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ON APPROXIMATION PROPERTIES OF GENERALIZED PARABOLIC POTENTIALS

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Abstract. Generalized parabolic potentials associated with the singular heat operators $\left(\frac{\partial}{\partial t}-\Delta_\nu\right)$ and $\left(\frac{\partial}{\partial t}+I-\Delta_\nu\right)$, where

$$\Delta_\nu(x)=\sum_{k=1}^{n}\left(\frac{\partial^2}{\partial x_k^2}+\frac{2\nu_k}{x_k}\frac{\partial}{\partial x_k}\right)$$

are introduced. These potentials are defined in terms of the relevant generalized translation operator. We obtained their approximation properties in this article.

1. Introduction

The classical Jones-Sampson parabolic potentials $\mathcal{H}^\alpha f$, $(\alpha>0)$ are defined in the Fourier terms by

$$F\left[H^\alpha f\right](x,t)=\left(|x|^2+it\right)^{-\alpha/2}F[f](x,t),$$

$$F\left[\mathcal{H}^\alpha f\right](x,t)=\left(1+|x|^2+it\right)^{-\alpha/2}F[f](x,t)$$

where $x\in\mathbb{R}^n$, $t\in\mathbb{R}^1$; F is the Fourier transform. These potentials are interpreted as negative (fractional) powers of the singular heat operators

$$\left(\frac{\partial}{\partial t}-\Delta\right),\quad \left(\frac{\partial}{\partial t}+I-\Delta\right),\quad \Delta=\sum_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}$$

where Δ is the Laplacian in $\mathbb{R}^n$ and I is an identity operator. Parabolic potentials were introduced by B.F. Jones and C.H. Sampson, various properties of these potentials and relevant anisotropic functions spaces of the Sobolev type were studied by R. Bagby, V.R. Gopala Rao, S. Chanillo, V.A. Nogin, B.S. Rubin, I.A. Aliev, A.D. Gadjiev, V.S. Guliyev and many other mathematicians (see, e.g., [2, 4, 5, 6, 10, 12, 14, 15, 17, 22, 23, 24].

One should note that the approximation properties of the classical parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$ as $\alpha\to 0^+$ have been studied by S.B Uyhan, A.D. Gadjiev, I.A. Aliev [28] and the classical Riesz and Bessel kernels as approximation of the identity have been studied by T. Kurokawa [19].

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The relevant singular parabolic potentials, associated with the singular heat operators $\left(\frac{\partial}{\partial t}-\Delta_B\right)$ and $\left(\frac{\partial}{\partial t}+I-\Delta_B\right)$, $\Delta_B=\sum\limits_{k=1}^{n}\frac{\partial^2}{\partial x_k^2}+\left(\frac{\partial}{\partial x_n}+\frac{2\nu}{x_n}\frac{\partial}{\partial x_n}\right)$, $(\nu>0)$ were introduced and studied by I.A. Aliev, B.S. Rubin[3, 4]. These potentials are defined in terms of the Fourier-Bessel transform F_B by

$$F_B\left[H_B^{\alpha}f\right](x,t)=\left(|x|^2+it\right)^{-\alpha/2}F_B[f](x,t),$$

and

$$F_B\left[\mathcal{H}_B^{\alpha}f\right](x,t)=\left(1+|x|^2+it\right)^{-\alpha/2}F_B[f](x,t),$$

where $x\in\mathbb{R}_+^n$, $t\in(0,\infty)$, $\alpha>0$ and Δ_B is the Laplace-Bessel differential operator which is known as an important differential operator in analysis and its applications (see, e.g., [7, 8, 13, 14, 15, 16, 18, 20, 21, 27]).

In this paper, generalized parabolic potentials $A_{\nu,\theta}^{\alpha}$ $(\theta=1,2)$ interpreted as the negative (fractional) powers of the heat operator $\left(\frac{\partial}{\partial t}+(\theta-1)\,I-\Delta_\nu\right)$, $(\theta=1,2)$, that is,

$$A_{\nu,\theta}^{\alpha}=\left(\frac{\partial}{\partial t}+(\theta-1)I-\Delta_\nu\right)^{-\alpha/2},$$

are introduced and the approximation properties are studied as $\alpha\to 0^+$. Here, $\Delta_\nu\equiv\Delta_\nu\left(x\right)$ is the Laplace-Bessel differential operator defined as

$$\Delta_\nu(x)=\sum_{k=1}^{n}\left(\frac{\partial^2}{\partial x_k^2}+\frac{2\nu_k}{x_k}\frac{\partial}{\partial x_k}\right),$$

where $\nu=(\nu_1,\nu_2,...,\nu_n)$, $\nu_1>0,...,\nu_n>0$, $|\nu|=\nu_1+\nu_2+...+\nu_n$.

We emphasize that the ideas of this work are originally introduced by A. D. Gadjiev, I. A. Aliev and A. Aral in [1, 9]. The plan of how we are to proceed is as follows. Necessary definitions and auxiliary facts are given in Section 2. Finally, in the last section we provide main theorems and their proofs.

