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HAUSDORFF OPERATORS WITHIN THE FRAMEWORK OF LINEAR CANONICAL DUNKL TRANSFORM

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Abstract. In the present paper, we introduce the linear canonical Dunkl operator $\mathcal{D}_\alpha^M f(x) = f'(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right) - i\frac{a}{b}xf(x)$, where f is a continuously differentiable function on $\mathbb{R}$. We prove the boundedness of the canonical Dunkl–Hausdorff operators on the space $L_\alpha^p(\mathbb{R})$, $1 \leq p < \infty$. We investigate the canonical Dunkl convolution operator, and obtain some useful results. The relation between the canonical Dunkl convolution operator and the canonical Dunkl–Hausdorff operator is also established. The properties of the adjoint canonical Dunkl–Hausdorff operators are further discussed. The harmonic analysis associated with the operator $\mathcal{D}_\alpha^M$ plays an important role in establishing the results of this paper.

1. Introduction

The study of Hausdorff operators, which originated from some classical summation methods, has a long history in real and complex analysis. In the one-dimensional setting, Hausdorff operators on the real line were introduced in [10] and studied on the Hardy space in [16]. The natural generalization in several dimensions was introduced and studied in [3, 4, 13]. Particularly, Hausdorff operator is an interesting operator in harmonic analysis [15]. It contains some important operators, such as Hardy operator, adjoint Hardy operator [5, 18], and the Cesàro operator [12] in one dimension. The Hardy–Littlewood–Pólya operator and the Riemann–Liouville fractional integral operator can also be derived from the Hausdorff operator [1, 2]. The modern study of general Hausdorff operators on $L^1(\mathbb{R})$ and the real Hardy space $H^1(\mathbb{R})$ over the real line was pioneered by Liflyand and Móricz in [16]. Many research papers have addressed the boundedness of the Hausdorff operator on Hardy spaces. For instance, Liflyand and his collaborators in [13, 14] proved, using more effective ways, that the Hausdorff operator has the same behavior on the Hardy space $H^1(\mathbb{R})$ as that in the Lebesgue space $L^1(\mathbb{R})$. Recently, Daher and Saadi in [6] investigated the Dunkl–Hausdorff operator on the Lebesgue space $L_\alpha^1(\mathbb{R})$ and on the Hardy space $H_\alpha^1(\mathbb{R})$. Subsequently, Mondal and Poria [17] studied Hausdorff operators associated with the

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Opdam–Cherednik operator. Furthermore, Tyr [24] studied the boundedness of q-Hausdorff operators on q-Hardy spaces. Another fundamental tool in harmonic analysis is the canonical Dunkl–Hausdorff operator, which is the main object of study in this paper.

Here, we denote by $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ an arbitrary matrix in $SL(2,\mathbb{R})$ such that $b \neq 0$. We define the canonical Dunkl operator $\mathcal{D}_\alpha^M$ on $\mathbb{R}$ by

$$\mathcal{D}_\alpha^M f(x) := f'(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right) - i\frac{a}{b}xf(x), \quad \alpha > -1/2,$$

where f is a continuously differentiable function.

Note that if $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the operator $\mathcal{D}_\alpha^M$ is reduced to the Dunkl operator $\mathcal{D}_\alpha$:

$$\mathcal{D}_\alpha f(x) := f'(x) + \frac{2\alpha+1}{x}\left(\frac{f(x)-f(-x)}{2}\right).$$

The classical Dunkl operator $\mathcal{D}_\alpha$ plays an important role in analysis [7]. In particular, the reference [22] investigates standard constructions of harmonic analysis, such as translation operators, convolution product, and Fourier transform, in connection with the operator $\mathcal{D}_\alpha$.

Using the Dunkl harmonic analysis [22], for all $y \in \mathbb{R}$, the system

$$\mathcal{D}_\alpha^M f(x) := -i\frac{y}{b}f(x), \quad f(0) = e^{i\frac{d}{2b}y^2},$$

admits a unique solution, denoted by E_α^M and given by

$$E_\alpha^M(x,y) = e^{\frac{i}{2b}(ax^2+dy^2)}e_\alpha\left(-\frac{ixy}{b}\right),$$

where e_α is the Dunkl kernel [8, 19].

We denote by μ_α the measure defined on $\mathbb{R}$ by

$$\mathrm{d}\mu_\alpha(x) := (2^{\alpha+1}\Gamma(\alpha+1))^{-1}|x|^{2\alpha+1}\mathrm{d}x,$$

and by $L_\alpha^p(\mathbb{R})$, $p \in [1,\infty)$, the space of measurable functions f on $\mathbb{R}$, such that

$$\|f\|_{L_\alpha^p(\mathbb{R})} := \left[\int_{\mathbb{R}} |f(x)|^p \mathrm{d}\mu_\alpha(x)\right]^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_{L_\alpha^\infty(\mathbb{R})} := \operatorname*{ess\,sup}_{x\in\mathbb{R}} |f(x)| < \infty.$$

The canonical Dunkl transform $\mathscr{F}_\alpha^M$ has been introduced by Ghazouani et al. in [11] for all $f \in L_\alpha^1(\mathbb{R})$ by

$$\mathscr{F}_\alpha^M(f)(y) := \frac{1}{(ib)^{\alpha+1}}\int_{\mathbb{R}} E_\alpha^M(x,y)f(x)\mathrm{d}\mu_\alpha(x), \qquad b \neq 0. \tag{1.1}$$

The canonical Dunkl transform $\mathscr{F}_\alpha^M$ can be regarded as a generalization of the Dunkl transform $\mathscr{F}_\alpha$, see [9, 20, 21]:

$$\mathscr{F}_\alpha(f)(y) := \int_{\mathbb{R}} e_\alpha(-ixy)f(x)\mathrm{d}\mu_\alpha(x). \tag{1.2}$$

Let $\varphi \in L^1(\mathbb{R})$. We define the canonical Dunkl–Hausdorff operator $H^M_{\alpha,\varphi}$ associated with the canonical Dunkl operator $\mathcal{D}^M_\alpha$ for $f \in L^1_\alpha(\mathbb{R})$ by

$$H^M_{\alpha,\varphi} f(x) := \int_{\mathbb{R}} \frac{e^{i\frac{a}{2b}x^2(\frac{1}{t^2}-1)}}{|t|^{2\alpha+2}} f\left(\frac{x}{t}\right)\varphi(t)\mathrm{d}t, \quad x \in \mathbb{R}.$$

