}_{n=1}^\infty$ which converges in $\mathbb{R} \times E$ to some $(\hat{\lambda}, w)$ (as $m \rightarrow \infty$) with $\|w\|_1 = 1$, since $\|w_{k,n_m,\sigma}^\nu\|_1 = 1$ for any $m \in \mathbb{N}$. Hence it follows from

$$\begin{cases} \ell(w_{k,n_m,\sigma}^\nu)(x) = \lambda_{k,n_m,\sigma}^\nu r(x) w_{k,n_m,\sigma}^\nu(x) + g_{k,n_m,\sigma}^\nu(x), & x \in (0, 1), \\ \alpha_0 w_{k,n_m,\sigma}^\nu(0) - \beta_0 (w_{k,n_m,\sigma}^\nu)'(0) = 0, \\ \alpha_1 w_{k,n_m,\sigma}^\nu(1) + \beta_1 (w_{k,n_m,\sigma}^\nu)'(1) = 0, \end{cases} \quad (3.25)$$

that $(\lambda_{k,n,\sigma}^\nu, w_{k,n,\sigma}^\nu) \rightarrow (\hat{\lambda}, w)$ as $m \rightarrow \infty$ in $\mathbb{R} \times C^2[0, 1]$. Now, using (3.22) and passing to the limit as $m \rightarrow \infty$ in (3.25) we obtain

$$\begin{cases} \ell(w)(x) = \hat{\lambda} r(x) w(x), & x \in (0, 1), \\ \alpha_0 w(0) - \beta_0 w'(0) = 0, \\ \alpha_1 w(1) + \beta_1 w'(1) = 0, \end{cases}$$

i.e., $(\hat{\lambda}, w)$ solves the linear problem (2.4). By the relation $\{(\lambda_{k,n_m,\sigma}^\nu, y_{k,n_m,\sigma}^\nu)\}_{m=1}^\infty \subset \mathcal{C}_{k,\sigma}^\nu$ we have $w_{k,n_m,\sigma}^\nu \in S_{k,\sigma}^\nu$, $m \in \mathbb{N}$, and consequently, $w \in S_{k,\sigma}^\nu \cup \partial S_{k,\sigma}^\nu$. Since $\|w\|_1 = 1$ it follows from Lemma 2.2 that $w \in S_{k,\sigma}^\nu$. Therefore, by Remark 2.1, we have $\hat{\lambda} = \lambda_k^\sigma$. The proof of Lemma 3.2 is complete.

By Lemmas 3.1 and 3.2, Theorem 3.1 is strengthened as follows.

Theorem 3.2. *For each $k \in \mathbb{N}$, each σ and each ν the component $\mathcal{C}_{k,\sigma}^\nu$ of nontrivial solutions of problem (1.1)-(1.3) is contained in $\mathbb{R}^\sigma \times S_{k,\sigma}^\nu$ and meets $(\lambda_k^\sigma, 0)$ and $(\lambda_k^\sigma, \infty)$ in $\mathbb{R} \times E$.*

Proof. By Theorem 3.1 for each $k \in \mathbb{N}$, each σ and each ν the component $\mathcal{C}_{k,\sigma}^\nu$ of nontrivial solutions of problem (1.1)-(1.3) meets $(\lambda_k^\sigma, 0)$, is contained in $\mathbb{R}^\sigma \times S_{k,\sigma}^\nu$ and either $\mathcal{C}_{k,\sigma}^\nu$ meets (λ, ∞) for some $\lambda \in \mathbb{R}^\sigma$, or its projection onto $\mathbb{R} \times \{0\}$ is unbounded. By Lemma 3.1 the natural projection of $\mathcal{C}_{k,\sigma}^\nu$ onto $\mathbb{R} \times \{0\}$ is bounded. Consequently, for each $k \in \mathbb{N}$, each σ and each ν the set $\mathcal{C}_{k,\sigma}^\nu$ meets $(\hat{\lambda}, \infty)$ for some $\hat{\lambda} \in \mathbb{R}^\sigma$ (see Remark 3.1). Then it follows from Lemma 3.2 that $\hat{\lambda} = \lambda_k^\sigma$, i.e. $\mathcal{C}_{k,\sigma}^\nu$ also meets $(\lambda_k^\sigma, \infty)$. The proof of Theorem 3.2 is complete.

