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POINTWISE AND INTEGRAL ESTIMATES FOR THE RIESZ POTENTIALS ON THE LAGUERRE HYPERGROUP

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Abstract. In this paper we consider the generalized shift operator, generated by Laguerre hypergroup, by means of which the maximal function , fractional maximal function and Riesz potential are investigated. We proved pointwise and integral estimates for Riesz potential in terms maximal and fractional maximal function on the Laguerre hypergroup.

1. Introduction

The theory of boundedness of classical operators of the real analysis, such as the maximal operator, fractional maximal operator, Riesz potential and the singular integral operators, etc, have been extensively investigated in various function spaces. Results on weak and strong type inequalities for operators of this kind in Lebesgue spaces are classical and can be found for example in ([3], [12] and [14]). These boundedness extended to several function spaces which are generalizations of L_p spaces, for example, Orlicz spaces, Morrey spaces, Lorentz spaces, Herz spaces, etc.

In this paper we investigate the maximal function, fractional maximal function and Riesz potential using harmonic analysis on the Laguerre hypergroup $\mathbb{K}$ which can be seen as a deformation of the hypergroup of radial functions on the Heisenberg group (see [1, 2, 8] and [13]). We get pointwise and integral estimate for Riesz potential in terms maximal and fractional maximal function on $\mathbb{K}$.

The paper is organized as follows. In Section 2, we present some definitions and auxiliary results. In Section 3, we give polar coordinates in Laguerre hypergroup and some lemmas. The main result of the paper is the estimates for Riesz potential in terms maximal and fractional maximal function on the Laguerre hypergroup, established in Section 4.

Finally, we mention that, C will be always used to denote a suitable positive constant that is not necessarily the same in each occurrence.

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2. Preliminaries

For $(x,t)\in]0,\infty[\times\mathbb{R}$ and $\alpha\in[0,\infty[$, we consider the following partial differential operators, for $(x,t)\in]0,\infty[\times\mathbb{R}$ and $\alpha\in[0,\infty[$:

$$\begin{cases} D_1 = \frac{\partial}{\partial t}, \\ \\ D_2 = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}. \end{cases}$$

For $\alpha = n-1$, $n\in\mathbb{N}\setminus\{0\}$, the operator D_2 is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}_n$.

For $(\lambda,m)\in\mathbb{R}\times\mathbb{N}$, the initial problem

$$\begin{cases} D_1 u = i\lambda u, \\ \\ D_2 u = -4|\lambda|\left(m+\frac{\alpha+1}{2}\right)u; \\ u(0,0)=1,\ \frac{\partial u}{\partial x}(0,t)=0 \text{ for all } t\in\mathbb{R}, \end{cases}$$

has a unique solution $\varphi_{\lambda,m}(x,t)$ given by

$$\varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathcal{L}_m^{(\alpha)}\left(|\lambda|x^2\right),\ (x,t)\in\mathbb{K},$$

where $\mathcal{L}_m^{(\alpha)}$ is the Laguerre functions defined on $\mathbb{R}_+$ by

$$\mathcal{L}_m^{(\alpha)}(x) = e^{-x/2}L_m^{(\alpha)}(x)/L_m^{(\alpha)}(0)$$

and $L_m^{(\alpha)}$ is the Laguerre polynomial of degree m and order α (see [1]).

Let $\alpha\geq 0$ be a fixed number and m_α be the weighted Lebesgue measure on $\mathbb{K}$, given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1}dxdt}{\pi\Gamma(\alpha+1)}.$$

We denote by $L_p(\mathbb{K})$, $1\leq p\leq\infty$, the spaces of complex-valued functions f, measurable on $\mathbb{K}$ such that

$$\|f\|_{L_p(\mathbb{K})} = \left(\int_{\hat{\mathbb{K}}} |f(x,t)|^p\, dm_\alpha(x,t)\right)^{1/p} < \infty \text{ if } 1\leq p<\infty$$

and

$$\|f\|_{L_\infty(\mathbb{K})} = ess\sup_{(x,t)\in\mathbb{K}}|f(x,t)| \text{ if } p=\infty.$$

