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DIRECT SPECTRAL PROBLEM FOR THE ONE-DIMENSIONAL STARK OPERATOR ON THE HALF-LINE WITH NEUMANN BOUNDARY CONDITION

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Abstract. The Stark operator $T = -\frac{d^2}{dx^2} + x + q(x)$ on the half-axis $0 \le x < \infty$ with a Neumann boundary condition at zero is considered. The direct spectral problem is studied. Asymptotic formulas for the spectral data are obtained.

1. Introduction

Consider the boundary value problem generated on the half-axis $0 \le x < \infty$ by the differential equation

$$-y'' + xy + q(x)y = \lambda y, \qquad \lambda \in \mathbb{C} \tag{1.1}$$

and the boundary condition

$$y'(0) = 0, \tag{1.2}$$

where $q(x)$ is a real-valued function and satisfies the conditions

$$q \in C[0,\infty), \qquad \int_0^\infty |x^4 q(x)|\, dx < \infty. \tag{1.3}$$

Equation (1.1) describes the effect of the potential on the electric field and is called the Stark equation (see [4], [8]). Interest in this equation grew significantly after it was used to integrate the cylindrical Korteweg–de Vries equation (see [2], [6]).

In [14], it is proven that under conditions (1.3), the boundary value problem (1.1)–(1.2) has a purely discrete spectrum consisting of simple eigenvalues λ_n $(n = 1, 2, \dots)$, where $\lambda_n \to \infty$ as $n \to \infty$. These eigenvalues $\lambda_1, \lambda_2, \dots$ are the zeros of the function $f'_x(0,\lambda)$, where $f(x,\lambda)$ is the solution of equation (1.1) satisfying the asymptotics

$$f(x,\lambda) = (1 + o(1))\,\mathrm{Ai}(x - \lambda), \qquad x \to \infty,$$

and $\mathrm{Ai}(z)$ is the Airy function of the first kind.

The collection of quantities

$$\{\lambda_n, \alpha_n\}_{n=1}^\infty, \quad \alpha_n > 0,$$

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where

$$\alpha_n = \left(\int_0^\infty |f(x,\lambda_n)|^2 \, dx \right)^{1/2},$$

is called the *spectral data* of problem (1.1)–(1.2). In the present work, the main properties of the spectral data of the boundary value problem (1.1)–(1.2) are studied. The main integral equation of the Marchenko type has been obtained, which indicates the relationship between the spectral data and the perturbation coefficient function $q(x)$. It is worth noting that spectral problems for equation (1.1) in various formulations have been investigated in [2], [4]–[16].

2. Properties of the spectral data

Let us first consider the unperturbed equation

$$-y'' + xy = \lambda y, \qquad 0 \le x < \infty, \quad \lambda \in \mathbb{C}. \tag{2.1}$$

It is known (see [1]) that equation (2.1) has a solution $f_0(x,\lambda)$ of the form

$$f_0(x,\lambda) = \mathrm{Ai}(x-\lambda), \tag{2.2}$$

where $\mathrm{Ai}(z)$ is the Airy function of the first kind. The function $\mathrm{Ai}(z)$ is an entire function (see [1], [17]) of order $3/2$ and type $2/3$. The following asymptotic formulas hold (see [1]) as $|z| \to \infty$:

$$\mathrm{Ai}(z) \sim \frac{1}{\sqrt{\pi} z^{1/4}} e^{-\zeta} \left[1 + O(\zeta^{-1})\right], \quad |\arg z| < \pi,$$
$$\mathrm{Ai}'(z) \sim -\frac{z^{1/4}}{\sqrt{\pi}} e^{-\zeta} \left[1 + O(\zeta^{-1})\right], \quad |\arg z| < \pi,$$

