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PROPERTIES OF THE RIESZ TRANSFORM OF A FINITE MEASURE WITH SINGULAR PART CONCENTRATED AT FINITELY MANY POINTS

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Abstract. In this paper, we investigate the asymptotic behavior of the distribution function of the Riesz transform for a finite measure with a singular part concentrated at finitely many points. Furthermore, using the notion of Q-integration introduced by Titchmarsh, we establish the Q-integrability of the Riesz transform for such measures and prove an analog of the classical Riesz equality.

1. Introduction

The j-th Riesz transform of a function $f \in L^p(\mathbb{R}^d)$ $(1 \le p < \infty,\ j = 1, \dots, d)$ is defined as the following singular integral:

$$R_j f(x) = \gamma(d) \cdot \lim_{\varepsilon \to 0^+} \int_{\{y \in \mathbb{R}^d : |x-y| > \varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}} f(y)\, dy,$$

where

$$\gamma(d) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}}$$

and $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t}\, dt$ is Euler's Gamma function.

It is well known (see [15]) that the Riesz transform plays an important role in the theory of harmonic functions. Indeed, if we consider the simplest non-trivial operators in $\mathbb{R}^d$ that are invariant with respect to rotations, translations, and dilations, we naturally arrive at the Riesz transforms. Furthermore, attempts to extend the classical results of complex-variable harmonic analysis to $\mathbb{R}^d$ also lead directly to the Riesz transform. In particular, the boundary values of conjugate harmonic functions in the upper half-space are interconnected by the Riesz transform.

From the theory of singular integrals (see [15]), it is well known that the Riesz transform is a bounded operator in the space $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. That is, if $f \in L^p(\mathbb{R}^d)$, then $R_j f \in L^p(\mathbb{R}^d)$ and

$$\|R_j f\|_{L^p(\mathbb{R}^d)} \le C_p \|f\|_{L^p(\mathbb{R}^d)}. \tag{1.1}$$

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In the case $f \in L^1(\mathbb{R}^d)$, the Riesz transform satisfies only the weak-type estimate

$$m\left(\{x \in \mathbb{R}^d : |R_j f(x)| > \lambda\}\right) \leq \frac{C_1}{\lambda}\|f\|_{L^1(\mathbb{R}^d)}, \quad \lambda > 0, \tag{1.2}$$

where m denotes the Lebesgue measure, and $C_p, C_1 > 0$ are constants independent of f.

From inequalities (1.1) and (1.2), it follows that the Riesz transform of a function $f \in L^1(\mathbb{R}^d)$ satisfies

$$m\left(\{x \in \mathbb{R}^d : |R_j f(x)| > \lambda\}\right) = o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to +\infty.$$

Note that the Riesz transform of a function $f \in L^1(\mathbb{R}^d)$ is not Lebesgue integrable in general. In [7], the authors show that the Riesz transform of a function $f \in L^1(\mathbb{R}^d)$ is Q-integrable on $\mathbb{R}^d$ and that an analogue of the Riesz equality holds.

In this paper, we study the asymptotic behavior of the distribution function of the Riesz transform of a finite measure whose singular part is concentrated at a finite number of points. Furthermore, using the notion of Q-integration introduced by Titchmarsh, we prove that the Riesz transform of this measure is Q-integrable and that an analog of the Riesz equality holds.

2. Asymptotic behavior of the distribution function of the Riesz transform of a finite measure whose singular part is concentrated at a finite number of points

In this section, we study the asymptotic behavior of the distribution function of the Riesz transform of a finite measure whose singular part is concentrated at a finite number of points.

Let ν be a finite measure on $\mathbb{R}^d$. The j-th Riesz transform of the measure ν $(j = 1, \ldots, d)$ is defined as the following singular integral:

$$R_j\nu(x) = \gamma(d) \cdot \lim_{\varepsilon \to 0^+} \int_{\{y \in \mathbb{R}^d : |x-y| > \varepsilon\}} \frac{x_j - y_j}{|x-y|^{d+1}}\, d\nu(y).$$

It is obvious that if the measure ν is a discrete measure concentrated at a finite number of points in $\mathbb{R}^d$, with

$$\operatorname{supp}\nu = X = \{x^{(k)}\}_{k \in J} \quad \text{and} \quad \nu(\{x^{(k)}\}) = \alpha_k, \quad k \in J,$$

then its j-th Riesz transform is given by the formula

$$R_j\nu(x) = \gamma(d) \sum_{k \in J} \alpha_k \frac{x_j - x_j^{(k)}}{|x - x^{(k)}|^{d+1}}, \quad x \notin X.$$

Theorem 2.1. *Let ν be a discrete measure concentrated at a finite number of points in $\mathbb{R}^d$. Then*

$$\lim_{\lambda \to +\infty} \lambda \cdot m\left(\{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\}\right) = \frac{2}{\pi d}\|\nu\|$$

holds, where $\|\nu\|$ is the total variation of the measure ν.