The main purpose of this paper is to extend some results of the classical Dunkl–Hausdorff operator given in [6] to the framework of canonical Dunkl theory, and to investigate the canonical Dunkl convolution operator. We prove the boundedness of canonical Dunkl–Hausdorff operator in space $L^p_\alpha(\mathbb{R})$, $1 \le p < \infty$. The relation between the canonical Dunkl convolution operator and the canonical Dunkl–Hausdorff operator is also established. Next, we introduce the adjoint operator $H^{M,*}_{\alpha,\varphi}$ on $L^2_\alpha(\mathbb{R})$ by

$$H^{M,*}_{\alpha,\varphi} f(x) := \int_{\mathbb{R}} e^{i\frac{a}{2b}x^2(t^2-1)} f(tx)\overline{\varphi(t)}\mathrm{d}t, \quad x \in \mathbb{R}.$$

We present the properties of the adjoint operator $H^{M,*}_{\alpha,\varphi}$, including its boundedness on $L^p_\alpha(\mathbb{R})$, $1 \le p < \infty$. We also establish a relation between the canonical Dunkl convolution operator and the adjoint operator $H^{M,*}_{\alpha,\varphi}$.

This paper is organized as follows. In Section 2, we investigate the properties of the canonical Dunkl transform $\mathscr{F}^M_\alpha$, the canonical Dunkl translation $\tau^M_{\alpha,x}$ and the canonical Dunkl convolution $*^M_\alpha$ associated with the operator $\mathcal{D}^M_\alpha$. In Section 3, we introduce the canonical Dunkl–Hausdorff operators $H^M_{\alpha,\varphi}$ and we establish their properties. In Section 4, we introduce the canonical Dunkl convolution operator and derive its relation with the operators $H^M_{\alpha,\varphi}$ and $H^{M,*}_{\alpha,\varphi}$. In the last section, we present some special cases of the canonical Hausdorff operators and study their boundedness on $L^p_\alpha(\mathbb{R})$.

2. Canonical Dunkl harmonic analysis

In this section we recall some results about the harmonic analysis associated with the canonical Dunkl operator $\mathcal{D}^M_\alpha$.

Theorem 2.1. *(See* [7, 9]*).*

(i) *Plancherel theorem for $\mathscr{F}_\alpha$. The Dunkl transform $\mathscr{F}_\alpha$ extends uniquely to an isometric isomorphism of $L^2_\alpha(\mathbb{R})$ onto $L^2_\alpha(\mathbb{R})$. In particular,*

$$\|\mathscr{F}_\alpha(f)\|_{L^2_\alpha(\mathbb{R})} = \|f\|_{L^2_\alpha(\mathbb{R})}.$$

(ii) *Inversion theorem for $\mathscr{F}_\alpha$. Let $f \in L^1_\alpha(\mathbb{R})$, such that $\mathscr{F}_\alpha(f) \in L^1_\alpha(\mathbb{R})$, then*

$$f(x) = \int_{\mathbb{R}} e_\alpha(ixy)\mathscr{F}_\alpha(f)(y)\mathrm{d}\mu_\alpha(y), \quad a.e. \quad x \in \mathbb{R}.$$

For $t \in \mathbb{R}$, the Dunkl kernel E_α satisfies the product formula [19]

$$E_\alpha(ixt)E_\alpha(iyt) = \int_{\mathbb{R}} E_\alpha(izt)\omega_\alpha(x,y,z)\mathrm{d}\mu_\alpha(z) \quad \text{for} \quad x,y \in \mathbb{R}, \tag{2.1}$$

where $\omega_\alpha(x,y,\cdot)$ is a signed measurable function on $\mathbb{R}$, with support

$$[-|x|-|y|,-||x|-|y||] \cup [||x|-|y||,|x|+|y|].$$

The kernel $\omega_\alpha(x,y,\cdot)$ satisfies also the following properties:

$$\begin{aligned}\int_{\mathbb{R}} |\omega_\alpha(x,y,z)|\mathrm{d}\mu_\alpha(z) &\leq 4,\\ \omega_\alpha(x,y,z) &= \omega_\alpha(y,x,z), \quad \text{for} \quad z\in\mathbb{R},\\ \omega_\alpha(x,y,z) &= \omega_\alpha(-x,z,y), \quad \text{for} \quad z\in\mathbb{R}.\end{aligned} \tag{2.2}$$

We now define the generalized translation operator induced by (2.1). For $f\in L^1_\alpha(\mathbb{R})$, the linear operator

$$\tau_{\alpha,x}(f)(y) := \int_{\mathbb{R}} f(z)\omega_\alpha(x,y,z)\mathrm{d}\mu_\alpha(z), \quad x,y\in\mathbb{R},$$

will be called Dunkl translation operator.
As a first remark, we note that the relation (2.2) means that

$$\tau_{\alpha,x}(f)(y) = \tau_{\alpha,y}(f)(x), \quad x,y\in\mathbb{R}.$$

Theorem 2.2. *(See* [22]*).*

(i) *For all $y\in\mathbb{R}$ and $f\in L^p_\alpha(\mathbb{R})$, $1\leq p\leq\infty$, we have*

$$\|\tau_{\alpha,x}(f)\|_{L^p_\alpha(\mathbb{R})} \leq 4\|f\|_{L^p_\alpha(\mathbb{R})}.$$

(ii) *For $f\in L^1_\alpha(\mathbb{R})\cup L^2_\alpha(\mathbb{R})$ and $y\in\mathbb{R}$, we have*

$$\mathscr{F}_\alpha\circ\tau_{\alpha,x}(f)(y) = e_\alpha(ixy)\mathscr{F}_\alpha(f)(y).$$

Let $f,g\in L^2_\alpha(\mathbb{R})$. The Dunkl convolution $f*_\alpha g$ of f and g is defined by

$$f*_\alpha g(x) := \int_{\mathbb{R}} \tau_{\alpha,x}f(-y)g(y)\mathrm{d}\mu_\alpha(y), \quad x\in\mathbb{R}.$$

The convolution $*_\alpha$ is commutative, associative and satisfies the Young inequality [22, 23]. Let $p,q,r\in[1,\infty]$ such that $1/p+1/q = 1+1/r$. Then for $f\in L^p_\alpha(\mathbb{R})$ and $g\in L^q_\alpha(\mathbb{R})$ we have

$$\|f*_\alpha g\|_{L^r_\alpha(\mathbb{R})} \leq 4\|f\|_{L^p_\alpha(\mathbb{R})}\|g\|_{L^q_\alpha(\mathbb{R})}.$$

The canonical Dunkl transform defined by (1.1) can be written as

$$\mathscr{F}^M_\alpha(f)(y) = \frac{e^{i\frac{d}{2b}y^2}}{(ib)^{\alpha+1}}\mathscr{F}_\alpha\left(e^{i\frac{a}{2b}(\cdot)^2}f(\cdot)\right)\left(\frac{y}{b}\right), \quad f\in L^1_\alpha(\mathbb{R}),$$

where $\mathscr{F}_\alpha$ is the Dunkl transform given by (1.2).
The canonical Dunkl kernel satisfies for all $x,y\in\mathbb{R}$:

$$|E^M_\alpha(x,y)|\leq 1.$$

The following properties of the canonical Dunkl transform are established in [11].

