Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Number 1, 2014, Pages 3–13

MORREY-TYPE BANACH SPACES, MAXIMAL OPERATOR AND FOURIER MULTIPLIERS

VAGIF S. GULIYEV, ZHIJIAN WU, AND YING XIAO

Abstract. Let $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ be a Banach space of complex-valued measurable functions on $\mathbb{R}^{n+1}_+$. In this paper, we consider the Morrey-type Banach space $\mathcal{M}_{\mathcal{F}}(p,\lambda)$ as well as its weak type $\mathcal{M}^*_{\mathcal{F}}(1,\lambda)$. We develop the theory of maximal operator and Fourier multipliers on these spaces.

1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space of points $x = (x_1, ..., x_n)$ with norm $|x| = \left(\sum_{i=1}^n x_i^2\right)^{1/2}$. For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ be the open ball in $\mathbb{R}^n$ centered at x with radius r.

For $f \in L_1^{loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal operator M is defined by

$$M(f)(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

For a domain $\Omega \subset \mathbb{R}^n$ denote by $WL_p(\Omega)$, the weak L_p space of locally integrable functions f on Ω with the finite quasi-norm

$$\|f\|_{WL_p(\Omega)} = \sup_{t>0} t \left|\{x \in \Omega \ : \ |f(x)| > t\}\right|^{1/p}.$$

Let $0 < p \leq \infty$ and $\lambda \in \mathbb{R}$. Denote by $\mathcal{M}(p,\lambda)$ the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathcal{M}(p,\lambda)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty.$$

Denote by $\mathcal{M}^*(p,\lambda)$ the space of all functions $f \in L_p^{loc}(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathcal{M}^*(p,\lambda)} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

If $0 < p \leq \infty$, then $\mathcal{M}(p,0) = L_p(\mathbb{R}^n)$ and $\mathcal{M}(p,n) = L_\infty(\mathbb{R}^n)$ isometrically.

The space $\mathcal{M}(p,\lambda)$, called the Morrey space, is first introduced by C. Morrey in 1938 in [9]. It plays an important role in the study of partial differential equations, especially the local behaviour of the solutions of elliptic partial differential

2010 *Mathematics Subject Classification.* 42B25, 31C15.

Key words and phrases. Morrey-type space, Hardy-Littlewood maximal function, Fourier multiplier.

equations. We refer the reader to the papers [1, 5, 10, 11] and the references therein. The following result is from [5], which is a model to our study.

Theorem 1.1 (see [5])**.** *Suppose* $1 \le p < \infty$ *and* $0 < \lambda < n$.
(1) *If* $p > 1$, *then the maximal operator* M *is bounded on* $\mathcal{M}(p,\lambda)$.
(2) *The maximal operator* M *is bounded from* $\mathcal{M}(1,\lambda)$ *to* $\mathcal{M}^*(1,\lambda)$.

Let $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ be a Banach space of complex-valued measurable functions on $\mathbb{R}^{n+1}_+$. In this paper, we consider the Morrey-type Banach space $\mathcal{M}_{\mathcal{F}}(p,\lambda)$ as well as the weak type $\mathcal{M}^*_{\mathcal{F}}(1,\lambda)$. We aim at developing the theory of maximal operator and Fourier multipliers acting on these spaces.

There are rich literatures about various Morrey-type spaces and operators acting on them. We refer the readers to [2, 3, 4, 7, 8]. It should be noted that our Morrey-type Banach space $\mathcal{M}_{\mathcal{F}}(p,\lambda)$ includes all the Morrey-type spaces studied in above papers. In particular, the Morrey-type space $\mathcal{M}_{p\theta,\lambda}$ introduced by D.R. Adams in [2] and used heavily by G. Lu in [8] for studying the embedding theorems for vector fields of Hörmander type is a special case of our Morrey-type Banach spaces.

2. Definitions and basic properties of Morrey-type spaces

Suppose $0 < p \le \infty$, $\lambda \in \mathbb{R}$ and $f \in L^{loc}_p(\mathbb{R}^n)$. For $x \in \mathbb{R}^n$ and $r > 0$, define

$$E_{p,\lambda}(f)(x,r) = r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} \quad \text{and} \quad E^*_{p,\lambda}(f)(x,r) = r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))}.$$

These two quantities can be viewed as functions of $(x,r) \in \mathbb{R}^{n+1}_+$.

Definition 2.1. Let $0 < p \le \infty$, $\lambda \in \mathbb{R}$ and $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ be a Banach space of complex-valued measurable functions on $\mathbb{R}^{n+1}_+$. Denote by $\mathcal{M}_{\mathcal{F}}(p,\lambda)$ the Morrey-type Banach space of all functions $f \in L^{loc}_p(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)} = \|E_{p,\lambda}(f)\|_{\mathcal{F}}.$$

Denote by $\mathcal{M}^*_{\mathcal{F}}(p,\lambda)$ the weak Morrey-type Banach space of all functions $f \in L^{loc}_p(\mathbb{R}^n)$ satisfying

$$\|f\|_{\mathcal{M}^*_{\mathcal{F}}(p,\lambda)} = \left\|E^*_{p,\lambda}(f)\right\|_{\mathcal{F}}.$$

It is clear that for the Banach space $L_\infty(\mathbb{R}^{n+1}_+)$, $\|\cdot\|_{\mathcal{M}_{L_\infty}(p,\lambda)} = \|\cdot\|_{\mathcal{M}(p,\lambda)}$.