Proof. Let ν be a finite singular complex discrete measure with support $\operatorname{supp}\nu = X = \{x^{(1)}, x^{(2)}, \dots, x^{(N)}\} \subset \mathbb{R}^d$, where $\nu(\{x^{(k)}\}) = \alpha_k \in \mathbb{C}\setminus\{0\}$ for $k = 1, \dots, N$. The total variation of ν is given by

$$\|\nu\| = \sum_{k=1}^{N} |\alpha_k|.$$

The j-th Riesz transform of ν at any point $x \in \mathbb{R}^d \setminus X$ is explicitly given by

$$R_j\nu(x) = \gamma(d) \sum_{k=1}^{N} \alpha_k \frac{x_j - x_j^{(k)}}{|x - x^{(k)}|^{d+1}}.$$

For $\lambda > 0$, we define the distribution set

$$E_\lambda = \left\{ x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda \right\}.$$

To analyze the measure $m(E_\lambda)$ as $\lambda \to +\infty$, we observe that the set E_λ concentrates in small disjoint neighborhoods around the atomic points $x^{(k)} \in \mathbb{R}^d$ $(k = 1, \dots, N)$. Choose $\varepsilon > 0$ sufficiently small such that the open balls $B(x^{(k)}, \varepsilon) \subset \mathbb{R}^d$ are pairwise disjoint for $k = 1, \dots, N$.

For $x \in B(x^{(k)}, \varepsilon)$ with $x \neq x^{(k)}$, the term corresponding to $x^{(k)}$ in the sum dominates. Specifically, we decompose $R_j\nu(x)$ near $x^{(k)}$ as

$$R_j\nu(x) = \gamma(d)\alpha_k \frac{x_j - x_j^{(k)}}{|x - x^{(k)}|^{d+1}} + H_k(x),$$

where

$$H_k(x) = \gamma(d) \sum_{l \neq k} \alpha_l \frac{x_j - x_j^{(l)}}{|x - x^{(l)}|^{d+1}}$$

is continuous and bounded on $B(x^{(k)}, \varepsilon)$. Hence, $|H_k(x)| \leq M_k$ for some constant $M_k > 0$.

Fix $k \in \{1, \dots, N\}$. By the triangle inequality, for $x \in B(x^{(k)}, \varepsilon)$, we have

$$\gamma(d)|\alpha_k| \frac{|x_j - x_j^{(k)}|}{|x - x^{(k)}|^{d+1}} - M_k \leq |R_j\nu(x)| \leq \gamma(d)|\alpha_k| \frac{|x_j - x_j^{(k)}|}{|x - x^{(k)}|^{d+1}} + M_k.$$

Consequently, for any $\delta > 0$, there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, the level set $E_\lambda \cap B(x^{(k)}, \varepsilon)$ is bounded between the sets

$$V_k^+(\lambda) = \left\{ x \in B(x^{(k)}, \varepsilon) : \gamma(d)|\alpha_k| \frac{|x_j - x_j^{(k)}|}{|x - x^{(k)}|^{d+1}} > \lambda - M_k \right\}$$

and

$$V_k^-(\lambda) = \left\{ x \in B(x^{(k)}, \varepsilon) : \gamma(d)|\alpha_k| \frac{|x_j - x_j^{(k)}|}{|x - x^{(k)}|^{d+1}} > \lambda + M_k \right\}.$$

Applying the change of variables $y = x - x^{(k)}$, we compute the Lebesgue measure of the domain

$$\Omega_k(\lambda) = \left\{ y \in \mathbb{R}^d : \gamma(d)|\alpha_k| \frac{|y_j|}{|y|^{d+1}} > \lambda \right\}.$$

Making the radial substitution $y = \left(\frac{\gamma(d)|\alpha_k|}{\lambda}\right)^{1/d} z$, we obtain

$$m(\Omega_k(\lambda)) = \left(\frac{\gamma(d)|\alpha_k|}{\lambda}\right) m\left(\left\{z \in \mathbb{R}^d : \frac{|z_j|}{|z|^{d+1}} > 1\right\}\right) = \frac{\gamma(d)|\alpha_k|}{\lambda} \cdot c_d,$$

where $c_d = \frac{2}{\pi d}$.

