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ON THE RIESZ-DUNKL POTENTIALS

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Abstract. In this paper we obtain pointwise and integral estimates for Riesz potentials in terms of maximal and fractional maximal functions associated with the Dunkl operators on $\mathbb{R}^d$.

1. Introduction

We consider $\mathbb{R}^d$ with the Euclidean inner product $\langle .,. \rangle$ and norm $\|y\| := \sqrt{\langle y,y\rangle}$. For $\alpha \in \mathbb{R}^d \backslash \{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α:

$$\sigma_\alpha y := y - \frac{2\langle \alpha, y\rangle}{\|\alpha\|^2}\alpha.$$

A finite set $\Re \subset \mathbb{R}^d \backslash \{0\}$ is called a root system, if $\Re \cap \mathbb{R}.\alpha = \{-\alpha, \alpha\}$ and $\sigma_\alpha \Re = \Re$ for all $\alpha \in \Re$. We assume that it is normalized by $\|\alpha\|^2 = 2$ for all $\alpha \in \Re$. For a root system $\Re$, the reflections σ_α, $\alpha \in \Re$ generate a finite group $G \subset O(d)$, the reflection group associated with $\Re$. All reflections in G, correspond to suitable pairs of roots. For a given $\beta \in \mathbb{R}^d \backslash \bigcup_{\alpha\in\Re} H_\alpha$, we fix the positive subsystem $\Re_+ := \{\alpha \in \Re : \langle \alpha, \beta\rangle > 0\}$. Then for each $\alpha \in \Re$ either $\alpha \in \Re_+$ or $-\alpha \in \Re_+$.

Let $k : \Re \to \mathbb{C}$ be a multiplicity function on $\Re$ (i.e. a function which is constant on the orbits under the action of G). For abbreviation, we introduce the index:

$$\gamma = \gamma_k := \sum_{\alpha\in\Re_+} k(\alpha).$$

Throughout this paper, we will assume that the multiplicity is non-negative, that is $k(\alpha) \geq 0$ for all $\alpha \in \Re$. Moreover, let w_k denote the weight function:

$$w_k(y) := \prod_{\alpha\in\Re_+} |\langle \alpha, y\rangle|^{2k(\alpha)}, \quad y \in \mathbb{R}^d,$$

which is G-invariant and homogeneous of degree 2γ.

We denote by μ_k the measure on $\mathbb{R}^d$ given by $\mathrm{d}\mu_k(y) := w_k(y)\mathrm{d}y$, and we introduce the Mehta-type constant c_k, by

$$c_k := \Big[\int_{\mathbb{R}^d} e^{-\|y\|^2/2}\mathrm{d}\mu_k(y)\Big]^{-1}.$$

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The Dunkl operators $\mathcal{D}_j(k)$; $j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group G and multiplicity function k are given for a function f of class C^1 on $\mathbb{R}^d$, by

$$\mathcal{D}_j(k)f(y) := \frac{\partial}{\partial y_j} f(y) + \sum_{\alpha \in \Re_+} k(\alpha)\alpha_j \frac{f(y) - f(\sigma_\alpha y)}{\langle \alpha, y \rangle}.$$

For $y \in \mathbb{R}^d$, the initial problem $\mathcal{D}_j(k)u(., y)(x) = y_j u(x, y)$; $j = 1, ..., d$, with $u(0, y) = 1$ admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $E_k(x, y)$ and called Dunkl kernel (see e.g., [1] and [3]). This kernel has the Laplace-type representation [11]:

$$E_k(x, z) = \int_{\mathbb{R}^d} e^{\langle y, z \rangle} \mathrm{d}\Gamma_x(y); \quad x \in \mathbb{R}^d, \, z \in \mathbb{C}^d,$$

where $\langle y, z \rangle := \sum_{i=1}^d y_i z_i$ and Γ_x is a probability measure on $\mathbb{R}^d$, such that

$$\operatorname{supp}(\Gamma_x) \subset \{y \in \mathbb{R}^d \, : \, \|y\| \leq \|x\|\}.$$

We denote by $L^p(\mu_k)$, $1 \leq p \leq \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^p(\mu_k)} := \Big[\int_{\mathbb{R}^d} |f(y)|^p \mathrm{d}\mu_k(y) \Big]^{1/p} < \infty, \quad 1 \leq p < \infty,$$
$$\|f\|_{L^\infty(\mu_k)} := \operatorname*{ess\,sup}_{y \in \mathbb{R}^d} |f(y)| < \infty.$$