(i) For all $f\in L^1_\alpha(\mathbb{R})$, we have $\mathscr{F}^M_\alpha(f)\in C_0(\mathbb{R})$ and

$$\|\mathscr{F}^M_\alpha(f)\|_{L^\infty_\alpha(\mathbb{R})} \leq \frac{1}{|b|^{\alpha+1}}\|f\|_{L^1_\alpha(\mathbb{R})}.$$

(ii) The canonical Dunkl transform has a unique extension to an isometric isomorphism on $L^2_\alpha(\mathbb{R})$, which is still denoted by $\mathscr{F}^M_\alpha$. In particular,

$$\|\mathscr{F}^M_\alpha(f)\|_{L^2_\alpha(\mathbb{R})} = \|f\|_{L^2_\alpha(\mathbb{R})}.$$

(iii) For all $f \in L^1_\alpha(\mathbb{R})$ with $\mathscr{F}^M_\alpha(f) \in L^1_\alpha(\mathbb{R})$, we have

$$\mathscr{F}^M_\alpha \circ \mathscr{F}^{M^{-1}}_\alpha(f)(x) = \mathscr{F}^{M^{-1}}_\alpha \circ \mathscr{F}^M_\alpha(f)(x) = f(x), \quad \text{a.e.} \quad x \in \mathbb{R}.$$

For all $x, y \in \mathbb{R}$ and f a continuous function, the canonical Dunkl translation operator $\tau^M_{\alpha,x}$ is defined in [25] by

$$\tau^M_{\alpha,x}(f)(y) := e^{-i\frac{a}{2b}(x^2+y^2)}\tau_{\alpha,x}\left(e^{i\frac{a}{2b}(\cdot)^2}f(\cdot)\right)(y) = \int_{\mathbb{R}} e^{i\frac{a}{b}z^2}f(z)\omega^M_\alpha(x,y,z)\mathrm{d}\mu_\alpha(z),$$

where

$$\omega^M_\alpha(x,y,z) = e^{-i\frac{a}{2b}(x^2+y^2+z^2)}\omega_\alpha(x,y,z).$$

Theorem 2.3. *(See* [25]*). The following properties are true.*

(i) *For $x, y, t \in \mathbb{R}$, we have*

$$E^{M^{-1}}_\alpha(t,x)E^{M^{-1}}_\alpha(t,y) = e^{-i\frac{d}{2b}t^2}\tau^M_{\alpha,x}\left(E^{M^{-1}}_\alpha(t,\cdot)\right)(y).$$

(ii) *For $x, y \in \mathbb{R}$, we have*

$$\tau^M_{\alpha,x}(f)(y) = \tau^M_{\alpha,y}(f)(x).$$

(iii) *If $f \in L^1_\alpha(\mathbb{R}) \cup L^2_\alpha(\mathbb{R})$, then*

$$\mathscr{F}^M_\alpha \circ \tau^M_{\alpha,x}(f)(y) = e^{-i\frac{a}{2b}x^2}e_\alpha\left(\frac{ixy}{b}\right)\mathscr{F}^M_\alpha(f)(y), \quad y \in \mathbb{R}.$$

(iv) *Let $f \in L^p_\alpha(\mathbb{R})$, $1 \le p \le \infty$. Then $\tau^M_{\alpha,x}(f) \in L^p_\alpha(\mathbb{R})$, and*

$$\|\tau^M_{\alpha,x}(f)\|_{L^p_\alpha(\mathbb{R})} \le 4\|f\|_{L^p_\alpha(\mathbb{R})}, \quad x \in \mathbb{R}.$$

Let $f, g \in L^1_\alpha(\mathbb{R})$. The canonical Dunkl convolution $f *^M_\alpha g$ of f and g is defined by

$$f *^M_\alpha g(x) := \frac{1}{(ib)^{\alpha+1}}\int_{\mathbb{R}} \tau^M_{\alpha,x}f(-y)e^{\frac{ia}{b}y^2}g(y)\mathrm{d}\mu_\alpha(y), \quad x \in \mathbb{R}. \tag{2.3}$$

Theorem 2.4. *(See* [25]*). The canonical Dunkl convolution $f *^M_\alpha g$ of f and g satisfies the following properties.*

(i) *Let $p, q, r \in [1, \infty]$ such that $1/p + 1/q = 1 + 1/r$. Then for $f \in L^p_\alpha(\mathbb{R})$ and $g \in L^q_\alpha(\mathbb{R})$ we have*

$$\|f *^M_\alpha g\|_{L^r_\alpha(\mathbb{R})} \le \frac{4}{|b|^{\alpha+1}}\|f\|_{L^p_\alpha(\mathbb{R})}\|g\|_{L^q_\alpha(\mathbb{R})}.$$

(ii) *For all $f \in L^1_\alpha(\mathbb{R})$ and $g \in L^1_\alpha(\mathbb{R}) \cup L^2_\alpha(\mathbb{R})$, we have the factorization identity*

$$\mathscr{F}^M_\alpha(f *^M_\alpha g)(y) = e^{-i\frac{d}{2b}y^2}\mathscr{F}^M_\alpha(f)(y)\mathscr{F}^M_\alpha(g)(y), \quad y \in \mathbb{R}.$$

(iii) *For $f, g \in L^2_\alpha(\mathbb{R})$ we have*

$$f *^M_\alpha g(x) := \mathscr{F}^{M^{-1}}_\alpha(e^{-i\frac{d}{2b}(\cdot)^2}\mathscr{F}^M_\alpha(f)\mathscr{F}^M_\alpha(g))(x), \quad x \in \mathbb{R}.$$

3. Canonical Dunkl–Hausdorff operator

In this section we define and study the canonical Hausdorff type operators associated with the canonical Dunkl transform $\mathscr{F}_\alpha^M$.