$$\mathrm{Ai}(-z) \sim \frac{1}{\sqrt{\pi} z^{1/4}} \sin\left(\zeta + \frac{\pi}{4}\right) \left[1 + O(\zeta^{-1})\right], \quad |\arg z| < \frac{2\pi}{3},$$
$$\mathrm{Ai}'(-z) \sim -\frac{z^{1/4}}{\sqrt{\pi}} \cos\left(\zeta + \frac{\pi}{4}\right) \left[1 + O(\zeta^{-1})\right], \quad |\arg z| < \frac{2\pi}{3},$$

where $\zeta = \frac{2}{3} z^{3/2}$. It is evident that the spectrum of problem (2.1), (1.2) is discrete and consists of the zeros of the function $f_0'(0,\lambda)$=$\mathrm{Ai}'(-\lambda)$. The function $\mathrm{Ai}'(-\lambda)$ has zeros $\hat{\lambda}_n$ $(n = 1, 2, \dots)$ on the positive half-axis only (see [1]), and the following asymptotic equality holds:

$$\hat{\lambda}_n = \mathrm{g}\left(\frac{3\pi(4n-3)}{8}\right), \tag{2.3}$$

where

$$\mathrm{g}(z) \sim z^{2/3}\left(1 - \frac{7}{48} z^{-2} + \frac{35}{288} z^{-4} - \frac{181223}{207360} z^{-6} + \cdots\right), \qquad z \to \infty. \tag{2.4}$$

Moreover, since the operator $\hat{T}_0$ is self-adjoint, the system of eigenfunctions $\left\{\frac{f_0(x,\hat{\lambda}_n)}{\hat{\alpha}_n}\right\}_{n=1}^{\infty}$ forms an orthonormal basis in the space $L_2(0,\infty)$, that is, the

following equality holds:

$$\sum_{n=1}^{\infty} \frac{f_0(x,\hat{\lambda}_n)}{\hat{\alpha}_n} \cdot \frac{f_0(y,\hat{\lambda}_n)}{\hat{\alpha}_n} = \delta(x-y),$$

where δ is the Dirac delta and the normalization constants $\hat{\alpha}_n$ $(n = 1, 2, \dots)$ are given by the formulas

$$\hat{\alpha}_n = \left(\int_0^{\infty} |f_0(x,\hat{\lambda}_n)|^2 \, dx \right)^{1/2}.$$

Let us now study the asymptotic behavior of the normalization constants $\hat{\alpha}_n$ as $n \to \infty$.

Theorem 2.1. *The following asymptotic relation holds:*

$$(\hat{\alpha}_n)^2 = \pi^{-1} \left(\frac{3\pi(4n-3)}{8} \right)^{1/3} + O(n^{\frac{-2}{3}}), \quad n \to \infty.$$

Proof. We adopt the convention that dots denote differentiation with respect to λ, and primes denote differentiation with respect to x: $u' = \frac{\partial u}{\partial x}, \dot{u} = \frac{\partial u}{\partial \lambda}$.

Consider the equation

$$-f_0''(x,\lambda) + xf(x,\lambda) = \lambda f(x,\lambda).$$

Differentiating this equation with respect to λ, we obtain:

$$-\dot{f}_0''(x,\lambda) + x\dot{f}(x,\lambda) = f(x,\lambda) + \lambda \dot{f}(x,\lambda).$$

Multiply the first equation by $\dot{f}(x,\lambda)$, the second by $f(x,\lambda)$ and subtract the first from the second. As a result, we get:

$$f_0^2(x,\lambda) = \{\dot{f}(x,\lambda), f(x,\lambda)\},$$

where $\{u,v\} = uv' - u'v$. Since both $f_0(x,\lambda)$ and $\dot{f}_0(x,\lambda)$ decrease exponentially as $x \to \infty$, it follows from the last identity that

$$\hat{\alpha}_n^2 = \int_0^{\infty} f_0^2(x,\hat{\lambda}_n)\, dx = \left\{ \dot{f}_0(x,\hat{\lambda}_n), f_0(x,\hat{\lambda}_n) \right\}_0^{\infty} =$$

$$= \dot{f}_0'(0,\hat{\lambda}_n) f_0(0,\hat{\lambda}_n) = \hat{\lambda}_n [\mathrm{Ai}(-\hat{\lambda}_n)]^2. \tag{2.5}$$