Since $m\left(E_\lambda \setminus \bigcup_{k=1}^N B(x^{(k)}, \varepsilon)\right) = o(1/\lambda)$ as $\lambda \to +\infty$, and the balls are pairwise disjoint in $\mathbb{R}^d$, summing over all $k = 1, \ldots, N$ yields

$$m(E_\lambda) = \sum_{k=1}^N m(E_\lambda \cap B(x^{(k)}, \varepsilon)) + o\left(\frac{1}{\lambda}\right) = \frac{c_d}{\lambda} \sum_{k=1}^N |\alpha_k| + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to +\infty.$$

Substituting $c_d = \frac{2}{\pi d}$ and $\sum_{k=1}^N |\alpha_k| = \|\nu\|$, we obtain

$$m(E_\lambda) = \frac{2}{\pi d\lambda}\|\nu\| + o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to +\infty.$$

Multiplying both sides by λ and taking the limit as $\lambda \to +\infty$, we conclude

$$\lim_{\lambda \to +\infty} \lambda \cdot m\left(\{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\}\right) = \frac{2}{\pi d}\|\nu\|.$$

This completes the proof. □

Theorem 2.2. *Let ν be a finite measure whose singular part is concentrated at a finite number of points in $\mathbb{R}^d$. Then*

$$\lim_{\lambda \to +\infty} \lambda \cdot m\left(\{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\}\right) = \frac{2}{\pi d}\|\nu_s\|$$

holds, where ν_s is a singular part of the measure ν.

Proof. By the Lebesgue decomposition theorem, the finite complex measure ν can be uniquely represented as the sum of an absolutely continuous part ν_a and a singular part ν_s:

$$\nu = \nu_a + \nu_s,$$

where $d\nu_a(x) = f(x)\,dx$ for some Radon-Nikodym derivative $f \in L^1(\mathbb{R}^d)$, and ν_s is a singular discrete measure concentrated at finitely many points $X = \{x^{(1)}, \ldots, x^{(N)}\} \subset \mathbb{R}^d$.

By the linearity of the Riesz transform, we write

$$R_j\nu(x) = R_jf(x) + R_j\nu_s(x).$$

Recall from (1.1) and (1.2) that for any function $f \in L^1(\mathbb{R}^d)$, the Riesz transform R_jf satisfies the weak-type $(1,1)$ estimate, which yields the asymptotic distribution property

$$m\left(\{x \in \mathbb{R}^d : |R_jf(x)| > \lambda\}\right) = o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to +\infty. \tag{2.1}$$

Furthermore, since ν_s is a discrete measure concentrated at finitely many points, Theorem 2.1 implies that

$$\lim_{\lambda \to +\infty} \lambda \cdot m\left(\{x \in \mathbb{R}^d : |R_j\nu_s(x)| > \lambda\}\right) = \frac{2}{\pi d}\|\nu_s\|. \tag{2.2}$$

To establish the required limit for $R_j\nu$, let $\eta \in (0,1)$ be arbitrary. For the upper estimate, observe the set inclusion

$$\{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\}$$

$$\subset \left\{x \in \mathbb{R}^d : |R_j f(x)| > \eta\lambda\right\} \cup \left\{x \in \mathbb{R}^d : |R_j\nu_s(x)| > (1-\eta)\lambda\right\}.$$

Applying the subadditivity of the Lebesgue measure m, we obtain

$$m\left(\{x : |R_j\nu(x)| > \lambda\}\right)$$

$$\leq m\left(\{x : |R_j f(x)| > \eta\lambda\}\right) + m\left(\{x : |R_j\nu_s(x)| > (1-\eta)\lambda\}\right).$$

Multiplying both sides by λ, taking the limit superior as $\lambda \to +\infty$, and using (2.1) together with (2.2), we derive

$$\limsup_{\lambda\to+\infty} \lambda \cdot m\left(\{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\}\right) \leq 0 + \frac{1}{1-\eta} \cdot \frac{2}{\pi d}\|\nu_s\|.$$

Since $\eta \in (0,1)$ is arbitrary, taking the limit as $\eta \to 0^+$ yields

$$\limsup_{\lambda\to+\infty} \lambda \cdot m\left(\{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\}\right) \leq \frac{2}{\pi d}\|\nu_s\|. \tag{2.3}$$