Let $f \in L_\alpha^p(\mathbb{R})$, $1 \le p < \infty$ and $t \neq 0$. We define the canonical dilation function f_t^M by

$$f_t^M(x) := \frac{e^{i\frac{a}{2b}x^2(\frac{1}{t^2}-1)}}{|t|^{2\alpha+2}} f\left(\frac{x}{t}\right), \quad x \in \mathbb{R}. \tag{3.1}$$

The canonical dilation function $f_t^M \in L_\alpha^p(\mathbb{R})$ and satisfies

$$\|f_t^M\|_{L_\alpha^p(\mathbb{R})} = \frac{1}{|t|^{(2\alpha+2)(1-1/p)}} \|f\|_{L_\alpha^p(\mathbb{R})}. \tag{3.2}$$

Let $\varphi \in L^1(\mathbb{R})$. We define the canonical Dunkl–Hausdorff operator $H_{\alpha,\varphi}^M$ associated for $f \in L_\alpha^1(\mathbb{R})$ by

$$H_{\alpha,\varphi}^M f(x) := \int_{\mathbb{R}} f_t^M(x)\varphi(t)\mathrm{d}t, \quad x \in \mathbb{R}, \tag{3.3}$$

where f_t^M is the canonical dilation defined by (3.1).

If $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $H_{\alpha,\varphi}^M = H_{\alpha,\varphi}$, where $H_{\alpha,\varphi}$ is the classical Dunkl–Hausdorff operator, see [6, 16]:

$$H_{\alpha,\varphi} f(x) := \int_{\mathbb{R}} f_t(x)\varphi(t)\mathrm{d}t, \quad x \in \mathbb{R},$$

where

$$f_t(x) := \frac{1}{t^{2\alpha+2}} f\left(\frac{x}{t}\right).$$

Let

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in SL(2,\mathbb{R}). \tag{3.4}$$

Theorem 3.1. *Let φ be a measurable function on $\mathbb{R}$ such that*

$$C_{\alpha,\varphi,p} := \int_{\mathbb{R}} \frac{|\varphi(t)|}{|t|^{(2\alpha+2)(1-1/p)}}\mathrm{d}t < \infty. \tag{3.5}$$

Then the canonical Dunkl–Hausdorff operator $H_{\alpha,\varphi}^M$ is bounded on $L_\alpha^p(\mathbb{R})$, $1 \le p < \infty$ with

$$\|H_{\alpha,\varphi}^M f\|_{L_\alpha^p(\mathbb{R})} \le C_{\alpha,\varphi,p}\|f\|_{L_\alpha^p(\mathbb{R})}. \tag{3.6}$$

Proof. Let φ be a measurable function on $\mathbb{R}$ satisfying the condition (3.5), and let $f \in L_\alpha^p(\mathbb{R})$, $1 \le p < \infty$. By using Minkowski's inequality for integrals, we

have

$$\begin{aligned}\|H^M_{\alpha,\varphi}f\|_{L^p_\alpha(\mathbb{R})} &= \left[\int_{\mathbb{R}}\left|\int_{\mathbb{R}} f_t^M(x)\varphi(t)\mathrm{d}t\right|^p \mathrm{d}\mu_\alpha(x)\right]^{1/p}\\ &\leq \left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f_t^M(x)||\varphi(t)|\mathrm{d}t\right)^p \mathrm{d}\mu_\alpha(x)\right]^{1/p}\\ &\leq \int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f_t^M(x)|^p|\varphi(t)|^p\mathrm{d}\mu_\alpha(x)\right)^{1/p}\mathrm{d}t\\ &= \int_{\mathbb{R}}\|f_t^M\|_{L^p_\alpha(\mathbb{R})}|\varphi(t)|\mathrm{d}t.\end{aligned}$$

Then by (3.2) we obtain

$$\|H^M_{\alpha,\varphi}f\|_{L^p_\alpha(\mathbb{R})} \leq C_{\alpha,\varphi,p}\|f\|_{L^p_\alpha(\mathbb{R})}.$$

Going back to the definition of

$$\left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}}|f_t^M(x)||\varphi(t)|\mathrm{d}t\right)^p \mathrm{d}\mu_\alpha(x)\right]^{1/p},$$

we deduce that the integral

$$H^M_{\alpha,\varphi}f(x) = \int_{\mathbb{R}} f_t^M(x)\varphi(t)\mathrm{d}t,$$

is absolutely convergent for almost all $x \in \mathbb{R}$. So $H^M_{\alpha,\varphi}f$ is well-defined and belongs to $L^p_\alpha(\mathbb{R})$. □

Theorem 3.2. *Let $\varphi \in L^1(\mathbb{R})$. Then for $f \in L^1_\alpha(\mathbb{R})$, we have*

$$\mathscr{F}^{M_1}_\alpha(H^{M_2}_{\alpha,\varphi}f)(y) = \int_{\mathbb{R}} \mathscr{F}^{M_1}_\alpha(f_t^{M_2})(y)\varphi(t)\mathrm{d}t, \quad x \in \mathbb{R},$$

where M_1 and M_2 are given by (3.4). In particular, if $M_1 = M$ and $M_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we get

$$\mathscr{F}^M_\alpha(H_{\alpha,\varphi}f)(y) = \int_{\mathbb{R}} \mathscr{F}^M_\alpha(f_t)(y)\varphi(t)\mathrm{d}t, \quad y \in \mathbb{R}.$$

Proof. Let $\varphi \in L^1(\mathbb{R})$, and let $f \in L^1_\alpha(\mathbb{R})$. Then by (3.3) we have

$$\begin{aligned}\mathscr{F}^{M_1}_\alpha(H^{M_2}_{\alpha,\varphi}f)(y) &= \frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}} H^{M_2}_{\alpha,\varphi}f(x)E^{M_1}_\alpha(x,y)\mathrm{d}\mu_\alpha(x)\\ &= \frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}}\left[\int_{\mathbb{R}} f_t^{M_2}(x)\varphi(t)\mathrm{d}t\right]E^{M_1}_\alpha(x,y)\mathrm{d}\mu_\alpha(x).\end{aligned}$$