2. Preliminaries

Let $\mathbb{R}_+^n=\{x=(x_1,...,x_n)\in\mathbb{R}^n;\ x_1>0,...,x_n>0\}$, $\mathbb{R}_+^{n+1}=\mathbb{R}_+^n\times(0,\infty)$. $L_{p,\nu}$ and the weighted $\mathcal{L}_{p,\nu}$ spaces, $1\le p\le\infty$ are defined as

$$L_{p,\nu}\equiv L_{p,\nu}(\mathbb{R}_+^n)=\{f:\ \|f\|_{L_{p,\nu}}=\left(\int\limits_{\mathbb{R}_+^n}|f(x)|^p\,x^{2\nu}dx\right)^{1/p}<\infty\},$$

$$\mathcal{L}_{p,\nu}\equiv\mathcal{L}_{p,\nu}(\mathbb{R}_+^{n+1})=\{g:\ \|g\|_{\mathcal{L}_{p,\nu}}=\left(\int\limits_{\mathbb{R}^1}\int\limits_{\mathbb{R}_+^n}|g(x,t)|^p\ x^{2\nu}dxdt\right)^{1/p}<\infty\}$$

, where $x^{2\nu}=x_1^{2\nu_1}...\ x_n^{2\nu_n}$; $dx=dx_1...\ dx_n$. In the case $p=\infty$ we identify $L_\infty\equiv L_\infty(R_+^{n+1})=\{f:\|f\|_\infty=\operatorname{ess\,sup}_{\mathbb{R}_+^{n+1}}|f(x,t)|<\infty\}$ and $C_0\equiv C_0(R_+^{n+1})$ denotes the class of all continuous functions on $\mathbb{R}_+^{n+1}$, vanishing at infinity. $C\equiv C(R_+^{n+1})$ is the class of all continuous functions on $\mathbb{R}_+^{n+1}$.

For $x, y \in \mathbb{R}_+^n,\ \ t, \tau \in (0,\infty),$ the generalized translation of $f\colon \mathbb{R}_+^{n+1} \to \mathbb{C}$ is defined by (see, e.g., [20, 21])

$$T^{y,\tau} f(x,t) = \pi^{-n/2} \prod_{k=1}^{n} \left(\Gamma(\nu_k + 1/2)\,\Gamma^{-1}(\nu_k)\right)$$
$$\times \int_0^\pi \cdots \int_0^\pi f\left(\sqrt{x_1^2 - 2x_1 y_1 \cos\xi_1 + y_1^2}, \ldots, \sqrt{x_n^2 - 2x_n y_n \cos\xi_n + y_n^2}, t-\tau\right)$$
$$\times \prod_{k=1}^{n} \sin^{2\nu_k - 1} \xi_k d\xi_1 \ldots d\xi_n.$$

It is known that for $1 \le p < \infty$, (see, e.g., [20])

$$\|T^{y,\tau} f\|_{\mathcal{L}_{p,\nu}} \le \|f\|_{\mathcal{L}_{p,\nu}} \quad \forall\ (y,\tau) \in \mathbb{R}_+^{n+1} \tag{2.1}$$

and

$$\|T^{y,\tau} f - f\|_{\mathcal{L}_{p,\nu}} \to 0 \quad \text{as} \quad (y,\tau) \to (0,0). \tag{2.2}$$

The generalized convolution associated with the generalized translation is

$$(f \otimes g)(x,t) = \int_{\mathbb{R}^1} \int_{\mathbb{R}_+^n} g(y,\tau)\ \left(T^{y,\tau} f(x,t)\right) y^{2\nu}\ dy\ d\tau. \tag{2.3}$$

By using (1) and Riesz-Thorin interpolation theorem, it is not difficult to prove the corresponding Young inequality

$$\|f \otimes g\|_{\mathcal{L}_{r,\nu}} \le \|f\|_{\mathcal{L}_{p,\nu}} \|g\|_{\mathcal{L}_{q,\nu}}, \quad 1 \le p,q,r \le \infty,\ \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1. \tag{2.4}$$

The Fourier-Bessel transform of a sufficiently good function $f : \mathbb{R}_+^n \to \mathbb{C}$ and its inverse are defined by

$$(F_\nu f)(x) = \int_{\mathbb{R}_+^n} f(y) \left(\prod_{k=1}^{n} j_{\nu_k - 1/2}(x_k y_k)\right) y^{2\nu} dy \tag{2.5}$$

$$(F_\nu^{-1} f)(x) = c_\nu(n)(F_\nu f)(x), \qquad c_\nu(n) = [2^{-2n} \prod_{k=1}^{n} \Gamma^2(\nu_k + 1/2)]^{-1}, \tag{2.6}$$

where $j_p(t) = 2^p \Gamma(p+1)\, t^{-p} J_p(t),\ \ p > -1/2$ is the normalized Bessel function such that $j_p(0) = 1$.

The Fourier-Bessel transform of a function $\varphi :\ \mathbb{R}_+^{n+1} \to \mathbb{C}$ and its inverse are defined as follows

$$(F_\nu \varphi)(x,t) = \int_{\mathbb{R}^1} \int_{\mathbb{R}_+^n} \varphi(y,\tau)\, e^{-it\tau} \left(\prod_{k=1}^{n} j_{\nu_k - 1/2}(x_k y_k)\right) y^{2\nu} dy d\tau, \tag{2.7}$$

$$(F_\nu^{-1} \varphi)(x,t) = \frac{c_\nu(n)}{2\pi} (F_\nu \varphi)(x,-t)$$

where $c_\nu(n)$ has been defined above (cf. (2.6)).

Now from the general theory we know that the Gauss-Weierstrass kernel is defined as the Fourier transformation of the function $e^{-\alpha|x|^2}$, $\alpha > 0$, (see [25, pp. 6–7]).