Conversely, for the lower estimate, we use the set inclusion

$$\{x \in \mathbb{R}^d : |R_j\nu_s(x)| > (1+\eta)\lambda\}$$

$$\subset \{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\} \cup \{x \in \mathbb{R}^d : |R_j f(x)| > \eta\lambda\}.$$

This implies

$$m\left(\{x : |R_j\nu(x)| > \lambda\}\right)$$

$$\geq m\left(\{x : |R_j\nu_s(x)| > (1+\eta)\lambda\}\right) - m\left(\{x : |R_j f(x)| > \eta\lambda\}\right).$$

Multiplying by λ and taking the limit inferior as $\lambda \to +\infty$, we get

$$\liminf_{\lambda\to+\infty} \lambda \cdot m\left(\{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\}\right) \geq \frac{1}{1+\eta} \cdot \frac{2}{\pi d}\|\nu_s\| - 0.$$

Taking $\eta \to 0^+$ yields

$$\liminf_{\lambda\to+\infty} \lambda \cdot m\left(\{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\}\right) \geq \frac{2}{\pi d}\|\nu_s\|. \tag{2.4}$$

Combining inequality (2.3) and inequality (2.4), we conclude that

$$\lim_{\lambda\to+\infty} \lambda \cdot m\left(\{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\}\right) = \frac{2}{\pi d}\|\nu_s\|,$$

which completes the proof. □

Theorem 2.3. *Let ν be a discrete measure concentrated at a finite number of points in $\mathbb{R}^d$. Then*

$$\lim_{\lambda\to 0^+} \lambda \cdot m\left(\{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\}\right) = \frac{2}{\pi d}|\nu(\mathbb{R}^d)|$$

holds, where $\nu(\mathbb{R}^d)$ is the total charge of the measure ν.

Proof. Let ν be a discrete measure with support

$$\operatorname{supp}\nu = X = \{x^{(1)}, x^{(2)}, \dots, x^{(N)}\} \subset \mathbb{R}^d,$$

where $\nu(\{x^{(k)}\}) = \alpha_k \in \mathbb{C} \setminus \{0\}$ for $k = 1, \dots, N$. Recall that the j-th Riesz transform of the measure ν is defined by

$$R_j\nu(x) = \gamma(d) \sum_{k=1}^{N} \alpha_k \frac{x_j - x_j^{(k)}}{|x - x^{(k)}|^{d+1}}, \quad x \in \mathbb{R}^d \setminus \{x^{(1)}, \dots, x^{(N)}\},$$

where $\gamma(d) = \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}}$.

As $\lambda \to 0^+$, the level set $E(\lambda) = \{x \in \mathbb{R}^d : |R_j\nu(x)| > \lambda\}$ expands towards infinity, which corresponds to the regime $|x| \to \infty$. For $|x| \gg \max_{1\le k\le N} |x^{(k)}|$, we apply the Taylor expansion to the kernel:

$$\frac{x_j - x_j^{(k)}}{|x - x^{(k)}|^{d+1}} = \frac{x_j}{|x|^{d+1}} + O\left(\frac{1}{|x|^{d+1}}\right), \quad \text{as } |x| \to \infty.$$

Substituting this into the expression for $R_j\nu(x)$, we obtain the principal asymptotic term as $|x| \to \infty$:

$$R_j\nu(x) = \gamma(d) \sum_{k=1}^{N} \alpha_k \left(\frac{x_j}{|x|^{d+1}} + O\left(\frac{1}{|x|^{d+1}}\right)\right) = \gamma(d)\nu(\mathbb{R}^d)\frac{x_j}{|x|^{d+1}} + O\left(\frac{1}{|x|^{d+1}}\right).$$

Consequently, as $\lambda \to 0^+$, the Lebesgue measure of the set $E(\lambda)$ is asymptotically equivalent to the measure of the set generated solely by the total mass $\nu(\mathbb{R}^d)$:

$$m\left(E(\lambda)\right) = m\left(\left\{x \in \mathbb{R}^d : \gamma(d)|\nu(\mathbb{R}^d)|\frac{|x_j|}{|x|^{d+1}} > \lambda\right\}\right) + o\left(\frac{1}{\lambda}\right).$$

Making the radial change of variables $x = \left(\frac{\gamma(d)|\nu(\mathbb{R}^d)|}{\lambda}\right)^{1/d} z$, we deduce:

$$m(E(\lambda)) = \frac{\gamma(d)|\nu(\mathbb{R}^d)|}{\lambda} m\left(\left\{z \in \mathbb{R}^d : \frac{|z_j|}{|z|^{d+1}} > 1\right\}\right) + o\left(\frac{1}{\lambda}\right).$$