Since

$$\int_{\mathbb{R}^2}|f_t^{M_2}(x)||\varphi(t)||E^{M_1}_\alpha(x,y)|\mathrm{d}t\mathrm{d}\mu_\alpha(x) \leq \|\varphi\|_{L^1(\mathbb{R})}\|f\|_{L^1_\alpha(\mathbb{R})} < \infty,$$

by Fubini's theorem we obtain

$$\begin{aligned}\mathscr{F}_\alpha^{M_1}(H_{\alpha,\varphi}^{M_2}f)(y) &= \frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}}\left[\int_{\mathbb{R}} f_t^{M_2}(x)E_\alpha^{M_1}(x,y)\mathrm{d}\mu_\alpha(x)\right]\varphi(t)\mathrm{d}t\\ &= \int_{\mathbb{R}}\mathscr{F}_\alpha^{M_1}(f_t^{M_2})(y)\varphi(t)\mathrm{d}t.\end{aligned}$$

The theorem is proved. □

Remark 3.1. Taking $M_1 = M_2 = M$, we get

$$\mathscr{F}_\alpha^{M}(H_{\alpha,\varphi}^{M}f)(y) = \int_{\mathbb{R}}\mathscr{F}_\alpha^{M}(f_t^{M})(y)\varphi(t)\mathrm{d}t, \quad y\in\mathbb{R}.$$

Using the fact that

$$\mathscr{F}_\alpha^{M}(f_t^{M})(y) = e^{i\frac{d}{2b}y^2(1-t^2)}\mathscr{F}_\alpha^{M}(f)(ty), \quad y\in\mathbb{R},$$

we obtain

$$\mathscr{F}_\alpha^{M}(H_{\alpha,\varphi}^{M}f)(y) = \int_{\mathbb{R}} e^{i\frac{d}{2b}y^2(1-t^2)}\mathscr{F}_\alpha^{M}(f)(ty)\varphi(t)\mathrm{d}t, \quad y\in\mathbb{R}.$$

Let $f, g\in L^2_\alpha(\mathbb{R})$, and let φ be a measurable function on $\mathbb{R}$ satisfying $C_{\alpha,\varphi,2} < \infty$. We define the canonical adjoint operator $H_{\alpha,\varphi}^{M,*}$ by the relation

$$\int_{\mathbb{R}} H_{\alpha,\varphi}^{M,*}f(x)\overline{g(x)}\mathrm{d}\mu_\alpha(x) = \int_{\mathbb{R}} f(x)\overline{H_{\alpha,\varphi}^{M}g(x)}\mathrm{d}\mu_\alpha(x).$$

Theorem 3.3. *Let $f\in L^2_\alpha(\mathbb{R})$, and let φ be a measurable function on $\mathbb{R}$ satisfying $C_{\alpha,\varphi,2}<\infty$. Then*

$$H_{\alpha,\varphi}^{M,*}f(x) = \int_{\mathbb{R}} f_t^{M,*}(x)\overline{\varphi(t)}\mathrm{d}t, \quad x\in\mathbb{R}, \tag{3.7}$$

where

$$f_t^{M,*}(x) := e^{i\frac{a}{2b}x^2(t^2-1)}f(tx). \tag{3.8}$$

Proof. Let $f, g\in L^2_\alpha(\mathbb{R})$, and let φ be a measurable function on $\mathbb{R}$ satisfying $C_{\alpha,\varphi,2}<\infty$. Using the change of variable $u = x/t$ and Fubini's theorem we have

$$\begin{aligned}\int_{\mathbb{R}} f(x)\overline{H_{\alpha,\varphi}^{M}g(x)}\mathrm{d}\mu_\alpha(x) &= \int_{\mathbb{R}} f(x)\left[\int_{\mathbb{R}}\overline{g_t^M(x)}\overline{\varphi(t)}\mathrm{d}t\right]\mathrm{d}\mu_\alpha(x)\\ &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}} f(x)\overline{g_t^M(x)}\mathrm{d}\mu_\alpha(x)\right]\overline{\varphi(t)}\mathrm{d}t\\ &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}} f_t^{M,*}(x)\overline{g(x)}\mathrm{d}\mu_\alpha(x)\right]\overline{\varphi(t)}\mathrm{d}t.\end{aligned}$$

Using (3.2), this calculation is justified by the fact that

$$\int_{\mathbb{R}^2}|f(x)||g_t^M(x)|\mathrm{d}\mu_\alpha(x)|\varphi(t)|\mathrm{d}t \le C_{\alpha,\varphi,2}\|f\|_{L^2_\alpha(\mathbb{R})}\|g\|_{L^2_\alpha(\mathbb{R})} < \infty.$$

Then according to Fubini's theorem we obtain

$$\begin{aligned}\int_{\mathbb{R}} f(x)\overline{H_{\alpha,\varphi}^{M}g(x)}\mathrm{d}\mu_\alpha(x) &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}} f_t^{M,*}(x)\overline{\varphi(t)}\mathrm{d}t\right]\overline{g(x)}\mathrm{d}\mu_\alpha(x)\\ &= \int_{\mathbb{R}} H_{\alpha,\varphi}^{M,*}f(x)\overline{g(x)}\mathrm{d}\mu_\alpha(x),\end{aligned}$$

where

$$H^{M,*}_{\alpha,\varphi}f(x)=\int_{\mathbb{R}} f^{M,*}_t(x)\overline{\varphi(t)}\mathrm{d}t.$$

This calculation is justified by the fact that

$$\int_{\mathbb{R}^2}|f^{M,*}_t(x)||g(x)|\mathrm{d}\mu_\alpha(x)|\varphi(t)|\mathrm{d}t\le C_{\alpha,\varphi,2}\|f\|_{L^2_\alpha(\mathbb{R})}\|g\|_{L^2_\alpha(\mathbb{R})}<\infty.$$

This completes the proof of the theorem. □

If $M=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ then $H^{M,*}_{\alpha,\varphi}=H^*_{\alpha,\varphi}$:

$$H^*_{\alpha,\varphi}f(x):=\int_{\mathbb{R}} f^*_t(x)\overline{\varphi(t)}\mathrm{d}t,\quad x\in\mathbb{R},$$

where

$$f^*_t(x):=f(tx).$$

Remark 3.2. Let $f\in L^p_\alpha(\mathbb{R})$ and $t\neq 0$. Then the function $f^{M,*}_t$ belongs to $L^p_\alpha(\mathbb{R})$ and satisfies

$$\|f^{M,*}_t\|_{L^p_\alpha(\mathbb{R})}=\frac{1}{|t|^{(2\alpha+2)/p}}\|f\|_{L^p_\alpha(\mathbb{R})}. \tag{3.9}$$

Therefore, the operator $H^{M,*}_{\alpha,\varphi}$ is bounded on $L^p_\alpha(\mathbb{R})$, $1\le p<\infty$ with

$$\|H^{M,*}_{\alpha,\varphi}f\|_{L^p_\alpha(\mathbb{R})}\le C^*_{\alpha,\varphi,p}\|f\|_{L^p_\alpha(\mathbb{R})}, \tag{3.10}$$

where

$$C^*_{\alpha,\varphi,p}:=\int_{\mathbb{R}}\frac{|\varphi(t)|}{|t|^{(2\alpha+2)/p}}\mathrm{d}t<\infty.$$

In the same way as in Theorem 3.2, we obtain the following result.