In this work, we need the Generalized Gauss-Weierstrass kernel defined as the Fourier-Bessel transformation of the function $e^{-\alpha|x|^2}$, $\alpha > 0$. Therefore, taking into account that the following equality for $\nu > -1$, $\alpha > 0$, $\beta > 0$

$$\int_0^\infty e^{-\alpha r^2} r^{\nu+1} J_\nu(\beta r)\, dr = \frac{\beta^\nu}{(2\alpha)^{\nu+1}} e^{-\beta^2/(4\alpha)}$$

where $J_\nu(t) = 2^{-\nu}(\Gamma(\nu+1))^{-1} t^\nu j_\nu(t)$ is the Bessel function of the first kind (see [11]) gives

$$\int_0^\infty e^{-\alpha r^2} r^{2\nu} j_{\nu-1/2}(\beta r)\, dr = \frac{1}{2\,\alpha^{\nu+1/2}} \Gamma\left(\nu + \frac{1}{2}\right) e^{-\beta^2/(4\alpha)}, \quad (\nu > -1/2)$$

in one dimension.

Thus we define the Generalized Gauss-Weierstrass kernel in $\mathbb{R}_+^{n+1}$ as

$$W_\nu(y,\tau) = \sqrt{c_\nu(n)}\, \tau^{-(n+2|\nu|)/2} e^{-|y|^2/(4\tau)} \quad , \tag{2.8}$$

where $c_\nu(n)$ is defined by (2.6) and $|\nu| = \nu_1 + \nu_2 + \dots + \nu_n$.

It is easy to prove for the Generalized Gauss-Weierstrass kernel that

$$(F_\nu W_\nu(\cdot,\tau))(x,t) = e^{-t|x|^2} \tag{2.9}$$

and

$$\int_{\mathbb{R}_+^n} W_\nu(y,\tau)\, y^{2\nu} dy = 1. \tag{2.10}$$

Generalized parabolic potentials $A^\alpha_{\theta,\nu}$, $(\theta = 1, 2)$ defined in terms of the Fourier-Bessel transform

$$F_\nu(A^\alpha_{\nu,\theta} f)(x,t) = \left((\theta - 1) + |x|^2 + it\right)^{-\alpha/2} (F_\nu f)(x,t), \quad \left((x,t) \in \mathbb{R}_+^{n+1},\ \alpha > 0\right)$$

have the following convolution type integral representation

$$A^\alpha_{\nu,1} f(x,t) = \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}_+^n} \int_0^\infty \tau^{\alpha/2-1}\ W_\nu(y,\tau)\ T^{y,\tau} f(x,t)\ y^{2\nu}\ d\tau dy, \tag{2.11}$$

$$A^\alpha_{\nu,2} f(x,t) = \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}_+^n} \int_0^\infty \tau^{\alpha/2-1}\ e^{-\tau}\ W_\nu(y,\tau)\ T^{y,\tau} f(x,t)\ y^{2\nu}\ d\tau dy. \tag{2.12}$$

To see this it suffices to apply Fourier-Bessel transform to both sides of (2.11) and (2.12) and make use of the statement (2.9). Then we have

$$A^\alpha_{\nu,1} f(x,t) = \frac{1}{\Gamma(\alpha/2)} \left(\tau^{\alpha/2-1}\ W_\nu(x,t) \otimes f\right)$$

and

$$A^\alpha_{\nu,2} f(x,t) = \frac{1}{\Gamma(\alpha/2)} \left(\tau^{\alpha/2-1} e^{-\tau}\ W_\nu(x,t) \otimes f\right).$$

The following theorem characterizes the behaviour of the operators $A^{\alpha}_{\nu,1}$ and $A^{\alpha}_{\nu,2}$ on $\mathcal{L}_{p,\nu}-$ spaces. Furthermore, it is an analogue of the Hardy-Littlewood-Sobolev theorem.

Theorem 2.1. ***a)*** *Let* $f \in L_{p,\nu}$, $1 < p < q < \infty$, $0 < \alpha < (n + 2|\nu| + 2)/p$. *The inequality*

$$\left\| A^{\alpha}_{\nu,1} f \right\|_{\mathcal{L}_{q,\nu}} \le c \left\| f \right\|_{\mathcal{L}_{p,\nu}}, \quad c = c(p, q, n, |\nu|)$$

holds if and only if $\alpha/(n + 2|\nu| + 2) = 1/p - 1/q$.

b) *If* $f \in L_{1,\nu}$, *then*

$$\mu \left\{ (x,t) \ : \ \left| A^{\alpha}_{\nu,1} f(x,t) \right| > \lambda \right\} \le \left(\frac{c \left\| f \right\|_{\mathcal{L}_{1,\nu}}}{\lambda} \right)^{q}$$

for any $\lambda > 0$ *and* $q > 1$, $(1/q + \alpha/(n + 2|\nu| + 2) = 1)$. *That is, the mapping* $f \to A^{\alpha}_{\nu,1} f$ *is of " weak-type"* $(1, q)$, *Here* $\mu \{E\} = \int\limits_E x^{2\nu} dx$ *is the Lebesgue measure for the* $E \subset R^{n+1}_{+}$.

c) *For any* $1 \le p \le \infty$ *and* $\alpha > 0$, $\left\| A^{\alpha}_{\nu,2} f \right\|_{\mathcal{L}_{p,\nu}} \le \left\| f \right\|_{\mathcal{L}_{p,\nu}}$.