For $0 < \theta, p \le \infty$, let $L(\theta,p)$ be the space of functions h on $\mathbb{R}^{n+1}_+$ satisfying

$$\|h\|_{L(\theta,p)} = \left\| \left\{ \int_0^\infty |h(x,s)|^\theta \frac{ds}{s} \right\}^{\frac{1}{\theta}} \right\|_{L_p(\mathbb{R}^n)} < \infty.$$

Clearly, $L(\theta,p)$ is a Banach space and $L(\infty,\infty) = L_\infty(\mathbb{R}^{n+1}_+)$. In fact

$$\mathcal{M}_{L(\theta,\infty)}(p,\lambda) = \mathcal{M}_{p\theta,\lambda}$$

(the space introduced by D.R. Adams in [2]). For $0 < p \le \infty$ and $1 \le \theta \le \infty$, denote by $\Lambda_n(p,\theta)$ the interval $\left(0, n - \frac{p}{\theta}\right)$ if $1 \le \theta < \infty$; the interval $[0,n]$ if $\theta = \infty$. It is easy to see that for $0 < p \le \infty$ and $1 \le \theta \le \infty$, $\mathcal{M}_{p\theta,\lambda}$ is trivial if $\lambda \notin \Lambda_n(p,\theta)$. Therefore the interesting case of λ for the space $\mathcal{M}_{p\theta,\lambda}$ is $\lambda \in \Lambda_n(p,\theta)$.

Lemma 2.1. *Let $1 \le p \le q \le \infty$ and $\mathcal{F}$ be a Banach space of measurable functions on $\mathbb{R}^{n+1}_+$. If $f \in \mathcal{M}_{\mathcal{F}}(q,\lambda)$, then $f \in \mathcal{M}_{\mathcal{F}}(p,(\lambda-n)\frac{p}{q}+n))$ and*

$$\|f\|_{\mathcal{M}_{\mathcal{F}}(p,(\lambda-n)\frac{p}{q}+n))} \le v_n^{\frac{1}{p}-\frac{1}{q}} \|f\|_{\mathcal{M}_{\mathcal{F}}(q,\lambda)}. \tag{2.1}$$

Here v_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof. For $x \in \mathbb{R}^n$ and $r > 0$, applying Hölder inequality, we have

$$\|f\|_{L_p(B(x,r))} \le \|f\|_{L_q(B(x,r))} |B(x,r)|^{\frac{1}{p}-\frac{1}{q}} = v_n^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L_q(B(x,r))} r^{\frac{n}{p}-\frac{n}{q}}.$$

This implies

$$E_{p,(\lambda-n)\frac{p}{q}+n}(f)(x,r) \le v_n^{\frac{1}{p}-\frac{1}{q}} E_{q,\lambda}(f)(x,r),$$

and hence (2.1) follows. □

Assume $(\mathcal{F}_i, \|\cdot\|_{\mathcal{F}_i}), i = 1,2$, and $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ are Banach spaces of complex-valued measurable functions on $\mathbb{R}^{n+1}_+$. The following two properties are considered also.

(I) $\mathcal{F}_1\mathcal{F}_2 \subseteq \mathcal{F}$, i.e., $fg \in \mathcal{F}$ if $f \in \mathcal{F}_1$ and $g \in \mathcal{F}_2$.

(II) If $\mathcal{F}_1\mathcal{F}_2 \subseteq \mathcal{F}$, then the Hölder inequality holds, i.e.,

$$\|fg\|_{\mathcal{F}} \le \|f\|_{\mathcal{F}_1} \|g\|_{\mathcal{F}_2}$$

holds for any $f \in \mathcal{F}_1$ and any $g \in \mathcal{F}_2$.

Lemma 2.2 (the Hölder inequality)**.** *Suppose $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ are Banach spaces of functions on $\mathbb{R}^{n+1}_+$ satisfying (I) and (II). Let $0 < p, p_1, p_2 \le \infty$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}^n$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ and $\frac{\lambda}{p} = \frac{\lambda_1}{p_1} + \frac{\lambda_2}{p_2}$. If $f \in \mathcal{M}_{\mathcal{F}_1}(p_1,\lambda_1)$ and $g \in \mathcal{M}_{\mathcal{F}_2}(p_2,\lambda_2)$, then $fg \in \mathcal{M}_{\mathcal{F}}(p,\lambda)$ and*

$$\|fg\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)} \le \|f\|_{\mathcal{M}_{\mathcal{F}_1}(p_1,\lambda_1)} \|g\|_{\mathcal{M}_{\mathcal{F}_2}(p_2,\lambda_2)}.$$