Theorem 3.4. *Let φ be a measurable function on $\mathbb{R}$ satisfying*

$$C^*_{\alpha,\varphi,1}:=\int_{\mathbb{R}}\frac{|\varphi(t)|}{|t|^{2\alpha+2}}\mathrm{d}t<\infty. \tag{3.11}$$

Then for $f\in L^1_\alpha(\mathbb{R})$, we have

$$\mathscr{F}^{M_1}_\alpha(H^{M_2,*}_{\alpha,\varphi}f)(y)=\int_{\mathbb{R}}\mathscr{F}^{M_1}_\alpha(f^{M_2,*}_t)(y)\overline{\varphi(t)}\mathrm{d}t,\quad y\in\mathbb{R},$$

where M_1, M_2 are given by (3.4). In particular, if $M_1=M$ and $M_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ we get

$$\mathscr{F}^{M}_\alpha(H^*_{\alpha,\varphi}f)(y)=\int_{\mathbb{R}}\mathscr{F}^{M}_\alpha(f^*_t)(y)\overline{\varphi(t)}\mathrm{d}t,\quad y\in\mathbb{R}.$$

Proof. Let φ be a measurable function on $\mathbb{R}$ satisfying the condition (3.11) and let $f\in L^1_\alpha(\mathbb{R})$. Then by (3.7) we have

$$\begin{aligned}\mathscr{F}^{M_1}_\alpha(H^{M_2,*}_{\alpha,\varphi}f)(y)&=\frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}}H^{M_2,*}_{\alpha,\varphi}f(x)E^{M_1}_\alpha(x,y)\mathrm{d}\mu_\alpha(x)\\&=\frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}}\left[\int_{\mathbb{R}}f^{M_2,*}_t(x)\overline{\varphi(t)}\mathrm{d}t\right]E^{M_1}_\alpha(x,y)\mathrm{d}\mu_\alpha(x).\end{aligned}$$

Since

$$\int_{\mathbb{R}^2} |f_t^{M_2,*}(x)||\varphi(t)||E_\alpha^{M_1}(x,y)|\mathrm{d}t\mathrm{d}\mu_\alpha(x) \le C^*_{\alpha,\varphi,1}\|f\|_{L^1_\alpha(\mathbb{R})} < \infty,$$

by Fubini's theorem we obtain

$$\begin{aligned}\mathscr{F}_\alpha^{M_1}(H^{M_2,*}_{\alpha,\varphi}f)(y) &= \frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}}\left[\int_{\mathbb{R}} f_t^{M_2,*}(x)E_\alpha^{M_1}(x,y)\mathrm{d}\mu_\alpha(x)\right]\overline{\varphi(t)}\mathrm{d}t\\ &= \int_{\mathbb{R}} \mathscr{F}_\alpha^{M_1}(f_t^{M_2,*})(y)\overline{\varphi(t)}\mathrm{d}t.\end{aligned}$$

The theorem is proved. □

Remark 3.3. Taking $M_1 = M_2 = M$, we get

$$\mathscr{F}_\alpha^{M}(H^{M,*}_{\alpha,\varphi}f)(y) = \int_{\mathbb{R}} \mathscr{F}_\alpha^{M}(f_t^{M,*})(y)\overline{\varphi(t)}\mathrm{d}t, \quad y\in\mathbb{R}.$$

Using the fact that

$$\mathscr{F}_\alpha^{M}(f_t^{M,*})(y) = \frac{e^{i\frac{d}{2b}y^2(1-\frac{1}{t^2})}}{t^{2\alpha+2}}\mathscr{F}_\alpha^{M}(f)\left(\frac{y}{t}\right), \quad y\in\mathbb{R},$$

we obtain

$$\mathscr{F}_\alpha^{M}(H^{M,*}_{\alpha,\varphi}f)(y) = \int_{\mathbb{R}} \frac{e^{i\frac{d}{2b}y^2(1-\frac{1}{t^2})}}{t^{2\alpha+2}}\mathscr{F}_\alpha^{M}(f)\left(\frac{y}{t}\right)\overline{\varphi(t)}\mathrm{d}t, \quad y\in\mathbb{R}.$$

4. Canonical Dunkl convolution operator

Let $f\in L^p_\alpha(\mathbb{R})$, $1\le p<\infty$ and let $g\in L^q_\alpha(\mathbb{R})$, with $1/p+1/q=1$. We define the canonical Dunkl convolution operator by

$$\mathscr{C}^M_{\alpha,g}(f) := f *^M_\alpha g,$$

where $*^M_\alpha$ is the canonical convolution product given by (2.3).
From Theorem 2.4 (i), we have

$$\|\mathscr{C}^M_{\alpha,g}(f)\|_{L^\infty_\alpha(\mathbb{R})} \le \frac{4}{|b|^{\alpha+1}}\|f\|_{L^p_\alpha(\mathbb{R})}\|g\|_{L^q_\alpha(\mathbb{R})}.$$

We obtain a relation between the canonical Dunkl convolution operator and the canonical Dunkl–Hausdorff operator.

Theorem 4.1. *Let $f\in L^p_\alpha(\mathbb{R})$, $1\le p<\infty$ and let $g\in L^q_\alpha(\mathbb{R})$, with $1/p+1/q=1$. Then for φ a measurable function on $\mathbb{R}$ satisfying $C_{\alpha,\varphi,p}<\infty$, we have*

$$\mathscr{C}^{M_1}_{\alpha,g}(H^{M_2}_{\alpha,\varphi}f)(s) = \int_{\mathbb{R}} \mathscr{C}^{M_1}_{\alpha,g}(f_t^{M_2})(s)\varphi(t)\mathrm{d}t, \quad s\in\mathbb{R},$$

where $f_t^{M_2}$ is the dilation of f given by (3.1) associated with the matrix M_2.