For $f \in L^1(\mu_k)$ the Dunkl transform is defined (see [2]) by

$$\mathcal{F}_k(f)(x) := c_k \int_{\mathbb{R}^d} E_k(-ix, y) f(y) \mathrm{d}\mu_k(y), \quad x \in \mathbb{R}^d.$$

The Dunkl transform allows us to define a generalized translation operator on $L^2(\mu_k)$ by setting $\mathcal{F}_k(\tau_x f)(y) = E_k(ix, y)\mathcal{F}_k(f)(y)$, $y \in \mathbb{R}^d$. It plays the role of the ordinary translation $\tau_x f = f(x + .)$ in $\mathbb{R}^d$, since the Euclidean Fourier transform satisfies $\mathcal{F}(\tau_x f)(y) = e^{ixy}\mathcal{F}(f)(y)$.

Let β be a real number such that $0 < \beta < 2\gamma + d$, the Dunkl-type Riesz potentials $I_{k,\beta} f$ are defined by (see [15]):

$$I_{k,\beta} f(x) := (d_{k,\beta})^{-1} \int_{\mathbb{R}^d} \frac{\tau_x f(y)}{\|y\|^{2\gamma+d-\beta}} \mathrm{d}\mu_k(y), \quad x \in \mathbb{R}^d,$$

where

$$d_{k,\beta} := 2^{-\gamma - \frac{d}{2} + \beta} \frac{\Gamma(\beta/2)}{\Gamma(\gamma + \frac{d-\beta}{2})}.$$

For $0 \leq \beta < 2\gamma + d$, the Dunkl-type fractional maximal functions $M_{k,\beta} f$ are defined by (see [10]):

$$M_{k,\beta} f(x) := \sup_{r > 0} \Big\{ \frac{1}{\mu_k(B_r)^{1 - \frac{\beta}{2\gamma + d}}} \int_{B_r} \tau_x |f|(y) \mathrm{d}\mu_k(y) \Big\}, \quad x \in \mathbb{R}^d,$$

where B_r denotes the ball with center the origin and radius r.

If $\beta = 0$, then $M_k = M_{k,0}$ is the Hardy-Littlewood-Paley function associated with the Dunkl operators on $\mathbb{R}^d$ (see [12, 13, 14]):

$$M_k f(x) := \sup_{r>0} \Big\{ \frac{1}{\mu_k(B_r)} \int_{B_r} \tau_x |f|(y) \mathrm{d}\mu_k(y) \Big\}, \quad x \in \mathbb{R}^d.$$

From [10, 14, 15], we have

$$\mu_k(B_r) = m_k r^{2\gamma+d},$$

where

$$m_k := \Big[c_k 2^{\gamma+d/2} \Gamma_k(\gamma + \frac{d}{2} + 1) \Big]^{-1}.$$

Note that the maximal function, associated with the Dunkl operator was first introduced and studied (L^p boundedness) by S. Thangavelu and Y. Xu in [14] 2005 and by Guliyev and Mammadov in [4] 2006 (on the real line) independently. Also the fractional maximal function and the Riesz potential associated with the Dunkl operator on the real line was first introduced and studied (($L^p; L^q$) boundedness) by Guliyev and Mammadov in [4] 2006, see also [5, 6, 7, 8, 9] and [15].

In this paper we study the pointwise estimates of Riesz potentials $I_{k,\beta}$ in terms of maximal and fractional maximal functions M_k and $M_{k,\beta}$. Throughout the paper c denotes a positive constant whose value may vary from line to line.

The results of this paper are given in the following theorems.

Theorem 1.1. (i) *Let* $0 < \beta < 2\gamma + d$. *For* $0 < \varepsilon < \min(\beta, 2\gamma + d - \beta)$,

$$I_{k,\beta}|f|(x) \le c\Big(M_{k,\beta-\varepsilon} f(x)\Big)^{1/2} \Big(M_{k,\beta+\varepsilon} f(x)\Big)^{1/2}, \quad x \in \mathbb{R}^d, \tag{1.1}$$

where $c = c(\varepsilon) > 0$ *is a constant independent of* f.

(ii) *Let* $0 < \beta < 2\gamma + d$. *For* $1 \le p < \lambda/\beta$,

$$I_{k,\beta}|f|(x) \le c\Big(M_{k,\frac{\lambda}{p}} f(x)\Big)^{\frac{\beta p}{\lambda}} \Big(M_k f(x)\Big)^{1-\frac{\beta p}{\lambda}}, \quad x \in \mathbb{R}^d, \tag{1.2}$$

where $c = c(p) > 0$ *is a constant independent of* f.