By (3.3) for any $x \in [0, 1]$ and any $(y, s) \in \mathbb{R}^2$ satisfying $|y| + |s| > \Delta_\varepsilon$ the relation

$$\frac{|g(x, y, s) y|}{|y| + |s|} < |g(x, y, s)| < \varepsilon$$

holds, whence implies that

$$g(x, y, s) y = o(|y| + |s|) \text{ as } |y| + |s| \rightarrow +\infty, \quad (3.26)$$

uniformly for $x \in [0, 1]$.

If only condition (1.5) is satisfied, then by condition $g \in C([0, 1] \times \mathbb{R}^2)$ there exists $N_1 > 0$ such that for any $x \in [0, 1]$ and any $(y, s) \in \mathbb{R}^2$ satisfying the relation $|y| + |s| \leq \Delta_1$ we have the following inequality:

$$|g(x, y, s)| \leq N_1. \quad (3.27)$$

Let $\mathcal{N}_1 = \max\{1, N_1\}$. Then by (3.3) (with $\varepsilon = 1$) and (3.27) we get

$$0 \leq \varphi(x) \leq \mathcal{N}_1, \quad x \in [0, 1]. \quad (3.28)$$

Then it follows from the proof of Lemma 3.1 that there exists $m_2 \in \mathbb{N}$ such that for any nontrivial solution $(\lambda, y) \in \mathbb{R} \times E$ of problem (1.1)-(1.3) the following relations hold:

$$\lambda \in [\lambda_k^+, \lambda_{k+m_2}^+) \text{ or } \lambda \in (\lambda_{k+m_2}^-, \lambda_k^-] \text{ and } y \in S_{k,\sigma}^\nu \text{ for some } k \in \mathbb{N}. \quad (3.29)$$

Remark 3.2. Recall that if $(\lambda, y) \in \mathbb{R} \times E$ is a nontrivial solution of problem (1.1)-(1.3), then λ is an eigenvalue and y is the corresponding eigenfunction of the linear problem (3.6) with a continuous function $\varphi(x)$ satisfying relation (3.28). If $y \in \partial S_{k,\sigma}^\nu$, then either there exists $\xi \in [0, 1]$ such that $y(\xi) = y'(\xi) = 0$, or $\int_0^1 r(x) y^2(x) dx = 0$, which are impossible due to the theorem on the uniqueness of the solution of the Cauchy problem and the following relation (see (2.8)), respectively,

$$\int_0^1 \{p(x) y'^2(x) + q(x) y^2(x) + \varphi(x) y^2(x)\} dx + N[y] > 0.$$

If only condition (1.5) is satisfied, then we have the following global bifurcation result from infinity for problem (1.1)-(1.3).

Theorem 3.3. *Let only the condition (1.5) be satisfied. Then for each $k \in \mathbb{N}$, each σ and each ν there exists a component $\mathcal{D}_{k,\sigma}^\nu$ of nontrivial solutions of problem (1.1)-(1.3) which meets $(\lambda_k^\sigma, \infty)$, is contained in $\mathbb{R}^\sigma \times S_{k,\sigma}^\nu$ and meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}^\sigma$.*