Proof. Let $f\in L^p_\alpha(\mathbb{R})$, $1\le p<\infty$ and let $g\in L^q_\alpha(\mathbb{R})$, with $1/p+1/q=1$. Since φ is a measurable function on $\mathbb{R}$ satisfying $C_{\alpha,\varphi,p}<\infty$, then $H^{M_2}_{\alpha,\varphi}f\in L^p_\alpha(\mathbb{R})$, $1\le p<\infty$ with

$$\|H^{M_2}_{\alpha,\varphi}f\|_{L^p_\alpha(\mathbb{R})} \le C_{\alpha,\varphi,p}\|f\|_{L^p_\alpha(\mathbb{R})}.$$

From relations (2.3) and (3.3) we get

$$\begin{aligned}\mathscr{C}_{\alpha,g}^{M_1}(H_{\alpha,\varphi}^{M_2}f)(s) &= \frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}} e^{i\frac{a_1}{b_1}x^2} H_{\alpha,\varphi}^{M_2}f(x)\tau_{\alpha,s}^{M_1}g(-x)\mathrm{d}\mu_\alpha(x)\\ &= \frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}} e^{i\frac{a_1}{b_1}x^2}\left[\int_{\mathbb{R}} f_t^{M_2}(x)\varphi(t)\mathrm{d}t\right]\tau_{\alpha,s}^{M_1}g(-x)\mathrm{d}\mu_\alpha(x).\end{aligned}$$

Using Hölder's inequality and (3.2), we have

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}} |f_t^{M_2}(x)||\tau_{\alpha,s}^{M_1}g(-x)||\varphi(t)|\mathrm{d}t\mathrm{d}\mu_\alpha(x)\\ &\qquad\le \int_{\mathbb{R}} \|f_t^{M_2}\|_{L_\alpha^p(\mathbb{R})}\|\tau_{\alpha,s}^{M_1}g\|_{L_\alpha^q(\mathbb{R})}|\varphi(t)|\mathrm{d}t\\ &\qquad\le 4C_{\alpha,\varphi,p}\|f\|_{L_\alpha^p(\mathbb{R})}\|g\|_{L_\alpha^q(\mathbb{R})} < \infty.\end{aligned}$$

Then according to Fubini's theorem we obtain

$$\begin{aligned}\mathscr{C}_{\alpha,g}^{M_1}(H_{\alpha,\varphi}^{M_2}f)(s) &= \frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}}\left[\int_{\mathbb{R}} e^{i\frac{a_1}{b_1}x^2} f_t^{M_2}(x)\tau_{\alpha,s}^{M_1}g(-x)\mathrm{d}\mu_\alpha(x)\right]\varphi(t)\mathrm{d}t\\ &= \int_{\mathbb{R}} \mathscr{C}_{\alpha,g}^{M_1}(f_t^{M_2})(s)\varphi(t)\mathrm{d}t.\end{aligned}$$

This ends the proof of the theorem. □

Remark 4.1. If $M_1 = M_2 = M$ then we get

$$\mathscr{C}_{\alpha,g}^{M}(H_{\alpha,\varphi}^{M}f)(s) = \int_{\mathbb{R}} \mathscr{C}_{\alpha,g}^{M}(f_t^{M})(s)\varphi(t)\mathrm{d}t, \quad s\in\mathbb{R}.$$

If $M_1 = M$ and $M_2 = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$ then we get

$$\mathscr{C}_{\alpha,g}^{M}(H_{\alpha,\varphi}f)(s) = \int_{\mathbb{R}} \mathscr{C}_{\alpha,g}^{M}(f_t)(s)\varphi(t)\mathrm{d}t, \quad s\in\mathbb{R}.$$

As in the same of Theorem 4.1, we obtain the following result.

Theorem 4.2. *Let $f\in L_\alpha^p(\mathbb{R})$, $1\le p<\infty$ and let $g\in L_\alpha^q(\mathbb{R})$, with $1/p+1/q=1$. Then for φ a measurable function on $\mathbb{R}$ satisfying $C_{\alpha,\varphi,p}^*<\infty$, we have*

$$\mathscr{C}_{\alpha,g}^{M_1}(H_{\alpha,\varphi}^{M_2,*}f)(s) = \int_{\mathbb{R}} \mathscr{C}_{\alpha,g}^{M_1}(f_t^{M_2,*})(s)\overline{\varphi(t)}\mathrm{d}t, \quad s\in\mathbb{R},$$

where $f_t^{M_2,*}$ is the function defined in (3.8) associated with the matrix M_2.

Proof. Let $f\in L_\alpha^p(\mathbb{R})$, $1\le p<\infty$ and let $g\in L_\alpha^q(\mathbb{R})$, with $1/p+1/q=1$. Since φ is a measurable function on $\mathbb{R}$ satisfying $C_{\alpha,\varphi,p}^*<\infty$, then $H_{\alpha,\varphi}^{M_2,*}f\in L_\alpha^p(\mathbb{R})$, $1\le p<\infty$ with

$$\|H_{\alpha,\varphi}^{M_2,*}f\|_{L_\alpha^p(\mathbb{R})} \le C_{\alpha,\varphi,p}^*\|f\|_{L_\alpha^p(\mathbb{R})}.$$

From relations (2.3) and (3.7) we get

$$\begin{aligned}\mathscr{C}_{\alpha,g}^{M_1}(H_{\alpha,\varphi}^{M_2,*}f)(s) &= \frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}} e^{i\frac{a_1}{b_1}x^2} H_{\alpha,\varphi}^{M_2,*}f(x)\tau_{\alpha,s}^{M_1}g(-x)\mathrm{d}\mu_\alpha(x)\\ &= \frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}} e^{i\frac{a_1}{b_1}x^2}\left[\int_{\mathbb{R}} f_t^{M_2,*}(x)\overline{\varphi(t)}\mathrm{d}t\right]\tau_{\alpha,s}^{M_1}g(-x)\mathrm{d}\mu_\alpha(x).\end{aligned}$$

Using Hölder's inequality and (3.9), we have

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}}|f_t^{M_2,*}(x)|\tau_{\alpha,s}^{M_1}g(x)||\varphi(t)|\mathrm{d}t\mathrm{d}\mu_\alpha(x)&\\ \leq \int_{\mathbb{R}}\|f_t^{M_2,*}\|_{L^p_\alpha(\mathbb{R})}\|\tau_{\alpha,s}^{M_1}g\|_{L^q_\alpha(\mathbb{R})}|\varphi(t)|\mathrm{d}t&\\ \leq 4C^*_{\alpha,\varphi,p}\|f\|_{L^p_\alpha(\mathbb{R})}\|g\|_{L^q_\alpha(\mathbb{R})}<\infty.&\end{aligned}$$