Proof. For $x \in \mathbb{R}^n$ and $r > 0$, applying Hölder inequality, we have

$$\|fg\|_{L_p(B(x,r))} \le \|f\|_{L_{p_1}(B(x,r))} \|g\|_{L_{p_2}(B(x,r))}.$$

Therefore,

$$\begin{aligned} E_{p,\lambda}(fg)(x,r) &= r^{-\frac{\lambda}{p}} \|fg\|_{L_p(B(x,r))} \\ &\le r^{-\frac{\lambda_1}{p_1}} r^{-\frac{\lambda_2}{p_2}} \|f\|_{L_{p_1}(B(x,r))} \|g\|_{L_{p_2}(B(x,r))} \\ &= E_{p_1,\lambda_1}(f)(x,r) E_{p_2,\lambda_2}(g)(x,r). \end{aligned}$$

Because of the assumption (II), the above estimate implies the desired result. □

3. Maximal operator on $\mathcal{M}_{\mathcal{F}}(p,\lambda)$

For a positive integer $k \ge 1$ and a measurable function h defined on $\mathbb{R}^{n+1}_+$, denote by V_k, the compression of h by a factor of 2^k on the $(n+1)^{th}$ variable, i.e.,

$$V_k(h)(x,y) = h(x, 2^k y), \qquad \text{for all } (x,y) \in \mathbb{R}^{n+1}_+.$$

Clearly V_k is a linear operator. A Banach space $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ of complex-valued measurable functions on $\mathbb{R}^{n+1}_+$ is said to be $\{V_k\}_{k\geq 1}$ admissible if there exists a $C>0$ such that

$$\|V_k(h)\|_{\mathcal{F}} \leq C\,\|h\|_{\mathcal{F}}\,, \qquad \text{for all} \;\; h\in\mathcal{F} \text{ and } k\geq 1\,.$$

It is easy to check that the space $L(\theta,p)$ is $\{V_k\}_{k\geq 1}$ admissible. In fact, it is $\{V_k\}_{k\geq 1}$ invariant, i.e., $\|V_k(h)\|_{L(\theta,p)} = \|h\|_{L(\theta,p)}$ for all $h\in L(\theta,p)$ and $k\geq 1$.

Inspired by the results in [3] about boundedness of the maximal operator in the local Morrey-type spaces, we establish the the following theorem, which clearly implies Theorem 1.1.

Theorem 3.1. *Let $1\leq p<\infty$, $\lambda<n$, and $\mathcal{F}$ be a $\{V_k\}_{k\geq 1}$ admissible Banach space of functions on $\mathbb{R}^{n+1}_+$. Then*

(1) *For $1<p<\infty$, the maximal operator M is bounded on $\mathcal{M}_{\mathcal{F}}(p,\lambda)$, i.e., there is a constant $C>0$ such that*

$$\|M(f)\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)} \leq C\|f\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)}, \qquad \textit{for all} \;\; f\in\mathcal{M}_{\mathcal{F}}(p,\lambda).$$

(2) *The maximal operator M is bounded from $\mathcal{M}_{\mathcal{F}}(1,\lambda)$ to $\mathcal{M}^*_{\mathcal{F}}(1,\lambda)$, i.e., there is a constant $C>0$ such that*

$$\|M(f)\|_{\mathcal{M}^*_{\mathcal{F}}(1,\lambda)} \leq C\|f\|_{\mathcal{M}_{\mathcal{F}}(1,\lambda)}, \qquad \textit{for all} \;\; f\in\mathcal{M}_{\mathcal{F}}(1,\lambda).$$

Instead of proofing Theorem 3.1, we prove the following generalized result for the Morrey-type Banach spaces of vector-valued functions.

Let $0<q\leq\infty$. If $f=\{f_j\}_{j=-\infty}^{\infty}$ is a sequence of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$, we write $f\in\mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)$, if

$$\|f\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)} = \Big\| \|f\|_{l_q} \Big\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)}.$$

Denote $M(f)=\{M(f_j)\}_{j=-\infty}^{\infty}$, if $f=\{f_j\}_{j=-\infty}^{\infty}$.

Theorem 3.2. *Let $1\leq p<\infty$, $1\leq q\leq\infty$, $\lambda<n$, and $\mathcal{F}$ be a $\{V_k\}_{k\geq 1}$ admissible Banach space of functions on $\mathbb{R}^{n+1}_+$. Then*

(1) *For $1<p<\infty$, the maximal operator M is bounded on $\mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)$, i.e., there is a constant $C>0$ such that*

$$\|M(f)\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)} \leq C\,\|f\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)}$$

holds for all $f\in\mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)$.

(2) *The maximal operator M is bounded from $\mathcal{M}_{\mathcal{F}}(1,\lambda)(l_q)$ to $\mathcal{M}^*_{\mathcal{F}}(1,\lambda)(l_q)$, i.e., there is a constant $C>0$ such that*

$$\|M(f)\|_{\mathcal{M}^*_{\mathcal{F}}(1,\lambda)(l_q)} \leq C\,\|f\|_{\mathcal{M}_{\mathcal{F}}(1,\lambda)(l_q)}$$

holds for all $f\in\mathcal{M}_{\mathcal{F}}(1,\lambda)(l_q)$.