Proof. Let only condition (1.5) be satisfied. Then relation (3.26) holds, and consequently, by following the arguments in Theorem 2.4 of [21] (see also [3, p. 6], [5, Theorem 4.1] and [22, Theorem 3.1]) we can show that for each $k \in \mathbb{N}$, each σ and each ν there exists a component $\mathcal{D}_{k,\sigma}^\nu$ of nontrivial solutions of problem (1.1)-(1.3) which meets $(\lambda_k^\sigma, \infty)$ and the neighborhood Q_k of $(\lambda_k^\sigma, \infty)$ such that (i) $\mathcal{D}_{k,\sigma}^\nu \cap Q_k \subset \mathbb{R} \times S_{k,\sigma}^\nu$; (ii) either $\mathcal{D}_{k,\sigma}^\nu \setminus Q_k$ meets $(\lambda_{k'}^\sigma, \infty)$ with respect to the set $\mathbb{R} \times S_{k',\sigma}^\nu$ for some $(k', \nu') \neq (k, \nu)$, or $\mathcal{D}_{k,\sigma}^\nu \setminus Q_k$ meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$, or the natural projection of $\mathcal{D}_{k,\sigma}^\nu \setminus Q_k$ onto $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ is unbounded. Then by Remark 3.2 we have $\mathcal{D}_{k,\sigma}^\nu \subset \mathbb{R} \times S_{k,\sigma}^\nu$. Since $S_{k',\sigma}^\nu \cap S_{k,\sigma}^\nu = \emptyset$ for any $(k', \nu') \neq (k, \nu)$ it follows that $\mathcal{D}_{k,\sigma}^\nu \setminus Q_k$ cannot meet $(\lambda_{k'}^\sigma, \infty)$ with respect to the set $\mathbb{R} \times S_{k',\sigma}^\nu$ for some $(k', \nu') \neq (k, \nu)$. Moreover, by (3.29) the natural projection of $\mathcal{D}_{k,\sigma}^\nu \setminus Q_k$ onto $\{(\lambda, 0) : \lambda \in \mathbb{R}\}$ is bounded. Therefore, the set $\mathcal{D}_{k,\sigma}^\nu \setminus Q_k$ meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$. Thus we show that $\mathcal{D}_{k,\sigma}^\nu \subset \mathbb{R} \times S_{k,\sigma}^\nu$ and this set meets $(\lambda, 0)$ for some $\lambda \in \mathbb{R}$. The proof of Theorem 3.3 is complete.

If both conditions (1.4) and (1.5) are satisfied, then we can improve Theorem 3.3 as follows.

Theorem 3.4. *Let conditions (1.4) and (1.5) both hold. Then for each $k \in \mathbb{N}$, each σ and each ν the set $\mathcal{D}_{k,\sigma}^\nu$ meets $(\lambda_k^\sigma, 0)$.*

The following theorem is the main result of this paper.

Theorem 3.5. *For each $k \in \mathbb{N}$, each σ and each ν the following relation holds:*

$$\mathcal{C}_{k,\sigma}^\nu = \mathcal{D}_{k,\sigma}^\nu.$$

Proof. By [16, Ch. IV, § 2, Theorem 2.1] and Theorem 3.1, for each $k \in \mathbb{N}$, each σ and each ν the point $(\lambda_k^\sigma, 0)$ is the unique bifurcation point of the nonlinear eigenvalue problem (1.1)-(1.3) with respect to the set $\mathbb{R}^\sigma \times S_{k,\sigma}^\nu$. According to [19, Lemma 1.24] and Dancer's construction [8, pp.1070-1071] to this point corresponds a unique continuous branch of non-trivial solutions. Moreover, by [16, Ch. IV, § 3, Theorem 3.1] and Theorem 3.3 for each $k \in \mathbb{N}$, each σ and each ν the point $(\lambda_k^\sigma, \infty)$ is the unique asymptotic bifurcation point of problem (1.1)-(1.3) with respect to the set $\mathbb{R}^\sigma \times S_{k,\sigma}^\nu$. According to the proof of Theorem 2.4 of [21, p. 469] and Dancer's construction [8, pp. 1070-1071], to this point corresponds a unique continuous branch of nontrivial solutions extending to infinity. Then, by Theorems 3.2, 3.4 and the proof of Lemma 3.2, it follows that

$$C_{k,\sigma}^\nu = D_{k,\sigma}^\nu.$$

The proof of Theorem 3.5 is complete.

Example 3.1. Consider the following nonlinear eigenvalue problem

$$\begin{cases} \ell(y)(x) = \lambda r(x)y(x) + \hat{g}(x, y(x), y'(x))\phi(\|y\|_1)y(x), & x \in (0, 1), \\ \alpha_0 y(0) - \beta_0 y'(0) = 0, \\ \alpha_1 y(1) + \beta_1 y'(1) = 0, \end{cases} \quad (3.30)$$

where

$$\hat{g}(x, y, s) = |\cos(x^2 + y^2 + s^2)^{\frac{1}{2}}|, \quad (3.31)$$

$$\phi(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \pi \\ \sin t & \text{if } \pi < t < 2\pi, \\ 0 & \text{if } t \geq 2\pi. \end{cases} \quad (3.32)$$

Note that all the results of Section 3 are true for problem (3.30) with a scalar function ϕ .