For $1\leq p<\infty$ we denote by $WL_p(\hat{\mathbb{K}})$, the weak $L_p(\hat{\mathbb{K}})$ spaces defined as the set of locally integrable functions $f(\lambda,m)$, $(\lambda,m)\in\hat{\mathbb{K}}$, with the finite norm

$$\|f\|_{WL_p(\hat{\mathbb{K}})} = \sup_{r>0} r\left(m_\alpha\left\{(x,t)\in\hat{\mathbb{K}}\ :\ |f(x,t)|>r\right\}\right)^{1/p}.$$

Let $|(x,t)|_{\mathbb{K}} = (x^4+4t^2)^{1/4}$ be the homogeneous norm of $(x,t)\in\mathbb{K}$. We will denote by $\delta_r(x,t) = (rx, r^2t)$, for $r>0$, the dilation of $(x,t)\in\mathbb{K}$, and by $B_r(x,t)$ the ball centered at (x,t) with radius r, i.e., the set of $B_r(x,t) = \{(y,s)\in\mathbb{K} : |(x-y,t-s)|_{\mathbb{K}} < r\}$, and by B_r the ball $B_r(0,0)$.

We denote by

$$f_r(x,t) = r^{-(2\alpha+4)}f\left(\delta_{\frac{1}{r}}(x,t)\right)$$

the dilated of the function f defined on $\mathbb{K}$ preserving the mean of f with respect to the measure dm_α, in the sense that

$$\int_{\mathbb{K}} f_r(x,t)dm_\alpha(x,t) = \int_{\mathbb{K}} f(x,t)dm_\alpha(x,t), \ \ \forall r>0 \text{ and } \ f \in L_1(\mathbb{K}).$$

The Fourier-Laguerre transform $\mathcal{F}$ is defined for $f \in L_1(\mathbb{K})$ by:

$$\mathcal{F}(f)(\lambda,m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t)f(x,t)dm_\alpha(x,t)$$

and we have (see [1, 9])

$$\|\mathcal{F}(f)\|_{L_\infty(\mathbb{K})} \le \|f\|_{L_1(\mathbb{K})}.$$

For $(x,t),(y,s) \in \mathbb{K}$ and $\theta \in [0,2\pi[$, $r \in [0,1[$ let

$$((x,t),(y,s))_{\theta,r} = \left(\left(x^2+y^2+2xyr\cos\theta\right)^{1/2}, t+s+xyr\sin\theta\right).$$

The generalized translation operator $T^{(\alpha)}_{(x,t)}$ defined on the Laguerre hypergroup is given for a suitable function f by

$$T^{(\alpha)}_{(x,t)}f(y,s) = \begin{cases} \frac{1}{2\pi}\int_0^{2\pi} f\left(((x,t),(y,s))_{\theta,1}\right)d\theta, \text{ if } \alpha=0, \\ \frac{\alpha}{\pi}\int_0^1 \left(\int_0^{2\pi} f\left(((x,t),(y,s))_{\theta,r}\right)d\theta\right) r(1-r^2)^{\alpha-1}dr, \text{ if } \alpha>0. \end{cases}$$

The satisfy the following properties (see [1, 9]):

$$T^{(\alpha)}_{(x,t)}f(y,s) = T^{(\alpha)}_{(y,s)}f(x,t), \ T^{(\alpha)}_{(0,0)}f(y,s) = f(y,s),$$

$$\|T^{(\alpha)}_{(x,t)}f\|_{L_p(\mathbb{K})} \le \|f\|_{L_p(\mathbb{K})} \text{ for all } f \in L_p(\mathbb{K}), \ 1 \le p \le \infty, \tag{2.1}$$

$$\mathcal{F}\big(T^{(\alpha)}_{(x,t)}f\big)(\lambda,m) = \mathcal{F}(f)(\lambda,m)\ \varphi_{\lambda,m}(x,t).$$