Remark 3.1. From the proof below, we will see that Theorem 3.2 remains true if the condition of "$\mathcal{F}$ is a $\{V_k\}_{k\geq 1}$ admissible Banach space of functions on $\mathbb{R}^{n+1}_+$" is replaced by following ($1<p<\infty$ for case (1) and $p=1$ for case (2), respectively)

$$\sum_{k=0}^{\infty}(2^k)^{-n/p+\lambda/p}\,\|V_k\|_{\mathcal{F}}<\infty.$$

Remark 3.2. In Theorem 3.2, let $\mathcal{F} = L(\theta, t)$ (which is clearly $\{V_k\}_{k\geq 1}$ admissible), especially for $\theta = t = \infty$ or $t = \infty$, we obtain results about the boundedness of the maximal operator on the classical Morrey spaces $\mathcal{M}(p, \lambda)(l_q)$ (see [12]), or on the space $\mathcal{M}_{p\theta,\lambda}(l_q)$ for $1 \leq p, \theta \leq \infty$ and $\lambda \in \Lambda_n(p, \theta)$.

Proof of Theorem 3.2. Let $1 \leq p < \infty$ and $f = \{f_j\}_{j=-\infty}^{\infty}$. For fixed $u \in \mathbb{R}^n$ and $r > 0$ denote

$$T^0_{u,r}(f) = \chi_{B(u,2r)} f = \{\chi_{B(u,2r)} f_j\}_{j=-\infty}^{\infty},$$

$$T^k_{u,r}(f) = \chi_{B(u,2^{k+1}r)\setminus B(u,2^k r)} f = \{\chi_{B(u,2^{k+1}r)\setminus B(u,2^k r)} f_j\}_{j=-\infty}^{\infty}, \quad k = 1, 2, \ldots .$$

Clearly $f = \sum_{k=0}^{\infty} T^k_{u,r}(f)$ and

$$\|M(f)(x)\|_{l_q} \leq \sum_{k=0}^{\infty} \left\|M(T^k_{u,r}(f))(x)\right\|_{l_q}, \quad \forall x \in \mathbb{R}^n .$$

This implies that

$$E_{p,\lambda}(\|M(f)\|_{l_q})(u,r) \leq \sum_{k=0}^{\infty} E_{p,\lambda}(\left\|M(T^k_{u,r}(f))\right\|_{l_q})(u,r), \tag{3.1}$$

$$E^*_{p,\lambda}(\|M(f)\|_{l_q})(u,r) \leq \sum_{k=0}^{\infty} E^*_{p,\lambda}(\left\|M(T^k_{u,r}(f))\right\|_{l_q})(u,r). \tag{3.2}$$

To prove Theorem 3.2 (1), we estimate $E_{p,\lambda}(\left\|M(T^0_{u,r}(f))\right\|_{l_q})(u,r)$ first. Recall the well known Fefferman-Stein maximal inequality (see [6])

$$\left\|\|M(f)\|_{l_q}\right\|_{L_p(\mathbb{R}^n)} \leq C \left\|\|f\|_{l_q}\right\|_{L_p(\mathbb{R}^n)},$$

where $C > 0$ is independent of the vector-valued function f. We have

$$\begin{aligned}
\left\|\|M(T^0_{u,r}(f))\|_{l_q}\right\|_{L_p(B(u,r))} &\leq \left\|\|M(T^0_{u,r}(f))\|_{l_q}\right\|_{L_p(\mathbb{R}^n)} \\
&\leq C \left\|\|T^0_{u,r}(f)\|_{l_q}\right\|_{L_p(\mathbb{R}^n)} \\
&= C \left\|\|f\|_{l_q}\right\|_{L_p(B(u,2r))},
\end{aligned}$$

where $C > 0$ is independent of $u \in \mathbb{R}^n$, $r > 0$ and the vector-valued function f. This yields

$$E_{p,\lambda}(\left\|M(T^0_{u,r}(f))\right\|_{l_q})(u,r) \leq C V_1(E_{p,\lambda}(\|f\|_{l_q}))(u,r).$$

It remains to estimate $E_{p,\lambda}(\left\|M(T^k_{u,r}(f))\right\|_{l_q})(u,r)$ for $k \geq 1$. Let $x \in B(u,r)$ and let

$$t_x = \inf\{t : B(x,t) \setminus B(u, 2^{k+1}r) \neq \emptyset\}.$$

It is easy to see that

$$t_x \asymp 2^k r \qquad \text{for all } \ x \in B(u,r).$$

Therefore for a complex-valued function $g \in L_p^{loc}(\mathbb{R}^n)$ and $x \in B(u,r)$, we have

$$\begin{aligned} M(T^k_{u,r}(g))(x) &= \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |T^k_{u,r}(g)(y)| dy \\ &\leq |B(x,t_x)|^{-1} \int_{\mathbb{R}^n} |T^k_{u,r}(g)(y)| dy \\ &\asymp C(2^k r)^{-n} \int_{\mathbb{R}^n} |T^k_{u,r}(g)(y)| dy. \end{aligned}$$