By Remark 2.1, the eigenfunction y_k^+ corresponding to the eigenvalue λ_k^+ of problem (2.4) is made unique by requiring that $y_k^+ \in S_{k,+}^+$ and $\|y_k^+\|_1 = 1$. Moreover, the eigenfunction y_k^- corresponding to the eigenvalue λ_k^- of problem (2.4) is also made unique by requiring that $y_k^- \in S_{k,-}^+$ and $\|y_k^-\|_1 = 1$.

It follows from (3.31) and (3.32) that

$$g(x, y, y') = \begin{cases} 0 & \text{if } \|y\|_1 \leq \pi, \\ |\cos(x^2 + y^2 + y'^2)^{\frac{1}{2}}| \cdot \sin(\|y\|_1) & \text{if } \pi < \|y\|_1 < 2\pi, \\ 0 & \text{if } \|y\|_1 \geq 2\pi. \end{cases} \quad (3.33)$$

Let $(\lambda, y) \in \mathbb{R} \times E$ be a solution to problem (3.30) such that $\|y\|_1 < \pi$ or $\|y\|_1 \geq 2\pi$ and $y \in S_{k,\sigma}^\nu$. Then, according to (3.33), the pair (λ, y) is also a solution to the linear problem (2.4). Since $y \in S_{k,\sigma}^\nu$, it follows that $\lambda = \lambda_k^\sigma$ and $y = \nu \varkappa y_k^\nu$, where $\varkappa \in (0, \pi)$ or $(2\pi, +\infty)$. If $(\lambda, y) \in \mathbb{R} \times E$ is a solution of problem (3.30) such that $\pi < \|y\|_1 < 2\pi$ and $y \in S_{k,\sigma}^\nu$, then by (3.33) we have $\lambda = \lambda_{k,\hat{\varphi}}^\sigma \in \mathbb{R}^\sigma$, where $\lambda_{k,\hat{\varphi}}^+$ and $\lambda_{k,\hat{\varphi}}^-$ are k -th positive and negative eigenvalues, respectively, of the linear problem

$$\begin{cases} \ell(y)(x) + \hat{\varphi}(x)y(x) = \lambda r(x)y(x), & x \in (0, 1), \\ \alpha_0 y(0) - \beta_0 y'(0) = 0, \\ \alpha_1 y(1) + \beta_1 y'(1) = 0, \end{cases}$$

with $\hat{\varphi}(x) = -g(x, y(x), y'(x)) \geq 0$, $x \in [0, 1]$. Moreover, it follows from (3.13) that there exists $\hat{m}_1 \in \mathbb{N}$ such that

$$\lambda \in [\lambda_k^+, \lambda_{k+\hat{m}_1}^+] \text{ if } \lambda = \lambda_{k,\hat{\varphi}}^+ \text{ and } \lambda \in [\lambda_{k+\hat{m}_1}^-, \lambda_k^-] \text{ if } \lambda = \lambda_{k,\hat{\varphi}}^-.$$

By the above arguments for each $k \in \mathbb{N}$ and each $\nu \in +, -$ we have

$$\begin{aligned} \mathcal{C}_{k,+}^\nu &= \mathcal{D}_{k,+}^\nu = \{(\lambda_k^+, \nu \varkappa y_k^+) : \varkappa \in (0, \pi) \cup (2\pi, +\infty)\} \cup \\ &\quad \{(\lambda, y_\lambda) \in \mathbb{R}^+ \times S_{k,+}^\nu : \lambda \in [\lambda_k^+, \lambda_{k+\hat{m}_1}^+]\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_{k,-}^\nu &= \mathcal{D}_{k,-}^\nu = \{(\lambda_k^-, \nu \varkappa y_k^-) : \varkappa \in (0, \pi) \cup (2\pi, +\infty)\} \cup \\ &\quad \{(\lambda, y_\lambda) \in \mathbb{R}^- \times S_{k,-}^\nu : \lambda \in [\lambda_{k+\hat{m}_1}^-, \lambda_k^-]\}. \end{aligned}$$

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