Proof. The necessity in **(a)** is a consequence of standart homogeneity argument (see [25, p. 118]). Further, let us write

$$k(x,t) = t^{(\alpha - n - 2|\nu|)/2 - 1} e^{-(|x|^2/(4t))}, \quad x \in \mathbb{R}^n_+, \quad t > 0.$$

We shall consider the transformation $f \to k(x,t) \otimes f$, instead of $f \to A^{\alpha}_{\nu,1} f$ from which it differs by a constant multiple. Let us decompose $k(x,t)$ as $k_1(x,t) + k_\infty(x,t)$, where

$$k_1(x,t) = \begin{cases} k(x,t), & t \le \beta \\ 0, & t > \beta \end{cases}, \quad k_\infty(x,t) = \begin{cases} 0, & t \le \beta \\ k(x,t), & t > \beta \end{cases}.$$

Now

$$\{x : |k \otimes f| > \lambda\} \subseteq \left\{ x : |k_1 \otimes f| > \frac{\lambda}{2} \right\} \cup \left\{ x : |k_\infty \otimes f| > \frac{\lambda}{2} \right\},$$

and

$$\mu \{x : |k \otimes f| > \lambda\} \le \mu \left\{ x : |k_1 \otimes f| > \frac{\lambda}{2} \right\} + \mu \left\{ x : |k_\infty \otimes f| > \frac{\lambda}{2} \right\},$$

Assume that $f \ge 0$ and $\|f\|_{\mathcal{L}_{1,\nu}} = 1$. Then

$$\begin{aligned} \mu \left\{ x : |k_1 \otimes f| > \frac{\lambda}{2} \right\} &\le \lambda^{-p} \left\| k_1 \otimes f \right\|^p_{\mathcal{L}_{p,\nu}} \\ &\le \lambda^{-p} \left\| k_1 \right\|^p_{\mathcal{L}_{1,\nu}} \le c_1 \beta^{\alpha p/2}. \end{aligned}$$

Further, by Hölder's inequality,

$$\begin{aligned}\|k_\infty \otimes f\|_\infty \le \|k_2\|_{\mathcal{L}_{p',\nu}} &= \left(\int_\beta^\infty \int_{\mathbb{R}^n_+} \left(t^{(\alpha-n-2|\nu|)/2-1} e^{-|x|^2/(4t)} \right)^{p'} x^{2\nu} dx dt \right)^{1/p'} \\ &= c_{p,n,|\nu|} \left(\int_\beta^\infty t^{(\alpha-n-2|\nu|-2)p'/2+(n+2|\nu|)/2} dt \right)^{1/p'} \\ &= c_2 \beta^\delta, \qquad (1/p'+1/p=1),\end{aligned}$$

$\delta = \left(\frac{\alpha-n-2|\nu|-2}{2}p' + \frac{n+2|v|}{2} + 1\right)\frac{1}{p'} = \frac{\alpha}{2} - \frac{n+2|\nu|+2}{2p} = -\frac{n+2|\nu|+2}{2q}$. Setting $c_2\beta^\delta = \lambda$,

we have $\beta = c_3 \lambda^{1/\delta}$. Therefore, $\|k_\infty \otimes f\|_\infty \le \lambda$, and so $\mu\left\{x : |k_\infty \otimes f| > \frac{\lambda}{2}\right\} = 0$. Finally,

$$\mu\{x : |k \otimes f| > \lambda\} \le c_1\ \beta^{\alpha p/2}\lambda^{-p} = c_4\ \lambda^{\alpha p/2\delta}\lambda^{-p} = c\ \lambda^{-q} = c\left(\frac{\|f\|_{\mathcal{L}_{1,\nu}}}{\lambda}\right)^q,$$

(since $\|f\|_{\mathcal{L}_{1,\nu}} = 1$) and **(b)** is proved. By the Marcinkiewicz interpolation theorem this implies the sufficiency in **(a)**. **(c)** is a consequence of Young's inequality. □

We will need the following " non-isotropic" Lipschitz class on $\mathbb{R}^{n+1}_+$.

$$\Lambda_\beta = \left\{ f \in L_\infty :\ \|T^{y,\tau} f(x,t) - f(x,t)\|_\infty \le c_f\ \left(|y|^2 + |\tau|\right)^{\beta/2} \right\} \tag{2.13}$$

$$\widetilde{\Lambda}_\beta = \left\{ f :\ |T^{y,\tau} f(x_0,t_0) - f(x_0,t_0)| \le c_f\ \left(|y|^2 + |\tau|\right)^{\beta/2} \right\} \tag{2.14}$$

Here c_f is constant, $x, x_0, y \in \mathbb{R}^n_+$; $t, t_0, \tau \in \mathbb{R}^1$ and $0 < \beta \le 1$.

3. Main Results and Proofs

Theorem 3.1. *Let* $f \in L_{p,\nu}(R^{n+1}_+)$, $1 \le p < \infty$.
(a) *If at a point* $(x,t) \in R^{n+1}_+$ *there exists the limit* $\lim\limits_{(z,s)\to(x,t)} f(z,s) = l$, $-\infty \le l \le \infty$, *then*

$$\lim_{\alpha\to 0^+} \left(A^\alpha_{\nu,\theta} f\right)(x,t) = l, \quad (\theta = 1,2).$$

If f *is continuous at the point* $(x,t) \in R^{n+1}_+$, *then* $\lim\limits_{\alpha\to 0^+}\left(A^\alpha_{\nu,\theta} f\right)(x,t) = f(x,t)$.
(b) *If* $f \in L_{p,\nu} \cap C_0$, *the convergence* $\lim\limits_{\alpha\to 0^+} A^\alpha_{\nu,\theta} f = f$ *is uniform on* R^{n+1}_+. *If* $f \in L_{p,\nu} \cap C$, *the convergence is uniform on any compact* $K \subset R^{n+1}_+$.