In [1] the translation operator $T^{(\alpha)}_{(x,t)}$ is defined by

$$T^{(\alpha)}_{(x,t)}f(y,s) = \int_{\mathbb{K}} f(z,v)W_\alpha((x,t),(y,s),(z,v))z^{2\alpha+1}dzdv,$$

where $dzdv$ is the Lebesgue measure on $\mathbb{K}$, and W_α is an appropriate kernel satisfying

$$\int_{\mathbb{K}} W_\alpha((x,t),(y,s),(z,v))z^{2\alpha+1}dzdv = 1.$$

For all $(\lambda,m) \in \mathbb{R}\times\mathbb{N}$, the function $\varphi_{\lambda,m}(x,t)$ satisfies the following product formula

$$\varphi_{\lambda,m}(x,t)\,\varphi_{\lambda,m}(y,s) = T^{(\alpha)}_{(x,t)}\varphi_{\lambda,m}(y,s).$$

Using the generalized translation operators $T^{(\alpha)}_{(x,t)}$, $(x,t)\in\mathbb{K}$, we define a generalized convolution product $*$ on $\mathbb{K}$ by

$$\left(\delta_{(x,t)} * \delta_{(y,s)}\right)(f) = T^{(\alpha)}_{(x,t)}f(y,s),$$

where $\delta_{(x,t)}$ is the Dirac measure at (x,t).

The convolution product on the space $M_b(\mathbb{K})$ of bounded Radon measures on $\mathbb{K}$ by

$$(\mu * \nu)(f) = \int_{\mathbb{K}\times\mathbb{K}} T^{(\alpha)}_{(x,t)} f(y,s)\, d\mu(x,t)\, d\nu(y,s).$$

When $\mu = h\cdot m_\alpha$ and $\nu = g\cdot m_\alpha$, with h and g in the space $L_1(\mathbb{K})$ of integrable functions on $\mathbb{K}$ with respect to the measure $dm_\alpha(x,t)$, we have

$$\mu * \nu = (h * \check{g})\cdot m_\alpha, \text{ with } \check{g}(y,s) = g(y,-s),$$

where $f * g$ is the convolution product of f and g defined by:

$$(f * g)(x,t) = \int_{\mathbb{K}} T^{(\alpha)}_{(x,t)} f(y,s)\, g(y,-s)\, dm_\alpha(y,s), \text{ for all } (x,t)\in\mathbb{K}.$$

$(M_b(\mathbb{K}), *, i)$ is an involutive Banach algebra, where i is the involution on $\mathbb{K}$ given by $i(x,t) = (x,-t)$ and the convolution product $*$ satisfies all the conditions of Jewett (see [7]). Hence $(\mathbb{K}, *, i)$ is a hypergroup in the sense of Jewett (see [1, 7]) and the functions $\varphi_{\lambda,m}$ are characters of $\mathbb{K}$. If $\beta = n-1$ is a nonnegative integer, then the Laguerre hypergroup $\mathbb{K}$ can be identified with the hypergroup of radial functions on the Heisenberg group $\mathcal{H}_n$.

3. Polar coordinates in Laguerre hypergroup and some lemmas

Let $\Sigma = \Sigma_2$ be the unit sphere in $\mathbb{K}$. We denote by ω_2 the surface area of Σ and by Ω_2 its volume (see [2, 4]). For $\xi = (x,t)\in\mathbb{K}$, consider the transformation given by

$$x = r(\cos\varphi)^{1/2},\ t = r^2\sin\varphi,$$

where $-\pi/2 \le \varphi \le \pi/2$, $r = |\xi|_{\mathbb{K}}$ and $\xi' = ((\cos\varphi)^{1/2}, \sin\varphi)\in\Sigma$.