In the paper [9] 2007 by Mammadov, it was obtained the analogue of Welland's theorem for the Riesz potentials associated with the Dunkl operator on the real line. The Theorem 1.1 was the analog of Mammadov's theorem for the Riesz potentials associated with the Dunkl operator on multidimensional case.

Theorem 1.2. *Let* $0 < \beta < 2\gamma + d$. *For* $0 < \theta < 1$,

$$I_{k,\beta\theta}|f|(x) \le c\Big(I_{k,\beta}|f|(x)\Big)^{\theta} \Big(M_k f(x)\Big)^{1-\theta}, \quad x \in \mathbb{R}^d, \tag{1.3}$$

$$I_{k,\beta\theta}|f|(x) \le c\Big(M_{k,\beta} f(x)\Big)^{\theta} \Big(M_k f(x)\Big)^{1-\theta}, \quad x \in \mathbb{R}^d, \tag{1.4}$$

where $c = c(\theta) > 0$ *is a constant independent of* f.

Theorem 1.2 was proved by Guliyev and Mammadov in [4] 2006, see also the papers [5] and [7] for the Riesz potentials associated with the Dunkl operator on the real line.

2. Proofs of the main results

Proof of Theorem 1.1. For any $r > 0$, we have

$$I_{k,\beta}|f|(x) = (d_{k,\beta})^{-1}\Big[\int_{B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y) + \int_{\mathbb{R}^d\setminus B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y)\Big]. \quad (2.1)$$

(i) For $0 < \varepsilon < \beta$, we have

$$\begin{aligned}\int_{B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y) &= \sum_{j=0}^{\infty}\int_{B_{2^{-j}r}\setminus B_{2^{-j-1}r}} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y)\\ &\le \sum_{j=0}^{\infty}\frac{1}{(2^{-j-1}r)^{2\gamma+d-\beta}}\int_{B_{2^{-j}r}}\tau_x|f|(y)\mathrm{d}\mu_k(y)\\ &= 2^{2\gamma+d-\beta}r^{\varepsilon}\sum_{j=0}^{\infty}\frac{2^{-j\varepsilon}}{(2^{-j}r)^{2\gamma+d-\beta+\varepsilon}}\int_{B_{2^{-j}r}}\tau_x|f|(y)\mathrm{d}\mu_k(y)\\ &= 2^{2\gamma+d-\beta}(m_k)^{1-\frac{\beta-\varepsilon}{2\gamma+d}}r^{\varepsilon}\sum_{j=0}^{\infty}\frac{2^{-j\varepsilon}}{\mu_k(B_{2^{-j}r})^{1-\frac{\beta-\varepsilon}{2\gamma+d}}}\int_{B_{2^{-j}r}}\tau_x|f|(y)\mathrm{d}\mu_k(y).\end{aligned}$$

Thus,

$$\int_{B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}d\mu_k(y) \le c_1 r^{\varepsilon} M_{k,\beta-\varepsilon}f(x), \quad (2.2)$$

where

$$c_1 := \frac{(2^{2\gamma+d}m_k)^{1-\frac{\beta-\varepsilon}{2\gamma+d}}}{2^{\varepsilon}-1}.$$

On the other hand, for $0 < \varepsilon < 2\gamma + d - \beta$, we have

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y) &= \sum_{j=0}^{\infty}\int_{B_{2^{j+1}r}\setminus B_{2^{j}r}} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y)\\ &\le \sum_{j=0}^{\infty}\frac{1}{(2^{j}r)^{2\gamma+d-\beta}}\int_{B_{2^{j+1}r}}\tau_x|f|(y)\mathrm{d}\mu_k(y)\\ &= 2^{2\gamma+d-\beta-\varepsilon}r^{-\varepsilon}\sum_{j=0}^{\infty}\frac{2^{-j\varepsilon}}{(2^{j+1}r)^{2\gamma+d-\beta-\varepsilon}}\int_{B_{2^{j+1}r}}\tau_x|f|(y)\mathrm{d}\mu_k(y)\\ &= (2^{2\gamma+d}m_k)^{1-\frac{\beta+\varepsilon}{2\gamma+d}}r^{-\varepsilon}\sum_{j=0}^{\infty}\frac{2^{-j\varepsilon}}{\mu_k(B_{2^{j+1}r})^{1-\frac{\beta+\varepsilon}{2\gamma+d}}}\int_{B_{2^{j+1}r}}\tau_x|f|(y)\mathrm{d}\mu_k(y).\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^d\setminus B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y) \le c_2 r^{-\varepsilon} M_{k,\beta+\varepsilon}f(x), \quad (2.3)$$

where

$$c_2 := \frac{(2^{2\gamma+d}m_k)^{1-\frac{\beta+\varepsilon}{2\gamma+d}}}{1-2^{-\varepsilon}}.$$