Proof. **(a)** We will proof the statements of the theorem for $\theta = 1$ (see Remark below about the $A^\alpha_{\nu,2}$). The following formula can be obtained from (2.11) by the use of Fubini theorem

$$A^\alpha_{\nu,1} f(x,t) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty \tau^{\alpha/2-1} \left(\int_{\mathbb{R}^n_+} W_\nu(y,\tau)\ T^{y,\tau} f(x,t) y^{2\nu} dy \right) d\tau. \tag{3.1}$$

Using the identity (2.10) we have

$$A^{\alpha}_{\nu,1}f(x,t)-l=\frac{1}{\Gamma(\alpha/2)}\int_0^\infty \tau^{\alpha/2-1}\int_{\mathbb{R}^n_+} W_\nu(y,\tau)\left(T^{y,\tau}f(x,t)-l\ e^{-\tau}\right)y^{2\nu}dyd\tau.$$

Given $\epsilon>0$, there exists $\delta>0$ such that

$$|T^{y,\tau}f(x,t)-l|<\epsilon \ \text{ and } \ (1-e^{-\tau})<\epsilon \tag{3.2}$$

for all $|y|<\sqrt{\delta}$ and $0<\tau<\delta$. We obtain that

$$\begin{aligned}
&|A^{\alpha}_{\nu,1}f(x,t)-l|\leq\\
&\leq \frac{1}{\Gamma(\alpha/2)}\left|\int_0^\delta \tau^{\alpha/2-1}\int_{|y|<\sqrt{\delta}} W_\nu(y,\tau)\left(T^{y,\tau}f(x,t)-l\ e^{-\tau}\right)y^{2\nu}dyd\tau\right|\\
&+ \frac{1}{\Gamma(\alpha/2)}\left|\int_0^\delta \tau^{\alpha/2-1}\int_{|y|\geq\sqrt{\delta}} W_\nu(y,\tau)\left(T^{y,\tau}f(x,t)-le^{-\tau}\right)y^{2\nu}dyd\tau\right|\\
&+ \frac{1}{\Gamma(\alpha/2)}\left|\int_\delta^\infty \tau^{\alpha/2-1}\int_{\mathbb{R}^n_+} W_\nu(y,\tau)\left(T^{y,\tau}f(x,t)-l\ e^{-\tau}\right)y^{2\nu}dyd\tau\right|\\
&= E_1(\alpha)+E_2(\alpha)+E_3(\alpha)
\end{aligned}\tag{3.3}$$

Estimate for $E_1(\alpha)$: We obtain the following estimate by using (3.2) and (2.10)

$$\begin{aligned}
E_1(\alpha) &\leq \frac{1}{\Gamma(\alpha/2)}\int_0^\delta \tau^{\alpha/2-1}\int_{|y|<\sqrt{\delta}} W_\nu(y,\tau)\ |T^{y,\tau}f(x,t)-l|\ y^{2\nu}dyd\tau\\
&+ \frac{|l|}{\Gamma(\alpha/2)}\int_0^\delta \tau^{\alpha/2-1}(1-e^{-\tau})\int_{|y|<\sqrt{\delta}} W_\nu(y,\tau)\ y^{2\nu}dyd\tau\\
&\leq \frac{1+|l|}{\Gamma(\alpha/2)}\,\epsilon\int_0^\delta \tau^{\alpha/2-1}\int_{\mathbb{R}^n_+} W_\nu(y,\tau)\ y^{2\nu}dyd\tau \overset{(10)}{=} \frac{(1+|l|)\delta^{\alpha/2}}{\Gamma(1+\alpha/2)}\ \epsilon.
\end{aligned}\tag{3.4}$$

Estimate for $E_2(\alpha)$: We can write that

$$\begin{aligned}
E_2(\alpha) &\leq \frac{1}{\Gamma(\alpha/2)}\int_0^\delta \tau^{\alpha/2-1}\int_{|y|\geq\sqrt{\delta}} W_\nu(y,\tau)\ |T^{y,\tau}f(x,t)|\ y^{2\nu}dyd\tau\\
&+ \frac{|l|}{\Gamma(\alpha/2)}\int_0^\delta \tau^{\alpha/2-1}\ e^{-\tau}\int_{|y|\geq\sqrt{\delta}} W_\nu(y,\tau)\ y^{2\nu}dyd\tau\\
&= E_2'+E_2''
\end{aligned}\tag{3.5}$$

Now, taking into account (2.8) and the Hölder inequality, for small $\alpha > 0$ we have

$$
\begin{aligned}
&E_2'(\alpha) \leq \\
\leq\ & \frac{c_1 \|f\|_{\mathcal{L}_{p,\nu}}}{\Gamma(\alpha/2)} \left(\int_0^\delta \tau^{(\alpha/2-1-n/2-2|\nu|)p'} d\tau \int_{|y|\geq\sqrt{\delta}} e^{-|y|^2 p'/(4\tau)}\, y^{2\nu} dy \right)^{1/p'} \\
=\ & \frac{c_2 \|f\|_{\mathcal{L}_{p,\nu}}}{\Gamma(\alpha/2)} \left(\int_0^\delta \tau^{(\alpha/2-1-n/2-2|\nu|)p'+n/2} d\tau \int_{|z|\geq\frac{1}{2}\sqrt{(\delta p')/\tau}} e^{-|z|^2} z^{2\nu} dz \right)^{1/p'} \\
=\ & (\text{ use } e^{-|z|^2} = e^{-|z|^2/2}\, e^{-|z|^2/2} \leq e^{-|z|^2/2} e^{-\delta p'/8\tau} \ \ for\ |z| \geq \frac{1}{2}\sqrt{(\delta p')/\tau}\ ) \\
\leq\ & \frac{c_3 \|f\|_{\mathcal{L}_{p,\nu}}}{\Gamma(\alpha/2)} \left(\int_0^\delta \tau^{\alpha/2-(1+n/2+2|\nu|)p'}\, e^{-\delta p'/8\tau}\, d\tau \right)^{1/p'} \left(\int_{\mathbb{R}_+^n} e^{-|z|^2/2} z^{2\nu} dz \right)^{1/p'} .
\end{aligned}
$$