Therefore by the Minkowski inequality, we have

$$\begin{aligned} \left\| M(T^k_{u,r}(f))(x) \right\|_{l_q} &\leq C(2^k r)^{-n} \left\| \left\{ \int_{\mathbb{R}^n} |T^k_{u,r}(f_j)(y)| dy \right\}_{j=-\infty}^{\infty} \right\|_{l_q} \\ &\leq C \left(2^k r\right)^{-n} \int_{\mathbb{R}^n} \left\| T^k_{u,r}(f)(y) \right\|_{l_q} dy \\ &= C \left(2^k r\right)^{-n} \int_{B(u,2^{k+1}r)\setminus B(u,2^k r)} \|f(y)\|_{l_q} dy \\ \text{(Hölder)} &\leq C(2^k r)^{-n/p} \left\| \|f\|_{l_q} \right\|_{L_p(B(u,2^{k+1}r))}. \end{aligned} \tag{3.3}$$

This yields

$$E_{p,\lambda}(\left\| M(T^k_{u,r}(f)) \right\|_{l_q})(u,r) \leq C(2^k)^{-n/p+\lambda/p} V_{k+1}(E_{p,\lambda}(\|f\|_{l_q}))(u,r)\,.$$

Combining the estimates above and using estimate (3.1), we obtain

$$\begin{aligned} E_{p,\lambda}(\|M(f)\|_{l_q})(u,r) &\leq \sum_{k=0}^{\infty} E_{p,\lambda}(\left\| M(T^k_{u,r}(f)) \right\|_{l_q})(u,r) \\ &\leq C \sum_{k=0}^{\infty} (2^k)^{-n/p+\lambda/p} V_{k+1}(E_{p,\lambda}(\|f\|_{l_q}))(u,r). \end{aligned} \tag{3.4}$$

Applying $\|\cdot\|_{\mathcal{F}}$ to both sides of the above estimate and using the fact that $\mathcal{F}$ is a $\{V_k\}_{k\geq 1}$ admissible Banach space of functions on $\mathbb{R}^{n+1}_+$ and the series $\sum_{k=0}^{\infty} (2^k)^{-\frac{n}{p}+\frac{\lambda}{p}}$ converges when $\lambda < n$, we can conclude the desired result.

The weak case can be proved similarly by using the weak type Fefferman-Stein maximal inequality [6]

$$\left\| \|M(f)\|_{l_q} \right\|_{WL_1(\mathbb{R}^n)} \leq C \left\| \|f\|_{l_q} \right\|_{L_1(\mathbb{R}^n)},$$

where $C > 0$ is independent of the vector-valued function f. Indeed, we have

$$\begin{aligned} \left\| \|M(T^0_{u,r}(f))\|_{l_q} \right\|_{WL_1(B(u,r))} &\leq \left\| \|M(T^0_{u,r}(f))\|_{l_q} \right\|_{WL_1(\mathbb{R}^n)} \\ &\leq C \left\| \|T^0_{u,r}(f)\|_{l_q} \right\|_{L_1(\mathbb{R}^n)} \\ &= C \left\| \|f\|_{l_q} \right\|_{L_1(B(u,2r))}, \end{aligned}$$

where $C > 0$ is independent of $u \in \mathbb{R}^n$, $r > 0$ and the vector-valued function f. This implies

$$E^*_{1,\lambda}(\left\|M(T^0_{u,r}(f))\right\|_{l_q})(u,r) \le CV_1(E_{1,\lambda}(\|f\|_{l_q}))(u,r)\,.$$

On the other hand, by (3.3) we have

$$E^*_{1,\lambda}(\left\|M(T^k_{u,r}(f))\right\|_{l_q})(u,r) \le C(2^k)^{-n+\lambda}V_{k+1}(E_{1,\lambda}(\|f\|_{l_q}))(u,r)\,.$$

Hence by (3.2), we obtain

$$\begin{aligned} E^*_{1,\lambda}(\|M(f)\|_{l_q})(u,r) &\le \sum_{k=0}^{\infty} E^*_{1,\lambda}(\left\|M(T^k_{u,r}(f))\right\|_{l_q})(u,r) \\ &\le C\sum_{k=0}^{\infty}(2^k)^{-n+\lambda}V_{k+1}(E_{1,\lambda}(\|f\|_{l_q}))(u,r). \end{aligned}$$

Applying $\|\cdot\|_{\mathcal{F}}$ to both sides of the above estimate and using the fact that $\mathcal{F}$ is a $\{V_k\}_{k\ge1}$ admissible Banach space of functions on $\mathbb{R}^{n+1}_+$ and the series $\sum_{k=0}^{\infty}(2^k)^{-n+\lambda}$ converges when $\lambda < n$, we can conclude the desired result. □

4. Fourier multipliers on $\mathcal{M}_{\mathcal{F}}(p,\lambda)$

In this section, we establish an application of our theorems in previous section for Fourier multipliers on $\mathcal{M}_{\mathcal{F}}(p,\lambda)$. Our approach has its root in [13].