Then according to Fubini's theorem we obtain

$$\begin{aligned}\mathscr{C}^{M_1}_{\alpha,g}(H^{M_2,*}_{\alpha,\varphi}f)(s)&=\frac{1}{(ib_1)^{\alpha+1}}\int_{\mathbb{R}}\left[\int_{\mathbb{R}}e^{i\frac{a_1}{b_1}x^2}f_t^{M_2,*}(x)\tau_{\alpha,s}^{M_1}g(-x)\mathrm{d}\mu_\alpha(x)\right]\overline{\varphi(t)}\mathrm{d}t\\ &=\int_{\mathbb{R}}\mathscr{C}^{M_1}_{\alpha,g}(f_t^{M_2,*})(s)\overline{\varphi(t)}\mathrm{d}t.\end{aligned}$$

Therefore, the theorem is proved. □

Remark 4.2. Taking $M_1=M_2=M$, we get

$$\mathscr{C}^{M}_{\alpha,g}(H^{M,*}_{\alpha,\varphi}f)(s)=\int_{\mathbb{R}}\mathscr{C}^{M}_{\alpha,g}(f_t^{M,*})(s)\overline{\varphi(t)}\mathrm{d}t,\quad s\in\mathbb{R}.$$

If $M_1=M$ and $M_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$ then we get

$$\mathscr{C}^{M}_{\alpha,g}(H^{*}_{\alpha,\varphi}f)(s)=\int_{\mathbb{R}}\mathscr{C}^{M}_{\alpha,g}(f_t^{*})(s)\overline{\varphi(t)}\mathrm{d}t,\quad s\in\mathbb{R}.$$

5. Some special examples

In this section, using relations (3.6) and (3.10), we establish the boundedness of several operators that arise as special cases of the canonical Dunkl–Hausdorff operator associated with the linear canonical Dunkl transform. The canonical Dunkl–Hausdorff operator unifies a broad class of integral operators corresponding to different choices of φ.

Example 5.1. We choose $\varphi(t):=|t|^{-a}\mathbf{1}_{|t|>1}(t)$, $a>1$. We obtain the canonical Dunkl–Hardy operator denoted by $\mathcal{H}^M_\alpha$:

$$\mathcal{H}^M_\alpha f(x):=\int_{|t|>1}f_t^M(x)|t|^{-a}\mathrm{d}t,\quad x\in\mathbb{R}.$$

It is well known that Hardy operators are important operators in harmonic analysis, for instance, see [5, 18]. The canonical Dunkl–Hardy operator $\mathcal{H}^M_\alpha$ is bounded on $L^p_\alpha(\mathbb{R})$, for $1\leq p<\infty$, with

$$C_{\alpha,\varphi,p}=\frac{2p}{(2\alpha+2)(p-1)+(a-1)p}.$$

The adjoint of the canonical Dunkl–Hardy operator given by

$$\mathcal{H}^{M,*}_\alpha f(x):=\int_{|t|>1}f_t^{M,*}(x)|t|^{-a}\mathrm{d}t,\quad x\in\mathbb{R},$$

is bounded on $L^p_\alpha(\mathbb{R})$ for $1\leq p<\infty$, with

$$C^*_{\alpha,\varphi,p}=\frac{2p}{2\alpha+2+p(a-1)}.$$

Example 5.2. We choose $\varphi(t) := |t|^{2\alpha+2}\mathbf{1}_{|t|<1}(t)$. We obtain the canonical Dunkl–Cesàro operator denoted by $\mathcal{C}_\alpha^M$:

$$\mathcal{C}_\alpha^M f(x) := \int_{-1}^{1} f_t^M(x)|t|^{2\alpha+2}\mathrm{d}t, \quad x \in \mathbb{R}.$$

A brief history of the study of Cesàro operator can be found in [12]. The canonical Dunkl–Cesàro operator $\mathcal{C}_{\alpha,s}^M$ is bounded on $L_\alpha^p(\mathbb{R})$, for $1 \le p < \infty$, with

$$C_{\alpha,\varphi,p} = \frac{2p}{2\alpha+2+p}.$$

The adjoint of the canonical Dunkl–Cesàro operator $\mathcal{C}_\alpha^{M,*}$ given by

$$\mathcal{C}_\alpha^{*M} f(x) := \int_{-1}^{1} f_t^{M,*}(x)|t|^{2\alpha+2}\mathrm{d}t, \quad x \in \mathbb{R},$$

is bounded on $L_\alpha^p(\mathbb{R})$, for $1 \le p < \infty$, with

$$C_{\alpha,\varphi,p}^* = \frac{2p}{(2\alpha+2)(p-1)+p}.$$

Example 5.3. We choose $\varphi(t) := |t|^{2\alpha+2}\mathbf{1}_{|t|<1}(t) + |t|^{-(2\alpha+2)}\mathbf{1}_{|t|>1}(t)$. We obtain the canonical Dunkl–Hardy–Littlewood–Pólya operator denoted by $\mathcal{P}_\alpha^M$:

$$\mathcal{P}_\alpha^M f(x) := \int_{-1}^{1} f_t^M(x)|t|^{2\alpha+2}\mathrm{d}t + \int_{|t|>1} f_t^M(x)|t|^{-(2\alpha+2)}\mathrm{d}t, \quad x \in \mathbb{R}.$$

The study of Hardy–Littlewood–Pólya operators can be found in [2]. The canonical Dunkl–Hardy–Littlewood–Pólya operator $\mathcal{P}_\alpha^M$ is bounded on $L_\alpha^p(\mathbb{R})$, for $1 \le p < \infty$, with

$$C_{\alpha,\varphi,p} = \frac{2p}{2\alpha+2+p} + \frac{2p}{(2\alpha+2)(p-1)+(2\alpha+1)p}.$$

The adjoint of the canonical Dunkl–Hardy–Littlewood–Pólya operator given by

$$\mathcal{P}_\alpha^{M,*} f(x) := \int_{-1}^{1} f_t^{M,*}(x)\mathrm{d}t + \int_{|t|>1} f_t^{M,*}(x)\frac{\mathrm{d}t}{|t|}, \quad x \in \mathbb{R},$$

is bounded on $L_\alpha^p(\mathbb{R})$, for $1 \le p < \infty$, with

$$C_{\alpha,\varphi,p}^* = \frac{2p}{(2\alpha+2)(p-1)+p} + \frac{2p}{2\alpha+2+(2\alpha+1)p}.$$

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