Using the known integral evaluation for the unit domain:

$$c_d^{(j)} = m\left(\left\{z \in \mathbb{R}^d : \frac{|z_j|}{|z|^{d+1}} > 1\right\}\right) = \frac{2}{\pi d\gamma(d)},$$

we arrive at:

$$m(E(\lambda)) = \frac{2}{\pi d}\frac{|\nu(\mathbb{R}^d)|}{\lambda} + o\left(\frac{1}{\lambda}\right), \quad \text{as } \lambda \to 0^+.$$

Multiplying both sides by λ and taking the limit as $\lambda \to 0^+$ completes the proof. □

3. On the additivity of the Q-integral and Q'-integral

For a measurable complex-valued function f on an interval $[a,b] \subset \mathbb{R}$, we set

$$[f(x)]_\lambda = \begin{cases} f(x), & |f(x)| \le \lambda, \\ \lambda \cdot \operatorname{sgn} f(x), & |f(x)| > \lambda, \end{cases}$$

and

$$[f(x)]'_\lambda = \begin{cases} f(x), & |f(x)| \le \lambda, \\ 0, & |f(x)| > \lambda, \end{cases}$$

where $\lambda > 0$, $\operatorname{sgn} z = \frac{z}{|z|}$ for $z \neq 0$, and $\operatorname{sgn} 0 = 0$.

In 1929, Titchmarsh [16] introduced the notions of Q- and Q'-integrals for measurable functions on $[a,b] \subset \mathbb{R}$.

Definition 3.1. If the finite limit

$$\lim_{\lambda \to +\infty} \int_a^b [f(x)]_\lambda \, dx \quad \left(\text{respectively, } \lim_{\lambda \to +\infty} \int_a^b [f(x)]'_\lambda \, dx \right)$$

exists, then f is said to be Q-integrable (respectively, Q'-integrable) on $[a,b]$, denoted by $f \in Q[a,b]$ (respectively, $f \in Q'[a,b]$). The value of this limit is called the Q-integral (respectively, Q'-integral) of f and is denoted by

$$(Q) \int_a^b f(x) \, dx \quad \left(\text{respectively, } (Q') \int_a^b f(x) \, dx \right).$$

In the same paper, Titchmarsh established that Q-integration yields natural results when studying conjugate Fourier series of L^1 functions. However, a significant drawback of Q- and Q'-integrals is the lack of additivity—the sum of two Q-integrable (or Q'-integrable) functions is not necessarily Q-integrable (or Q'-integrable).

If we impose the asymptotic condition

$$m\left(\{x \in [a,b] : |f(x)| > \lambda\}\right) = o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to +\infty, \tag{3.1}$$

where m denotes the Lebesgue measure, then $Q[a,b] = Q'[a,b]$, the Q- and Q'-integrals coincide, and the resulting integral becomes additive.

Definition 3.2. If $f \in Q'[a,b]$ (or $f \in Q[a,b]$) and condition (3.1) holds, then f is said to be A-integrable on $[a,b]$ ($f \in A[a,b]$), and the corresponding limit is denoted by

$$(A) \int_a^b f(x) \, dx.$$

Similarly, we extend these definitions to measurable complex-valued functions on $\mathbb{R}^d$. For a measurable function f on $\mathbb{R}^d$ and $0 < \delta < \lambda$, we set

$$[f(x)]_{\delta,\lambda} = \begin{cases} f(x), & \delta \le |f(x)| \le \lambda, \\ \lambda \cdot \operatorname{sgn} f(x), & |f(x)| > \lambda, \\ 0, & |f(x)| < \delta, \end{cases}$$

and

$$[f(x)]'_{\delta,\lambda} = \begin{cases} f(x), & \delta \le |f(x)| \le \lambda, \\ 0, & |f(x)| > \lambda \text{ or } |f(x)| < \delta. \end{cases}$$

Definition 3.3. If the finite limit

$$\lim_{\substack{\delta\to 0^+ \\ \lambda\to+\infty}} \int_{\mathbb{R}^d} [f(x)]_{\delta,\lambda}\, dx \quad \left(\text{respectively, } \lim_{\substack{\delta\to 0^+ \\ \lambda\to+\infty}} \int_{\mathbb{R}^d} [f(x)]'_{\delta,\lambda}\, dx \right)$$

exists, then f is said to be Q-integrable (respectively, Q'-integrable) on $\mathbb{R}^d$, denoted by $f \in Q(\mathbb{R}^d)$ (respectively, $f \in Q'(\mathbb{R}^d)$).