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all rapidly decreasing infinitely differential complex-valued functions on $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ is the space of all complex-valued tempered distributions on $\mathbb{R}^n$. Let

$$(F\phi)(\xi) = (2\pi)^{n/2}\int_{\mathbb{R}^n}\phi(x)e^{-ix\cdot\xi}dx$$

and let F^{-1} denote the Fourier transform and its inverse on $\mathcal{S}'(\mathbb{R}^n)$, respectively.

If $s \in \mathbb{R}$, we write

$$H^s_2(\mathbb{R}^n) = \left\{f \in \mathcal{S}'(\mathbb{R}^n)\ :\ \|f\|_{H^s_2} = \left\|(1+|x|^2)^{s/2}Ff(x)\right\|_{L_2} < \infty\right\}.$$

If Ω is a compact set of $\mathbb{R}^n$, we write

$$L_{p,\Omega} = \left\{f \in \mathcal{S}'(\mathbb{R}^n)\ :\ \operatorname{supp} Ff \subset \Omega,\ \|f\|_{L_p} < \infty\right\}.$$

For $p \ge 1$ and $0 < s < p$, it is proved in [13] (page 22) that if $g \in L_{p,B(0,1)}$, then

$$\frac{|g(x-z)|}{1+|z|^{n/s}} \le C\left[M\left(|g|^s\right)(x)\right]^{1/s}, \qquad \text{for all}\ \ x,z\in\mathbb{R}^n, \tag{4.1}$$

where the constant $C > 0$ is independent of x, z, and g.

For $d, s > 0$, consider the following maximal operator $N_{d,s}$ defined by

$$N_{d,s}(g)(x) = \sup_{y\in\mathbb{R}^n}\frac{|g(x-y)|}{1+(d|y|)^{n/s}}.$$

We have the following result.

Theorem 4.1. *Suppose $\mathcal{F}$ is a $\{V_k\}_{k\geq 1}$ admissible Banach space of functions on $\mathbb{R}^{n+1}_+$. Let $1 \leq p < \infty$, $\lambda < n$, Ω be a compact set, d be the radius of Ω, $0 < s < p$ and $f \in L_{p,\Omega}$. Then there is a constant $C > 0$ such that*

(1) *For $1 < p < \infty$*

$$\|N_{d,s}(f)\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)} \leq C\,\|f\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)}$$

holds for all $f \in \mathcal{M}_{\mathcal{F}}(p,\lambda)$.

(2) *For $p = 1$*

$$\|N_{d,s}(f)\|_{\mathcal{M}^*_{\mathcal{F}}(1,\lambda)} \leq C\,\|f\|_{\mathcal{M}_{\mathcal{F}}(1,\lambda)}$$

holds for all $f \in \mathcal{M}_{\mathcal{F}}(1,\lambda)$.

Instead of proofing Theorem 4.1, we prove the following generalized result for vector-valued functions. Assume $0 < p,\, q \leq \infty$ and $\lambda \in \mathbb{R}^n$. If $\Omega = \{\Omega_j\}_{j=-\infty}^{\infty}$ is a sequence of compact sets on $\mathbb{R}^n$, we denote $f = \{f_j\}_{j=-\infty}^{\infty} \in \mathcal{M}_{\mathcal{F}}(p,\lambda,\Omega)(l_q)$, If $f_j \in L_{p,\Omega_j}$ for $j \in \mathbb{Z}$ and f is in $\mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)$.

Theorem 4.2. *Suppose $\mathcal{F}$ is a $\{V_k\}$ admissible Banach space of functions on $\mathbb{R}^{n+1}_+$, $1 \leq p, q < \infty$, $\lambda < n$, $\Omega = \{\Omega_j\}_{j=-\infty}^{\infty}$ is a sequence of compact sets, and d_j is the radius of Ω_j for $j \in \mathbb{Z}$. If $f = \{f_j\}_{j=-\infty}^{\infty} \in \mathcal{M}_{\mathcal{F}}(p,\lambda,\Omega)(l_q)$ and $0 < s < \min\{p, q\}$, then exists a constant C such that*

(1) *For $1 < p < \infty$*

$$\left\|\{N_{d_j,s}(f_j)\}\right\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)} \leq C\,\|f\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)}$$

holds for all $f \in \mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)$.

(2) *For $p = 1$*

$$\left\|\{N_{d_j,s}(f_j)\}\right\|_{\mathcal{M}^*_{\mathcal{F}}(1,\lambda)(l_q)} \leq C\,\|f\|_{\mathcal{M}_{\mathcal{F}}(1,\lambda)(l_q)}$$

holds for all $f \in \mathcal{M}_{\mathcal{F}}(1,\lambda)(l_q)$.

Proof. Let $1 \leq p < \infty$ and $f = \{f_j\}_{j=-\infty}^{\infty} \in \mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)$, and y^j be the center of Ω_j. Define $h_j(x) = e^{-ix\cdot y^j} f_j(x)$ for $j \in \mathbb{Z}$. We have clearly $|h_j(x)| = |f_j(x)|$, $F(h_j)(x) = F(f_j)(x)(x + y^j)$ and therefore $\operatorname{supp} F(h_j) \subset \Omega_j - y^j$. Thus, without lost of generality, we assume that $0 \in \Omega_j$, and $\Omega_j = B(0, d_j)$ for all $j \in \mathbb{Z}$.