Now using asymptotic formulas (2.3)-(2.5), as well as the known relation (see, [1])

$$\mathrm{Ai}(-\hat{\lambda}_n) = (-1)^{n-1}\, \mathrm{g}_1 \left(\frac{3\pi(4n-3)}{8} \right), \tag{2.6}$$

where

$$\mathrm{g}_1(z) \sim \pi^{-1/2} z^{-1/6} \left(1 - \frac{7}{96} z^{-2} + \frac{1673}{6144} z^{-4} - \frac{84394709}{26542080} z^{-6} + \cdots \right), \quad z \to \infty, \tag{2.7}$$

we derive from (2.5), (2.6) that

$$(\hat{\alpha}_n)^2 = \hat{\lambda}_n \left[\mathrm{g}_1 \left(\frac{3\pi(4n-3)}{8} \right) \right]^2.$$

The theorem follows directly from (2.3), (2.4), (2.6), and (2.7).

The theorem is proven. □

Now, let us consider the perturbed equation (1.1). It is known (see,[8], [14], [15]) that under condition (1.3), equation (1.1) has a solution $f(x,\lambda)$, expressed in the form

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^\infty K(x,t)\, f_0(t,\lambda)\, dt, \tag{2.8}$$

where the kernel $K(x,t)$ is a continuously differentiable function satisfying the following relations:

$$|K(x,t)| \le C\sigma_0\left(\frac{x+t}{2}\right) \tag{2.9}$$

$$\left|\frac{\partial K(x_1,x_2)}{\partial x_j} + \frac{1}{2}q\left(\frac{x_1+x_2}{2}\right)\right| \le C[1+(x_2-x_1)^2]\sigma_0\left(\frac{x_1+x_2}{2}\right), \tag{2.10}$$

$$K(x,x) = \frac{1}{2}\int_x^\infty q(t)\, dt,$$

where $\sigma_0 = \int_x^\infty |q(t)|\, dt$, while C from here on denotes, in general, various constants. From (2.8) and (2.9), it follows that for each fixed λ, the function $f(x,\lambda)$ belongs to the space $L_2(0,\infty)$. Consequently, the spectrum of the problem (1.1)-(1.2) coincides with the set of zeros of the function $f'(0,\lambda)$, i.e., the relations $f'(0,\lambda_n) = 0$ $(n = 1,2,\dots)$ hold. As shown in [15], [18],in compliance with conditions (1.3), the eigenvalues λ_n admit the following asymptotic formula:

$$\lambda_n = \left(\frac{3\pi(4n-3)}{8}\right)^{2/3} + O(n^{-2/3}), \quad n \to \infty. \tag{2.11}$$

We introduce the normalization constants α_n by setting

$$\hat{\alpha}_n = \left(\int_0^\infty |f(x,\lambda_n)|^2\, dx\right)^{1/2}.$$

Then, the eigenfunctions $\left\{\frac{f(x,\lambda_n)}{\alpha_n}\right\}_{n=1}^\infty$ of problem (1.1)-(1.2) form an orthonormal basis in $L_2(0,\infty)$, that is, the following formula holds:

$$\sum_{n=1}^\infty \frac{f(x,\lambda_n)}{\alpha_n}\cdot\frac{f_(y,\lambda_n)}{\alpha_n} = \delta(x-y).$$

We will need the asymptote of the normalization constants α_n as $n \to \infty$.