Note that, as in the case of an interval, the Q- and Q'-integrals of measurable functions on $\mathbb{R}^d$ do not satisfy the additivity property. If one adds the conditions

$$m\left(\{x \in \mathbb{R}^d : |f(x)| > \delta\}\right) = o\left(\frac{1}{\delta}\right) \quad \text{as } \delta \to 0^+, \tag{3.2}$$

$$m\left(\{x \in \mathbb{R}^d : |f(x)| > \lambda\}\right) = o\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to +\infty, \tag{3.3}$$

to the definition of Q-integrability (or Q'-integrability), then the Q-integral and the Q'-integral coincide ($Q(\mathbb{R}^d) = Q'(\mathbb{R}^d)$) and these integrals become additive.

Definition 3.4. If $f \in Q'(\mathbb{R}^d)$ (or $f \in Q(\mathbb{R}^d)$) and conditions (3.2) and (3.3) hold, then f is said to be A-integrable on $\mathbb{R}^d$, denoted by $f \in A(\mathbb{R}^d)$. In this case, the limit

$$\lim_{\substack{\delta\to 0^+ \\ \lambda\to+\infty}} \int_{\mathbb{R}^d} [f(x)]_{\delta,\lambda}\, dx \quad \left(\text{or } \lim_{\substack{\delta\to 0^+ \\ \lambda\to+\infty}} \int_{\mathbb{R}^d} [f(x)]'_{\delta,\lambda}\, dx \right)$$

is denoted by $(A)\int_{\mathbb{R}^d} f(x)\, dx$.

The properties of the Q and Q'-integrals were investigated in [2, 6, 8–12, 16], and applications of the A-, Q-, and Q'-integrals in the theory of functions of real and complex variables are given in [1, 3–7, 10, 13, 14, 17–19].

We denote by $M(\mathbb{R}^d;\mathbb{C})$ the class of measurable complex-valued functions f on $\mathbb{R}^d$ for which the finite limits

$$\lim_{\delta\to 0^+} \delta \cdot m\left(\{x \in \mathbb{R}^d : |f(x)| > \delta\}\right) \quad \text{and} \quad \lim_{\lambda\to+\infty} \lambda \cdot m\left(\{x \in \mathbb{R}^d : |f(x)| > \lambda\}\right)$$

exist.

From Theorems (2.1) and (2.3) it follows that the Riesz transform of a Lebesgue integrable function belongs to $M(\mathbb{R}^d;\mathbb{C})$. In [7] it was proven that the Q- and Q'-integrals coincide on the function class $M(\mathbb{R}^d;\mathbb{C})$ and become additive with the A-integral.

Theorem 3.1. *The Q- and Q'-integrals coincide on the function class $M(\mathbb{R}^d;\mathbb{C})$, that is, if $f \in M(\mathbb{R}^d;\mathbb{C})$, then for the existence of the integral $(Q)\int_{\mathbb{R}^d} f(x)\, dx$ it is necessary and sufficient that the integral $(Q')\int_{\mathbb{R}^d} f(x)\, dx$ exists, and in that case*

$$(Q)\int_{\mathbb{R}^d} f(x)\, dx = (Q')\int_{\mathbb{R}^d} f(x)\, dx.$$

Theorem 3.2. *If the function $f \in M(\mathbb{R}^d;\mathbb{C})$ is Q'-integrable on $\mathbb{R}^d$ and the function g is A-integrable on $\mathbb{R}^d$, then their sum $f+g \in M(\mathbb{R}^d;\mathbb{C})$ is Q'-integrable on $\mathbb{R}^d$, and the following equality holds:*

$$(Q')\int_{\mathbb{R}^d}[f(x)+g(x)]\,dx = (Q')\int_{\mathbb{R}^d} f(x)\,dx + (A)\int_{\mathbb{R}^d} g(x)\,dx.$$

Corollary 3.1. *If the function $f \in M(\mathbb{R}^d;\mathbb{C})$ is Q-integrable on $\mathbb{R}^d$ and the function g is A-integrable on $\mathbb{R}^d$, then their sum $f+g \in M(\mathbb{R}^d;\mathbb{C})$ is Q-integrable on $\mathbb{R}^d$, and the following equality holds:*

$$(Q)\int_{\mathbb{R}^d}[f(x)+g(x)]\,dx = (Q)\int_{\mathbb{R}^d} f(x)\,dx + (A)\int_{\mathbb{R}^d} g(x)\,dx.$$