Consequently, according to (2.1), (2.2) and (2.3), for any $0 < \varepsilon < \min(\beta, 2\gamma + d - \beta)$, we obtain

$$I_{k,\beta}|f|(x) \leq (d_{k,\beta})^{-1}\Big(c_1 r^{\varepsilon} M_{k,\beta-\varepsilon}f(x) + c_2 r^{-\varepsilon} M_{k,\beta+\varepsilon}f(x)\Big). \tag{2.4}$$

Minimizing (2.4) at $r = \left(\frac{c_2}{c_1}\frac{M_{k,\beta+\varepsilon}f(x)}{M_{k,\beta-\varepsilon}f(x)}\right)^{1/\varepsilon}$, we obtain (1.1) with $c = 2(d_{k,\beta})^{-1}\sqrt{c_1c_2}$.

(ii) Firstly, we have

$$\begin{aligned}
\int_{B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y) &= \sum_{j=0}^{\infty}\int_{B_{2^{-j}r}\setminus B_{2^{-j-1}r}} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y) \\
&\leq \sum_{j=0}^{\infty}\frac{1}{(2^{-j-1}r)^{2\gamma+d-\beta}}\int_{B_{2^{-j}r}} \tau_x|f|(y)\mathrm{d}\mu_k(y) \\
&= 2^{2\gamma+d-\beta}r^{\beta}\sum_{j=0}^{\infty}\frac{2^{-j\beta}}{(2^{-j}r)^{2\gamma+d}}\int_{B_{2^{-j}r}} \tau_x|f|(y)\mathrm{d}\mu_k(y) \\
&= 2^{2\gamma+d-\beta}m_k r^{\beta}\sum_{j=0}^{\infty}\frac{2^{-j\beta}}{\mu_k(B_{2^{-j}r})}\int_{B_{2^{-j}r}} \tau_x|f|(y)\mathrm{d}\mu_k(y).
\end{aligned}$$

Thus,

$$\int_{B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y) \leq c_3 r^{\beta} M_k f(x), \tag{2.5}$$

where

$$c_3 := \frac{2^{2\gamma+d}m_k}{2^{\beta}-1}.$$

Secondly, we have

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y) &= \sum_{j=0}^{\infty}\int_{B_{2^{j+1}r}\setminus B_{2^{j}r}} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y) \\
&\leq \sum_{j=0}^{\infty}\frac{1}{(2^{j}r)^{2\gamma+d-\beta}}\int_{B_{2^{j+1}r}} \tau_x|f|(y)\mathrm{d}\mu_k(y) \\
&= 2^{2\gamma+d-\frac{\lambda}{p}}r^{\beta-\frac{\lambda}{p}}\sum_{j=0}^{\infty}\frac{2^{j(\beta-\frac{\lambda}{p})}}{(2^{j+1}r)^{2\gamma+d-\frac{\lambda}{p}}}\int_{B_{2^{j+1}r}} \tau_x|f|(y)\mathrm{d}\mu_k(y) \\
&= (2^{2\gamma+d}m_k)^{1-\frac{\lambda}{p(2\gamma+d)}}r^{\beta-\frac{\lambda}{p}}\sum_{j=0}^{\infty}\frac{2^{j(\beta-\frac{\lambda}{p})}}{\mu_k(B_{2^{j+1}r})^{1-\frac{\lambda}{p(2\gamma+d)}}}\int_{B_{2^{j+1}r}} \tau_x|f|(y)\mathrm{d}\mu_k(y).
\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^d\setminus B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}}\mathrm{d}\mu_k(y) \leq c_4 r^{\beta-\frac{\lambda}{p}} M_{k,\frac{\lambda}{p}}f(x), \tag{2.6}$$

where

$$c_4 := \frac{(2^{2\gamma+d}m_k)^{1-\frac{\lambda}{p(2\gamma+d)}}}{1-2^{\beta-\frac{\lambda}{p}}}.$$

Consequently, according to (2.1), (2.5) and (2.6), we obtain

$$I_{k,\beta}|f|(x) \le (d_{k,\beta})^{-1}\Big(c_3 r^{\beta} M_k f(x) + c_4 r^{\beta-\frac{\lambda}{p}} M_{k,\frac{\lambda}{p}} f(x)\Big). \tag{2.7}$$