For $j \in \mathbb{Z}$, let $g_j(x) = f_j({d_j}^{-1}x)$. Then $F(g_j)(x) = d_j^n F(f_j)(d_j x)$ and

$$\operatorname{supp} F(g_j) \subset B(0,1).$$

From (4.1), we obtain

$$N_{d_j,s}(f_j)(x) \leq C\,[M\left(|f_j|^s\right)(x)]^{\frac{1}{s}} \quad \text{for all} \quad x,\, z \in \mathbb{R}^n, \tag{4.2}$$

where the constant $C > 0$ is independent of x, j, and f_j.

The above estimate implies

$$\left\|\{N_{d_j,s}(f_j)(x)\}\right\|_{l_q} \leq C \left\|\{[M(|f_j|^s)(x)]^{\frac{1}{s}}\}\right\|_{l_q} = C\,\|\{M(|f_j|^s)(x)\}\|^{\frac{1}{s}}_{l_{\frac{q}{s}}},$$

and therefore

$$E_{p,\lambda}(\left\|\{N_{d_j,s}(f_j)\}\right\|_{l_q})(u,r) \leq C \left(E_{p/s,\lambda}(\|\{M(|f_j|^s)\}\|_{l_{\frac{q}{s}}})(u,r)\right)^{\frac{1}{s}}. \tag{4.3}$$

Since $\frac{q}{s}, \frac{p}{s} > 1$, by (3.4), we have

$$\begin{aligned} & E_{p/s,\lambda}(\|\{M(|f_j|^s)\}\|_{l_{\frac{q}{s}}})(u,r) \\ & \qquad \leq C\sum_{k=0}^{\infty}\left(2^{\frac{n-\lambda}{p/s}}\right)^{-k} V_{k+1}(E_{p/s,\lambda}(\|\{|f_j|^s\}\|_{l_{\frac{q}{s}}}))(u,r) \\ & \qquad = C\sum_{k=0}^{\infty}\left(2^{\frac{n-\lambda}{p/s}}\right)^{-k} \left(V_{k+1}(E_{p,\lambda}(\|f\|_{l_q}))(u,r)\right)^s . \end{aligned}$$

If $s \geq 1$, then

$$\left(E_{p/s,\lambda}(\|\{M(|f_j|^s)\}\|_{l_{\frac{q}{s}}})(u,r)\right)^{\frac{1}{s}} \leq C\sum_{k=0}^{\infty}\left(2^{\frac{n-\lambda}{p}}\right)^{-k} V_{k+1}(E_{p,\lambda}(\|f\|_{l_q}))(u,r).$$

If $0 < s < 1$, by Hölder inequality, we have

$$\begin{aligned} & \left(E_{p/s,\lambda}(\|\{M(|f_j|^s)\}\|_{l_{\frac{q}{s}}})(u,r)\right)^{\frac{1}{s}} \\ & \qquad \leq C\left(\sum_{k=0}^{\infty}\left(2^{\frac{n-\lambda}{p/s}}\right)^{-k}\right)^{\frac{1}{s}-1} \sum_{k=0}^{\infty}\left(2^{\frac{n-\lambda}{p/s}}\right)^{-k} V_{k+1}(E_{p,\lambda}(\|f\|_{l_q}))(u,r). \end{aligned}$$

Together with (4.3), we conclude

$$\left\|\{N_{d_j,s}(f_j)\}\right\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)} \leq C\,\|f\|_{\mathcal{M}_{\mathcal{F}}(p,\lambda)(l_q)} .$$

For the weak estimation, it is easy to see that if

$$f = \{f_j\}_{j=-\infty}^{\infty} \in W\mathcal{M}_{\mathcal{F}}(1,\lambda)(l_q)$$

then

$$E^*_{1,\lambda}(\left\|\{N_{d_j,s}(f_j)\}\right\|_{l_q})(u,r) \leq C\left(E^*_{1/s,\lambda}(\|\{M(|f_j|^s)\}\|_{l_{\frac{q}{s}}})(u,r)\right)^{\frac{1}{s}} .$$

Since $\frac{q}{s}, \frac{1}{s} > 1$, we can continue the above estimation by

$$\begin{aligned} & \leq C\left(E_{1/s,\lambda}(\|\{M(|f_j|^s)\}\|_{l_{\frac{q}{s}}}))(u,r)\right)^{\frac{1}{s}} \\ & \leq C\left(\sum_{k=0}^{\infty}\left(2^{\frac{n-\lambda}{1/s}}\right)^{-k}\left(V_{k+1}(E_{1,\lambda}(\|f\|_{l_q}))(u,r)\right)^s\right)^{\frac{1}{s}} \\ & \leq C\left(\sum_{k=0}^{\infty}\left(2^{\frac{n-\lambda}{1/s}}\right)^{-k}\right)^{\frac{1}{s}-1} \sum_{k=0}^{\infty}\left(2^{\frac{n-\lambda}{1/s}}\right)^{-k} V_{k+1}(E_{1,\lambda}(\|f\|_{l_q}))(u,r). \end{aligned}$$

We conclude

$$\left\|\{N_{d_j,s}(f_j)\}\right\|_{W\mathcal{M}_{\mathcal{F}}(1,\lambda)(l_q)} \leq C\,\|f\|_{\mathcal{M}_{\mathcal{F}}(1,\lambda)(l_q)} .$$

□

By Theorem 4.2 and the proof of p. 31–32 in [13], it is easy to obtain the following theorem.