We get the following inequality by using the estimate $\frac{1}{\Gamma(\alpha/2)} \sim \frac{1}{2}\alpha$ as $\alpha \to 0^+$

$$
E_2'(\alpha) \leq c_4\,(\delta)\,\alpha\,\|f\|_{\mathcal{L}_{p,\nu}}\,. \tag{3.6}
$$

Similarly, the second integral $E_2''\,(\alpha)$ is majorized by

$$
E_2''\,(\alpha) \leq c_5\,(\delta)\,\alpha|l|. \tag{3.7}
$$

If we substitute (3.6) and (3.7) in (3.5) we get

$$
E_2\,(\alpha) \leq \Big(c_4\,(\delta)\,\|f\|_{\mathcal{L}_{p,\nu}} + c_5\,(\delta)\,|l| \Big)\,\alpha = c_6\,(\delta)\,\alpha \quad as \quad \alpha \to 0^+. \tag{3.8}
$$

Estimate for $E_3(\alpha)$: We have

$$
\begin{aligned}
E_3\,(\alpha) \ &\leq\ \frac{1}{\Gamma\,(\alpha/2)} \int_\delta^\infty \tau^{\alpha/2-1} \int_{\mathbb{R}_+^n} W_\nu\,(y,\tau)\,|T^{y,\tau} f\,(x,t)\,|y^{2\nu} dy d\tau \\
&+\ \frac{|l|}{\Gamma\,(\alpha/2)} \int_\delta^\infty \tau^{\alpha/2-1} e^{-\tau} \int_{\mathbb{R}_+^n} W_\nu\,(y,\tau)\,y^{2\nu} dy d\tau \\
&=\ E_3'\,(\alpha) + E_3''(\alpha).
\end{aligned} \tag{3.9}
$$

Using Hölder's inequality, for $\alpha + 4|\nu| < \frac{n+2}{p}$ we have

$$\begin{aligned} E_3'(\alpha) &\leq \frac{\|f\|_{\mathcal{L}_{p,\nu}}}{\Gamma(\alpha/2)} \left(\int\limits_\delta^\infty \tau^{(\alpha/2-1)p'} d\tau \int\limits_{\mathbb{R}^n_+} (W_\nu(y,\tau))^{p'} y^{2\nu} dy \right)^{1/p'} \\ &\leq \frac{c_7 \|f\|_{\mathcal{L}_{p,\nu}}}{\Gamma(\alpha/2)} \left(\int\limits_\delta^\infty \tau^{(\alpha/2-1-n/2-2|\nu|)p'} d\tau \int\limits_{\mathbb{R}^n_+} e^{-|y|^2 p'/4\tau} y^{2\nu} dy \right)^{1/p'} \\ &= \frac{c_8 \|f\|_{\mathcal{L}_{p,\nu}}}{\Gamma(\alpha/2)} \left(\int\limits_\delta^\infty \tau^{(\alpha/2-1-n/2-2|\nu|)p'+n/2} d\tau \int\limits_{\mathbb{R}^n_+} e^{-|z|^2} dz \right)^{1/p'} \\ &\leq c_9(\delta) \|f\|_{\mathcal{L}_{p,\nu}} \alpha. \end{aligned} \tag{3.10}$$

Now the estimate

$$E_3''(\alpha) \leq \frac{|l|}{\Gamma(\alpha/2)} \int\limits_\delta^\infty \tau^{\alpha/2-1} e^{-\tau} d\tau \leq c_{10}(\delta) |l| \alpha \quad \text{as} \quad \alpha \to 0^+ \tag{3.11}$$

is obtained by noticing (3.9). Thus from (3.9) and (3.10) we get

$$E_3(\alpha) \leq \left(c_9(\delta) \|f\|_{\mathcal{L}_{p,\nu}} + c_{10}(\delta) |l| \right) \alpha = c_{11}(\delta) \alpha \quad \text{as} \quad \alpha \to 0^+. \tag{3.12}$$

Finally, substituting (3.4), (3.8) and (3.12) in (3.3) gives

$$\left| A^\alpha_{\nu,1} f(x,t) - l \right| \leq \frac{(1+|l|)\delta^{\alpha/2}}{\Gamma(1+\alpha/2)} \epsilon + c_6(\delta)\alpha + c_{11}(\delta)\alpha.$$

The last estimate yields

$$\limsup_{\alpha \to 0^+} \left| A^\alpha_{\nu,1} f(x,t) - l \right| \leq (1+|l|)\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get

$$\lim_{\alpha \to 0^+} \left| A^\alpha_{\nu,1} f(x,t) - l \right| = 0.$$

Let now $l = +\infty$, i.e. $\lim\limits_{(y,\tau)\to(x,t)} f(y,\tau) = +\infty$ (the case of $l = -\infty$ is examined analogously). For a given $M > 0$ there exists $\delta > 0$ such that $|T^{y,\tau} f(x,t)| > M$

for any $|y| < \sqrt{\delta}$, $0 < \tau < \delta$. We have

$$\begin{aligned} A^{\alpha}_{\nu,1} f(x,t) &= \frac{1}{\Gamma(\alpha/2)} \int_0^{\delta} \tau^{\alpha/2-1} \int_{|y|<\sqrt{\delta}} W_{\nu}(y,\tau)\, T^{y,\tau} f(x,t) y^{2\nu} dy d\tau \\ &+ \frac{1}{\Gamma(\alpha/2)} \int_0^{\delta} \tau^{\alpha/2-1} \int_{|y|\geq\sqrt{\delta}} W_{\nu}(y,\tau)\, T^{y,\tau} f(x,t) y^{2\nu} dy d\tau \\ &+ \frac{1}{\Gamma(\alpha/2)} \int_{\delta}^{\infty} \tau^{\alpha/2-1} \int_{\mathbb{R}^n_+} W_{\nu}(y,\tau)\, T^{y,\tau} f(x,t) y^{2\nu} dy d\tau \\ &\equiv F_1(\alpha) + F_2(\alpha) + F_3(\alpha). \end{aligned} \tag{3.13}$$