Theorem 2.2. *The following relation holds:*

$$\alpha_n^{-2} = \hat{\alpha}_n^{-2}\left[1 + O(n^{-2/3})\right], \quad n \to \infty. \tag{2.12}$$

Proof. Analogously to (2.5), we have

$$\alpha_n^2 = \int_0^\infty f^2(x,\lambda_n)\, dx = \left\langle \dot{f}(x,\lambda_n), f(x,\lambda_n)\right\rangle_0^\infty = \dot{f}'(0,\lambda_n) f(0,\lambda_n). \tag{2.13}$$

Therefore, the eigenvalues of the boundary value problem (1.1)-(1.2) are simple. Furthermore, from (2.3)-(2.6), (2.8)-(2.10), and (2.11), it follows that

$$f(0,\lambda_n) = f_0(0,\hat{\lambda}_n)\left[1 + O(n^{-2/3})\right],$$

$$\dot{f}'(0,\lambda_n) = \dot{f}_0'(0,\hat{\lambda}_n)\left[1 + O(n^{-2/3})\right].$$

Then, from (2.13), relation (2.12) follows. The theorem is proven. □

Consider the sequence of functions

$$\Phi_N(x,y)=\sum_{n=1}^{N}\left(\frac{f(x,\lambda_n)f(y,\lambda_n)}{\alpha_n^2}\right)-\sum_{n=1}^{N}\left(\frac{f_0(x,\hat{\lambda}_n)f_0(y,\hat{\lambda}_n)}{\hat{\alpha}_n^2}\right). \tag{2.14}$$

According to the results of [13], the sequence $\Phi_N(x,y)$ converges boundedly; that is, it converges at each point (x,y), remaining bounded in every finite region of the (x,y)-plane. Moreover, from the known properties of transformation operators (see, for example, [16]) and from (2.8), (2.9), it follows that

$$f_0(x,\lambda)=f(x,\lambda)+\int_x^{\infty}\hat{K}(x,t)\,f(t,\lambda)\,dt, \tag{2.15}$$

where the kernel $\hat{K}(x,y)$ satisfies the equation

$$\hat{K}(x,y)+K(x,y)+\int_x^{y}\hat{K}(x,t)\,K(t,y)\,dt=0. \tag{2.16}$$

From the last equation and (2.9), (2.10) it follows that

$$|\hat{K}(x,y)|\le C\,\sigma_0\left(\frac{x+y}{2}\right), \tag{2.17}$$

$$\left|\frac{\partial\hat{K}(x_1,x_2)}{\partial x_j}+\frac{1}{2}q\left(\frac{x_1+x_2}{2}\right)\right|\le C[1+(x_2-x_1)^2]\sigma_0\left(\frac{x_1+x_2}{2}\right).$$

From this and from (1.3) we find that the kernel $\hat{K}(x,y)$ satisfies the estimate

$$\int_x^{\infty}\left[\left|\hat{K}'_y(x,y)\right|^2+y^2\left|\hat{K}(x,y)\right|^2\right]dy<\infty. \tag{2.18}$$

Because $|x|$ is an even function, the eigenvalues v_k, $(k=1,2,\dots)$ of the latter problem coincide with the zeros of the function $\Delta(\lambda)=\mathrm{Ai}(-\lambda)\mathrm{Ai}'(-\lambda)$. Since the zeros of the functions $\mathrm{Ai}(-\lambda)$ and $\mathrm{Ai}'(-\lambda)$ alternate (see [1]), the equalities $\mathrm{Ai}(-v_{2n})=0$ and $\mathrm{Ai}'(-v_{2n-1})=0$ hold. Moreover, the eigenfunction corresponding to the eigenvalue v_k has the form

$$\psi(x,v_k)=\begin{cases}\mathrm{Ai}(x-v_k), x\ge 0,\\ (-1)^{k-1}\mathrm{Ai}(-x-v_k), x<0.\end{cases}$$