Theorem 4.3. *Let $1 \le p < \infty$, $1 \le q \le \infty$, $1 \le \theta < \infty$, $0 < \lambda < n - \frac{p}{\theta}$. Let also $\Omega = \{\Omega_j\}_{j=-\infty}^{\infty}$ is a sequence of compact sets on $\mathbb{R}^n$, $f_j \in L_{p,\Omega_j}$ for $j \in \mathbb{Z}$, and d_j be the radius of Ω_j. If $\nu > n/2 + n/\min\{p, q\}$, then exists a constant C such that*

$$\left\| \left\{F^{-1}G_jFf_j\right\}_j \right\|_{M_{p\theta,\lambda}(l_q)} \le C \sup_j \|G_j(d_j\cdot)\|_{H_2^\nu} \left\| \{f_j\}_j \right\|_{M_{p\theta,\lambda}(l_q)}$$

and

$$\left(\sum_{j=\infty}^{\infty} \|F^{-1}G_jFf_j\|^q_{M_{p\theta,\lambda}} \right)^{1/q} \le C \sup_j \|G_j(d_j\cdot)\|_{H_2^\nu} \left(\sum_{j=\infty}^{\infty} \|f_j\|^q_{M_{p\theta,\lambda}} \right)^{1/q}.$$

for any sequence $\{G_j\}_j \in H_2^\nu(\mathbb{R}^n)$.

Acknowledgements

The authors, correspondingly, were partially supported by the grant of Science Development Foundation under the President of the Republic of Azerbaijan project EIF-2010-1(1)-40/06-1 and the Scientific and Technological Research Council of Turkey (TUBITAK Project No: 110T695), a CRDF grant, and a youth research grant of the Ministry of Education of China (Grant No. 12YJC880123) and a humanities and social science research projects of the China University of Political Science and Law.

References

[1] D. R. Adams, A note on Riesz potentials, *Duke Math. J.* **42** (1975), 765–778.
[2] D. R. Adams, *Lectures on L^p–potential theory,* Umeå University, Report no. 2, 1981.
[3] V. I. Burenkov and H. V. Guliyev, Necessary and sufficient conditions for boundedness of the maximal operator in local Morrey-type spaces, *Studia Math.* **163** (2004), 157–176.
[4] V. I. Burenkov, V. S. Guliyev, A. Serbetci and T. V. Tararykova, Necessary and sufficient conditions for the boundedness of genuine singular integral operators in local Morrey-type spaces, *Eurasian Math. J.* **1** (2010), 32–53.
[5] F. Chiarenza and M. Frasca, Morrey spaces and Hardy–Littlewood maximal function, *Rend. Mat. Appl. (7)* **7** (1987), 273–279.
[6] C. Fefferman and E. M. Stein, Some maximal inequalities, *Amer. J. Math.* **93** (1971), 107–115.
[7] V. S. Guliyev, *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$ (Russian), Doctor's degree dissertation, Math. Inst. Steklov, Moscow, 1994.
[8] G. Lu, Embedding theorems on Campanato–Morrey spaces for vector fields and applications, *C. R. Acad. Sci. Paris Sér. I Math.* **320** (1995), 429–434.
[9] C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, *Trans. Amer. Math. Soc.* **43** (1938), 126–166.
[10] J. Peetre, On the theory of $\mathcal{L}_{p,\lambda}$ spaces, *J. Functional Analysis* **4** (1969), 71–87.
[11] S. Spanne, Sur l'interpolation entre les espaces $\mathcal{L}_k^{p\Phi}$, *Ann. Scuola Norm. Sup. Pisa (3)* **20** (1966), 625–648.
[12] L. Tang and J. Xu, Some properties of Morrey type Besov-Triebel spaces, *Math. Nachr.* **278** (2005), 904–917.
[13] H. Triebel, *Theory of function spaces,* Birkhäuser Verlag, Basel, 1983.

Vagif S. Guliyev
Department of Mathematics, Baku State University, Baku, Azerbaijan.
E-mail address: vagif@guliyev.com

Zhijian Wu
Department of Mathematics, The University of Alabama, Tuscaloosa, AL 35487, USA.
E-mail address: zwu@as.ua.edu

Ying Xiao
Department of Science and Technology, China University of Political Science and Law, Beijing, 102249, China.
E-mail address: yingx@cupl.edu.cn

Received: January 9, 2014; Accepted: February 7, 2014