It is clear that

$$\begin{aligned} F_1(\alpha) &\geq \frac{M}{\Gamma(\alpha/2)} \int_0^{\delta} \tau^{\alpha/2-1} \int_{|y|<\sqrt{\delta}} W_{\nu}(y,\tau)\, T^{y,\tau} f(x,t) y^{2\nu} dy d\tau \\ &= \frac{c_1 M}{\Gamma(\alpha/2)} \int_0^{\delta} \tau^{\alpha/2-1-n/2-2|\nu|} \int_{|y|<\sqrt{\delta}} e^{-|y|^2/4\tau} y^{2\nu} dy d\tau \\ &= \frac{c_1 M}{\Gamma(\alpha/2)} \int_0^{\delta} \tau^{\alpha/2-1-2|\nu|} \int_{|x|<\frac{1}{2}\sqrt{\delta/\tau}} e^{-|x|^2} x^{2\nu} dx d\tau \end{aligned} \tag{3.14}$$

$$\begin{aligned} &= \frac{c_2 M}{\Gamma(\alpha/2)} \int_0^{\delta} \tau^{\alpha/2-1-n/2-2|\nu|} \int_0^{\frac{1}{2}\sqrt{\delta/\tau}} e^{-r^2} r^{n-1} dr d\tau \\ &\geq \frac{c_2 M}{\Gamma(\alpha/2)} \int_0^{\delta} \tau^{\alpha/2-1-n/2-2|\nu|} d\tau (\int_0^1 e^{-r^2} r^{n-1} dr) \\ &= c_2 \frac{M}{\Gamma(\alpha/2)} \frac{2}{\alpha} \delta^{\alpha/2} = c_3 \frac{M}{\Gamma(1+\alpha/2)} \delta^{\alpha/2}, \qquad (c_3 > 0). \end{aligned} \tag{3.15}$$

Further, by making use of the estimates for $E_2'(\alpha)$ and $E_3'(\alpha)$ (see (3.6) and (3.10), respectively) we have

$$|F_2(\alpha)| \leq c_1(\delta) \|f\|_{\mathcal{L}_{p,\nu}} \alpha, \quad |F_3(\alpha)| \leq c_2(\delta) \|f\|_{\mathcal{L}_{p,\nu}} \alpha. \tag{3.16}$$

Thus, substituting (3.14) and (3.15) in (3.13) gives

$$A^{\alpha}_{\nu,1} f(x,t) \geq c_3 \frac{M}{\Gamma(1+\alpha/2)} \delta^{\alpha/2} - c_1(\delta) \|f\|_{\mathcal{L}_{p,\nu}} \alpha - c_2(\delta) \|f\|_{\mathcal{L}_{p,\nu}} \alpha$$

and therefore, $\lim\inf_{\alpha\to 0^+} A^{\alpha}_{\nu,1} f(x,t) \geq c_3 M$, $(c_3 > 0)$. Since $M > 0$ is arbitrary, the last estimate yields that $\lim_{\alpha\to 0^+} A^{\alpha}_{\nu,1} f(x,t) = \infty$. **(b)** Now, let $f \in \mathcal{L}_{p,\nu} \cap C_0$.

For any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sup_{(x,t)\in\mathbb{R}^{n+1}_+} \left|T^{y,\tau} f(x,t) - f(x,t)\right| < \epsilon \text{ and } \left(1 - e^{-\tau}\right) < \epsilon \tag{3.17}$$

for all $|y| < \sqrt{\delta}$ and $0 < \tau < \delta$. Setting $l = f(x,t)$ in (3.3) and using (3.17), we have

$$\begin{aligned} \left\|A^{\alpha}_{\nu,1} f - f\right\|_{\infty} &\leq \frac{\left(1 + \|f\|_{\infty}\right)\delta^{\alpha/2}}{\Gamma\left(1+\alpha/2\right)}\epsilon + \left(c_4\left(\delta\right)\|f\|_{\mathcal{L}_{p,\nu}} + c_5\left(\delta\right)\|f\|_{\infty}\right)\alpha \\ &+ \left(c_9\left(\delta\right)\|f\|_{\mathcal{L}_{p,\nu}} + c_{10}\left(\delta\right)\|f\|_{\infty}\right)\alpha, \quad \alpha \to 0^+, \end{aligned}$$

(see (3.3), (3.4), (3.8), (3.12)). The latter estimate yields that

$$\limsup_{\alpha\to 0^+} \left\|A^{\alpha}_{\nu,1} f - f\right\|_{\infty} \leq \left(1 + \|f\|_{\infty}\right)\epsilon,$$

and therefore $\lim\limits_{\alpha\to 0^+} \left\|A^{\alpha}_{\nu,1} f - f\right\|_{\infty} = 0.$ □