Then, according to (2.18), for each $x\ge 0$ the coefficient $\tau_k=\int_x^{\infty}\hat{K}(x,t)\frac{\psi(t,v_k)}{\beta_k}\,dx$, where $\beta_k^2=\int_{-\infty}^{\infty}|\psi(t,v_k)|^2\,dt$, satisfies the inequality (see [3], Chapter II, formula (3.20))

$$\sum_{k=1}^{\infty}v_k|\tau_k|^2<\infty.$$

It is obvious that $v_{2n-1}=\hat{\lambda}_n$, $\beta_{2n-1}^2=2\hat{\alpha}_n^2$. Therefore, for each $x\ge 0$ the coefficient

$$\hat{\tau}_n=\int_x^{\infty}\hat{K}(x,t)\frac{f_0(t,\hat{\lambda}_n)}{\hat{\alpha}_n}\,dt \tag{2.19}$$

satisfies the estimate

$$\sum_{n=1}^{\infty} \hat{\lambda}_n |\hat{\tau}_n|^2 < \infty. \tag{2.20}$$

Next, using formula (2.3) and Theorem 2.1, we find that for large values of n, uniformly with respect to x taken from any finite interval $[0, b]$, the following estimate holds:

$$\left| \frac{f_0(x, \hat{\lambda}_n)}{\hat{\alpha}_n} \right| \leq \frac{C}{n^{1/3}}, \tag{2.21}$$

where $C = C(b)$. Now, let us reformulate (2.15) as $f_0(y, \lambda) = \left(I_y + \hat{K}_y\right) f(y, \lambda)$. Applying the operator $\left(I_x + \hat{K}_x\right)\left(I_y + \hat{K}_y\right)$ to the equality (2.14), we obtain

$$\begin{aligned}
\left(I_x + \hat{K}_x\right)\left(I_y + \hat{K}_y\right) \Phi_N(x, y) = \sum_{n=1}^{N} \frac{f_0(x, \lambda_n) f_0(y, \lambda_n)}{\alpha_n^2} - \\
- \sum_{n=1}^{N} \left(I_x + \hat{K}_x\right)\left(I_y + \hat{K}_y\right) \frac{f_0(x, \hat{\lambda}_n) f_0(y, \hat{\lambda}_n)}{\hat{\alpha}_n^2} = \\
= \left(\sum_{n=1}^{N} \frac{f_0(x, \lambda_n) f_0(y, \lambda_n)}{\alpha_n^2} - \sum_{n=1}^{N} \frac{f_0(x, \hat{\lambda}_n) f_0(y, \hat{\lambda}_n)}{\hat{\alpha}_n^2} \right) - \\
- \sum_{n=1}^{N} \left(\int_x^{\infty} \hat{K}(x, t) \frac{f_0(t, \hat{\lambda}_n)}{\hat{\alpha}_n} \, dt \right) \frac{f_0(y, \hat{\lambda}_n)}{\hat{\alpha}_n} - \\
- \sum_{n=1}^{N} \left(\int_y^{\infty} \hat{K}(y, t) \frac{f_0(t, \hat{\lambda}_n)}{\hat{\alpha}_n} \, dt \right) \frac{f_0(x, \hat{\lambda}_n)}{\hat{\alpha}_n} - \\
- \sum_{n=1}^{N} \left(\int_x^{\infty} \hat{K}(x, t) \frac{f_0(t, \hat{\lambda}_n)}{\hat{\alpha}_n} \, dt \right) \left(\int_y^{\infty} \hat{K}(y, t) \frac{f_0(t, \hat{\lambda}_n)}{\hat{\alpha}_n} \, dt \right).
\end{aligned} \tag{2.22}$$

The left-hand side of identity (2.22) converges boundedly. Therefore, the same is true for the right-hand side. However, due to (2.19)–(2.21), the second, third, and fourth terms converge boundedly. Consequently, the first term on the right-hand side in (2.22), i.e., the sequence of functions $F_N(x, y) = \sum_{n=1}^{N} \frac{f_0(x, \lambda_n) f_0(y, \lambda_n)}{\alpha_n^2} - \sum_{n=1}^{N} \frac{f_0(x, \hat{\lambda}_n) f_0(y, \hat{\lambda}_n)}{\hat{\alpha}_n^2}$ also converges boundedly.