4. Duality formula for the Riesz transform of a finite measure whose singular part is concentrated at finitely many points

From the standard properties of singular integrals it follows that if $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, where $p>1,\ q>1$, and $1/p+1/q=1$, then for any $j=1,\dots,d$, the classical duality identity holds:

$$\int_{\mathbb{R}^d} g(x)(R_j f)(x)\,dx = -\int_{\mathbb{R}^d} f(x)(R_j g)(x)\,dx. \tag{4.1}$$

In [7], the authors showed that the Riesz transform $R_j f$ of a function $f \in L^1(\mathbb{R}^d)$ is A-integrable on $\mathbb{R}^d$ when paired with an appropriate test function, establishing the following analog:

Theorem 4.1. *Let $f \in L^1(\mathbb{R}^d)$ and let $g \in L^p(\mathbb{R}^d)$, $1<p<\infty$, be a bounded function on $\mathbb{R}^d$ such that its Riesz transforms $R_j g$ $(j=1,\dots,d)$ are also bounded on $\mathbb{R}^d$. Then for any $j=1,\dots,d$, the product $g\cdot R_j f$ is A-integrable on $\mathbb{R}^d$ and*

$$(A)\int_{\mathbb{R}^d} g(x)(R_j f)(x)\,dx = -\int_{\mathbb{R}^d} f(x)(R_j g)(x)\,dx. \tag{4.2}$$

In this section, we prove that the Riesz transform of a finite measure on $\mathbb{R}^d$, whose singular part is concentrated at a finite number of points, is Q-integrable and an analog of (4.1) and (4.2) holds.

Theorem 4.2. *Let ν be a finite complex measure on $\mathbb{R}^d$, whose singular part is concentrated at a finite number of points. Let g be a Hölder continuous function on $\mathbb{R}^d$ belonging to $L^p(\mathbb{R}^d)$ for some $p \ge 1$. Then for any $j=1,\dots,d$, the product function $g(x)(R_j\nu)(x)$ is Q'-integrable on $\mathbb{R}^d$, and the duality identity*

$$(Q')\int_{\mathbb{R}^d} g(x)(R_j\nu)(x)\,dx = -\int_{\mathbb{R}^d}(R_j g)(x)\,d\nu(x)$$

holds.

Proof. Let $\nu \in \mathcal{M}(\mathbb{R}^d)$ be a finite complex measure with Lebesgue decomposition:

$$d\nu(x) = f(x)\,dx + d\nu_s(x),$$

where $f \in L^1(\mathbb{R}^d)$ is the Radon–Nikodym derivative of the absolutely continuous part of ν, and ν_s is the singular part concentrated at a finite set of points $\operatorname{supp}\nu_s =$

$\{x^{(1)}, x^{(2)}, \dots, x^{(n)}\} \subset \mathbb{R}^d$ with $\nu_s(\{x^{(k)}\}) = \alpha_k \in \mathbb{C} \setminus \{0\}$ for $k = 1, \dots, n$, so that:

$$\nu_s = \sum_{k=1}^{n} \alpha_k \delta_{x^{(k)}}.$$

We first prove the duality for the discrete singular measure ν_s, that is:

$$(Q') \int_{\mathbb{R}^d} g(x)(R_j\nu_s)(x)\,dx = -\int_{\mathbb{R}^d} (R_j g)(x)\,d\nu_s(x) =$$

$$= -\sum_{k=1}^{n} \alpha_k (R_j g)(x^{(k)}). \tag{4.3}$$

Since g is a bounded Hölder continuous function on $\mathbb{R}^d$, there exist constants $M > 0$, $C_g > 0$, and $\gamma \in (0, 1]$ such that for all $x, y \in \mathbb{R}^d$:

$$|g(x)| \le M \quad \text{and} \quad |g(x) - g(y)| \le C_g |x - y|^{\gamma}.$$

The j-th Riesz transform of ν_s is given pointwise for $x \notin \operatorname{supp} \nu_s$ by:

$$G_j(x) := g(x)(R_j\nu_s)(x) = c_d g(x) \sum_{k=1}^{n} \alpha_k \frac{x_j - x_j^{(k)}}{|x - x^{(k)}|^{d+1}}.$$

Let

$$\delta_0 = \frac{1}{2} \min_{i \neq j} |x^{(i)} - x^{(j)}| > 0.$$

Near each singular point $x^{(k)}$, as $|x - x^{(k)}| \to 0$, the behavior of $|G_j(x)|$ is dominated by $c_d |\alpha_k| |g(x^{(k)})| |x - x^{(k)}|^{-d}$.