Remark 3.1. The proof of the statements of Theorem 3.1 for $A^{\alpha}_{\nu,2}$ follows the same lines and is based on the equality

$$\begin{aligned} A^{\alpha}_{\nu,2} f(x,t) - l &= \frac{1}{\Gamma\left(\alpha/2\right)} \int_0^{\infty} \tau^{(\alpha/2)-1} e^{-\tau} \int_{\mathbb{R}^n_+} W_{\nu}\left(y,\tau\right) T^{y,\tau} f(x,t) y^{2\nu} dy d\tau \\ &= \frac{1}{\Gamma\left(\alpha/2\right)} \int_0^{\delta} \tau^{(\alpha/2)-1} e^{-\tau} \int_{|y|\leq\sqrt{\delta}} W_{\nu}\left(y,\tau\right) T^{y,\tau} f(x,t) y^{2\nu} dy d\tau \\ &+ \int_0^{\delta} \tau^{(\alpha/2)-1} e^{-\tau} \int_{|y|>\sqrt{\delta}} W_{\nu}\left(y,\tau\right) T^{y,\tau} f(x,t) y^{2\nu} dy d\tau \\ &+ \int_{\delta}^{\infty} \tau^{(\alpha/2)-1} e^{-\tau} \int_{\mathbb{R}^n_+} W_{\nu}\left(y,\tau\right) T^{y,\tau} f(x,t) y^{2\nu} dy d\tau. \end{aligned}$$

The next theorem gives an estimation of the order of approximation of the "Lipschitz functions" by the families $A^{\alpha}_{\nu,\theta} f$.

Theorem 3.2. *Let $f \in L_{p,\nu} \cap \Lambda_{\beta}$, $1 \leq p < \infty$, $0 < \beta \leq 1$, where Λ_{β} is the non-isotropic Lipschitz class defined as in (2.13). Then*

$$\left\|A^{\alpha}_{\nu,\theta} f - f\right\|_{\infty} = O\left(1\right)\alpha, \ (\theta = 1,2) \quad \text{as} \quad \alpha \to 0^+. \tag{3.18}$$

Moreover if $f \in L_{p,\nu} \cap \widetilde{\Lambda_{\beta}}$, $1 \leq p < \infty$, $0 < \beta \leq 1$, where $\widetilde{\Lambda_{\beta}}$ is the non-isotropic Lipschitz class defined as in (2.14), then

$$\left(A^{\alpha}_{\nu,\theta} f\right)\left(x_0,t_0\right) - f\left(x_0,t_0\right) = O\left(1\right)\alpha \quad (\theta = 1,2) \quad \text{as} \quad \alpha \to 0^+. \tag{3.19}$$

Proof. We will prove the theorem only for $\theta = 1$. In the case of $A^{\alpha}_{\nu,2}$, the statements are proved by the similar way (see Remark). Let $f \in \mathcal{L}_{p,\nu} \cap \Lambda_{\beta}$. Setting $l = f(x,t)$ in (3.3) we have

$$\left|A^{\alpha}_{\nu,1} f(x,t) - f(x,t)\right| \leq i_1\left(\alpha\right) + i_2\left(\alpha\right) + i_3\left(\alpha\right) \tag{3.20}$$

and using (3.8) and (3.12) we get

$$i_2(\alpha) \le \left(c_1(\delta)\|f\|_{\mathcal{L}_{p,\nu}} + c_2(\delta)\|f\|_\infty\right)\alpha \equiv c_3(\delta)\alpha, \tag{3.21}$$

$$i_3(\alpha) \le \left(c_4(\delta)\|f\|_{\mathcal{L}_{p,\nu}} + c_5(\delta)\|f\|_\infty\right)\alpha \equiv c_6(\delta)\alpha. \tag{3.22}$$

Now, let us estimate $i_1(\alpha)$. We have

$$\begin{aligned} i_1(\alpha) &\le \frac{1}{\Gamma(\alpha/2)}\int_0^\delta \tau^{(\alpha/2)-1}\int_{|y|<\sqrt{\delta}} W_\nu(y,\tau)\left\|T^{y,\tau}f(x,t)-f(x,t)\right\|_\infty dyd\tau \\ &+ \frac{\|f\|_\infty}{\Gamma(\alpha/2)}\int_0^\delta \tau^{(\alpha/2)-1}\left(1-e^{-\tau}\right)\int_{|y|<\sqrt{\delta}} W_\nu(y,\tau)\,dyd\tau. \end{aligned}$$

Taking into account that $1-e^{-\tau}=\tau+O(1)\tau^2$, as $\tau\to 0^+$ and

$$\left\|T^{y,\tau}f(x,t)-f(x,t)\right\|_\infty \le c_f\left(|y|^2+|\tau|\right)^{\beta/2},$$

we get

$$\begin{aligned} i_1(\alpha) &\le \frac{c}{\Gamma(\alpha/2)}\int_0^\delta \tau^{(\alpha/2)-1}\int_{\mathbb{R}_n^+} W_\nu(y,\tau)\left(|y|^2+|\tau|\right)^{\beta/2} dyd\tau \\ &+ \frac{c}{\Gamma(\alpha/2)}\int_0^\delta \tau^{(\alpha/2)-1+1}d\tau\int_{\mathbb{R}_n^+} W_\nu(y,\tau)dy \\ &\equiv i_1'(\alpha)+i_2'(\alpha). \end{aligned}$$

After changing y with $\sqrt{\tau}\ z$, a simple calculation leads to $i_1'(\alpha)=O(1)\alpha$ as $\alpha\to 0^+$. Further, using (3.10) we have

$$i_2'(\alpha)=\frac{c}{\Gamma(\alpha/2)}\int_0^\delta \tau^{(\alpha/2)}d\tau = O(1)\alpha \quad \text{as} \quad \alpha\to 0^+,$$

and therefore, $i_1(\alpha)=O(1)\alpha$ as $\alpha\to 0^+$. Now from (3.20), (3.21) and (3.22) it follows that $\|A^\alpha_{\nu,1}f(x,t)-f(x,t)\|_\infty = O(1)\alpha$ as $\alpha\to 0^+$. The proof of (3.19) can be carried out analogously, by replacing the expression $\|...\|_\infty$ with $|...|$. □

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