Minimizing (2.7) at $r = \Big(\frac{c_4}{c_3}(\frac{\lambda}{\beta p} - 1)\frac{M_{k,\frac{\lambda}{p}} f(x)}{M_k f(x)}\Big)^{p/\lambda}$, we obtain (1.2) with

$$c = (d_{k,\beta})^{-1}\lambda\Big(\frac{c_4}{\beta p}\Big)^{\frac{\beta p}{\lambda}}\Big(\frac{c_3}{\lambda-\beta p}\Big)^{1-\frac{\beta p}{\lambda}}.$$

□

Proof of Theorem 1.2. Firstly, we prove the inequality (1.3). Since $0 < \theta < 1$, then $\beta\theta - \beta < 0$ and $\|y\|^{\beta\theta-\beta} \le r^{\beta\theta-\beta}$ for any $y \in \mathbb{R}^d \backslash B_r$. Hence,

$$\int_{\mathbb{R}^d\backslash B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta\theta}} d\mu_k(y) \le r^{\beta\theta-\beta}\int_{\mathbb{R}^d\backslash B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}} \mathrm{d}\mu_k(y).$$

Then

$$\int_{\mathbb{R}^d\backslash B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta\theta}} \mathrm{d}\mu_k(y) \le d_{k,\beta} r^{\beta\theta-\beta} I_{k,\beta}|f|(x). \tag{2.8}$$

According to (2.1), (2.5) and (2.8), we have

$$I_{k,\beta\theta}|f|(x) \le (d_{k,\beta\theta})^{-1}\Big(c_3 r^{\beta\theta} M_k f(x) + d_{k,\beta} r^{\beta\theta-\beta} I_{k,\beta}|f|(x)\Big). \tag{2.9}$$

Minimizing (2.9) at $r = \Big(\frac{(1-\theta)d_{k,\beta}}{\theta c_3}\frac{I_{k,\beta}|f|(x)}{M_k f(x)}\Big)^{1/\beta}$, we obtain (1.3) with

$$c = (d_{k,\beta\theta})^{-1}\frac{c_3}{1-\theta}\Big(\frac{(1-\theta)d_{k,\beta}}{\theta c_3}\Big)^{\theta}.$$

Secondly, we prove the inequality (1.4). We have

$$\begin{aligned}
\int_{\mathbb{R}^d\backslash B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta\theta}} \mathrm{d}\mu_k(y) &= \sum_{j=0}^{\infty}\int_{B_{2^{j+1}r}\backslash B_{2^j r}} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta\theta}} \mathrm{d}\mu_k(y) \\
&\le \sum_{j=0}^{\infty}\frac{1}{(2^j r)^{2\gamma+d-\beta\theta}}\int_{B_{2^{j+1}r}} \tau_x|f|(y)\mathrm{d}\mu_k(y) \\
&= 2^{2\gamma+d-\beta} r^{\beta\theta-\beta}\sum_{j=0}^{\infty}\frac{2^{j(\beta\theta-\beta)}}{(2^j r)^{2\gamma+d-\beta}}\int_{B_{2^{j+1}r}} \tau_x|f|(y)\mathrm{d}\mu_k(y) \\
&= (2^{2\gamma+d}m_k)^{1-\frac{\beta}{2\gamma+d}} r^{\beta\theta-\beta}\sum_{j=0}^{\infty}\frac{2^{j(\beta\theta-\beta)}}{\mu_k(B_{2^{j+1}r})^{1-\frac{\beta}{2\gamma+d}}}\int_{B_{2^{j+1}r}} \tau_x|f|(y)\mathrm{d}\mu_k(y).
\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^d\backslash B_r} \frac{\tau_x|f|(y)}{\|y\|^{2\gamma+d-\beta}} \mathrm{d}\mu_k(y) \le c_5 r^{\beta\theta-\beta} M_{k,\beta} f(x), \tag{2.10}$$

where

$$c_5 := \frac{(2^{2\gamma+d}m_k)^{1-\frac{\beta}{2\gamma+d}}}{1-2^{\beta\theta-\beta}}.$$

Taking into account (2.1), (2.5) and (2.10), we have

$$I_{k,\beta}|f|(x) \le (d_{k,\beta\theta})^{-1}\Big(c_3 r^{\beta\theta} M_k f(x) + c_5 r^{\beta\theta-\beta} M_{k,\beta} f(x)\Big). \tag{2.11}$$

Minimizing (2.11) at $r = \left(\frac{(1-\theta)c_5}{\theta c_3}\frac{M_{k,\beta}f(x)}{M_k f(x)}\right)^{1/\beta}$, we obtain (1.4) with

$$c = (d_{k,\beta\theta})^{-1}\frac{c_3}{1-\theta}\Big(\frac{(1-\theta)c_5}{\theta c_3}\Big)^{\theta}.$$

□

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