Consider the function

$$F(x, y) = \sum_{n=1}^{\infty} \left\{ \frac{f_0(x, \lambda_n) f_0(y, \lambda_n)}{\alpha_n^2} - \frac{f_0(x, \hat{\lambda}_n) f_0(y, \hat{\lambda}_n)}{\hat{\alpha}_n^2} \right\} \tag{2.23}$$

From the reasoning given above, it follows that the sequence of functions $F_N(x, y)$ converges boundedly to $F(x, y)$.

In solving the inverse problem, an integral equation of the Marchenko type plays an important role.

Theorem 2.3. *For each fixed x, the function $K(x,y)$ appearing in the representation (2.8) satisfies the integral equation*

$$F(x,y)+K(x,y)+\int_x^\infty K(x,t)F(t,y)\,dt=0, y>x. \tag{2.24}$$

Proof. Consider representation (2.15). Since the eigenfunctions $\left\{\frac{f(x,\lambda_n)}{\alpha_n}\right\}_{n=0}^\infty$ of problem (1.1)–(1.2) form an orthonormal basis in the space $L_2(0,\infty)$, it follows that, for $y>x$,

$$\sum_{n=1}^\infty \frac{f(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}=\sum_{n=1}^\infty \frac{f(x,\lambda_n)}{\alpha_n}\frac{f(y,\lambda_n)}{\alpha_n}+$$

$$+\int_y^\infty \hat K(y,t)\left\{\sum_{n=1}^\infty \frac{f(x,\lambda_n)}{\alpha_n}\frac{f_0(t,\lambda_n)}{\alpha_n}\right\}dt=\delta(x-y)+\int_y^\infty \hat K(y,t)\delta(x-t)\,dt=$$

$$=\delta(x-y)+\hat K(y,x)=\delta(x-y).$$

Using (2.8) and the fact that the system of functions $\left\{\frac{f_0(x,\hat\lambda_n)}{\hat\alpha_n}\right\}_{n=1}^\infty$ forms an orthonormal basis in the space $L_2(0,\infty)$, we obtain

$$\sum_{n=1}^\infty \frac{f(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}=\sum_{n=1}^\infty \frac{f_0(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}+$$

$$+\int_x^\infty K(x,t)\left\{\sum_{n=1}^\infty \frac{f_0(t,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}\right\}dt=\sum_{n=1}^\infty \frac{f_0(x,\hat\lambda_n)}{\hat\alpha_n}\frac{f_0(y,\hat\lambda_n)}{\hat\alpha_n}+$$

$$+\sum_{n=1}^\infty\left\{\frac{f_0(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}-\frac{f_0(x,\hat\lambda_n)}{\hat\alpha_n}\frac{f_0(y,\hat\lambda_n)}{\hat\alpha_n}\right\}+$$

$$+\int_x^\infty K(x,t)\left\{\sum_{n=1}^\infty \frac{f_0(t,\hat\lambda_n)}{\hat\alpha_n}\frac{f_0(y,\hat\lambda_n)}{\hat\alpha_n}\right\}dt+$$

$$+\int_x^\infty K(x,t)\left\{\sum_{n=1}^\infty\left\{\frac{f_0(t,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}-\frac{f_0(t,\hat\lambda_n)}{\hat\alpha_n}\frac{f_0(y,\hat\lambda_n)}{\hat\alpha_n}\right\}\right\}dt=$$

$$=\delta(x-y)+F(x,y)+K(x,y)+\int_x^\infty K(x,t)F(t,y)\,dt.$$

Comparing the last two equalities, we obtain the main integral equation (2.24). The theorem is proven. □

Note that the obtained results can be used in the study of the inverse problem for the boundary value problem (1.1)–(1.2).

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