Define the upper level sets (balls around $x^{(k)}$) for $\lambda > \lambda_1$ (where $\lambda_1 > 0$ is sufficiently large):

$$K_{1,\lambda} = \bigcup_{k=1}^{n} B\left(x^{(k)}, r_{1,k}(\lambda)\right), \quad r_{1,k}(\lambda) = \left(\frac{c_d n M |\alpha_k|}{\lambda}\right)^{1/d},$$

$$K_{2,\lambda} = \bigcup_{k=1}^{n} B\left(x^{(k)}, r_{2,k}(\lambda)\right), \quad r_{2,k}(\lambda) = \left(\frac{c_d |\alpha_k| |g(x^{(k)})|}{\lambda + \lambda_1}\right)^{1/d}.$$

For $\lambda > \lambda_1$, we have the inclusions:

$$\{x \in \mathbb{R}^d : |G_j(x)| > \lambda\} \subset K_{1,\lambda},$$

and on the set $K_{1,\lambda} \setminus K_{2,\lambda}$, applying the Hölder continuity of g at each $x^{(k)}$:

$$\int_{K_{1,\lambda} \setminus K_{2,\lambda}} |G_j(x)|\,dx \le C \sum_{k=1}^{n} |\alpha_k| \int_{r_{2,k}(\lambda)}^{r_{1,k}(\lambda)} \frac{|g(x) - g(x^{(k)})|}{|x - x^{(k)}|^d}\,dx + o(1) \longrightarrow 0 \quad \text{as } \lambda \to +\infty.$$

Taking the limit as $\lambda \to +\infty$ and $\delta \to 0^+$ for the truncated Q'-integral:

$$(Q') \int_{\mathbb{R}^d} g(x)(R_j\nu_s)(x)\,dx = \lim_{\substack{\delta \to 0^+ \\ \lambda \to +\infty}} \int_{\delta \le |G_j(x)| \le \lambda} G_j(x)\,dx$$

$$= \sum_{k=1}^{n} \alpha_k \lim_{\varepsilon \to 0^+} c_d \int_{|x - x^{(k)}| > \varepsilon} g(x) \frac{x_j - x_j^{(k)}}{|x - x^{(k)}|^{d+1}}\,dx$$

$$= -\sum_{k=1}^{n} \alpha_k (R_j g)(x^{(k)}) = -\int_{\mathbb{R}^d} (R_j g)(x)\, d\nu_s(x),$$

which proves equation (4.3).

Now, combining the singular part ν_s with the absolutely continuous part $f \in L^1(\mathbb{R}^d)$, by Theorem (4.1), $g \cdot R_j f$ is A-integrable on $\mathbb{R}^d$ and:

$$(A)\int_{\mathbb{R}^d} g(x)(R_j f)(x)\, dx = -\int_{\mathbb{R}^d} f(x)(R_j g)(x)\, dx.$$

In a manner analogous to Theorems (2.1) and (2.3), it can be shown that the product $g \cdot R_j \nu_s$ also belongs to the space $M(\mathbb{R}^d; \mathbb{C})$. Then by Theorem (3.2), the sum $gR_j\nu = gR_j\nu_s + gR_j f$ is Q'-integrable on $\mathbb{R}^d$, and:

$$(Q')\int_{\mathbb{R}^d} g(x)(R_j\nu)(x)\, dx = (Q')\int_{\mathbb{R}^d} g(x)(R_j\nu_s)(x)\, dx + (A)\int_{\mathbb{R}^d} g(x)(R_j f)(x)\, dx$$

$$= -\int_{\mathbb{R}^d} (R_j g)(x)\, d\nu_s(x) - \int_{\mathbb{R}^d} f(x)(R_j g)(x)\, dx = -\int_{\mathbb{R}^d} (R_j g)(x)\, d\nu(x).$$

This completes the proof of Theorem (4.2). □

Corollary 4.1. *Let ν be a finite complex measure on $\mathbb{R}^d$, whose singular part is concentrated at a finite number of points. Let g be a Hölder continuous function on $\mathbb{R}^d$ belonging to $L^p(\mathbb{R}^d)$ for some $p \geq 1$. Then for any $j = 1, \ldots, d$, the product function $g(x)(R_j\nu)(x)$ is Q'-integrable on $\mathbb{R}^d$, and the duality identity*

$$(Q')\int_{\mathbb{R}^d} g(x)(R_j\nu)(x)\, dx = -\int_{\mathbb{R}^d} (R_j g)(x)\, d\nu(x)$$

holds.

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