The Jacobian of the above transformation is $r^{2\alpha+3}(\cos\varphi)^\alpha$, if f is integrable in $\mathbb{K}$, then

$$\int_{\mathbb{K}} f(x,t)\, dm_\alpha(x,t) = \frac{1}{2\pi\Gamma(\alpha+1)}\int_{-\pi/2}^{\pi/2}\int_0^\infty f\left(r(\cos\varphi)^{1/2}, r^2\sin\varphi\right) r^{2\alpha+3}(\cos\varphi)^\alpha dr d\varphi.$$

We write

$$\frac{1}{2\pi\Gamma(\alpha+1)}\int_{-\pi/2}^{\pi/2}(\cos\varphi)^\alpha d\varphi = \int_\Sigma d\xi',$$

and thus

$$\int_{\mathbb{K}} f(x,t)\, dm_\alpha(x,t) = \int_\Sigma\int_0^\infty r^{2\alpha+3} f(\delta_r\xi')\, dr d\xi'. \tag{3.1}$$

Here $d\xi'$ is the surface area element on Σ.

Lemma 3.1. [2, 4] *The following equalities are valid*

$$\omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{2\sqrt{\pi}\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)},$$

and

$$\Omega_2 = \frac{\Gamma(\frac{\alpha+1}{2})}{4\sqrt{\pi}(\alpha+2)\Gamma(\alpha+1)\Gamma(\frac{\alpha}{2}+1)}.$$

Note that for any $(x,t) \in \mathbb{K}$ and $r > 0$, the area of the sphere $S_r(x,t)$ of center (x,t) and radius r is equal to $r^{2\alpha+3}\omega_2$ and its volume is equal to $r^{2\alpha+4}\Omega_2 = r^{2\alpha+4}\frac{\omega_2}{2\alpha+4}$.

Lemma 3.2. [2, 4] *The function $f(x,t) = |(x,t)|_{\mathbb{K}}^{\lambda}$ is integrable in any neighborhood of the origin if and only if $\lambda > -2\alpha-4$, and f is integrable in the complement of any neighborhood of the origin if and only if $\lambda < -2\alpha-4$.*

4. Estimates of fractional integrals on the Laguerre hypergroup

Now on the Laguerre hypergroup we define the fractional maximal function by (see [6])

$$M_\beta f(x,t) = \sup_{r>0} \big(m_\alpha B_r\big)^{\frac{\beta}{2\alpha+4}-1} \int_{B_r} T_{(x,t)}^{(\alpha)} |f(y,s)|\, dm_\alpha(y,s), \; 0 \le \beta < 2\alpha+4.$$

If $\beta = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator on the Laguerre hypergroup (see [2]).

In [2] the following theorem was proved.

Theorem 4.1. *1. If $f \in L_1(\mathbb{K})$, then for every $\delta > 0$*

$$m_\alpha\{(x,t) \in \mathbb{K} : Mf(x,t) > \delta\} \le \frac{C}{\delta} \int_{\mathbb{K}} |f(x,t)| dm_\alpha(x,t),$$

where $C > 0$ is independent of f.

2. If $f \in L_p(\mathbb{K})$, $1 < p \le \infty$, then $Mf \in L_p(\mathbb{K})$ and

$$\|Mf\|_{L_p(\mathbb{K})} \le C\|f\|_{L_p(\mathbb{K})},$$

where $C > 0$ is independent of f.

For the fractional maximal operator M_β the following theorem is valid (see [6]).

Theorem 4.2. *Let $0 < \beta < 2\alpha+4$, $\frac{1}{p} - \frac{1}{q} = \frac{\beta}{2\alpha+4}$, $1 \le p \le \frac{2\alpha+4}{\beta}$.*

1) If $f \in L_1(K)$, then for all $\theta > 0$

$$\int\limits_{\{(x,t)\in K : M_\beta f(x,t) > \theta\}} dm_\alpha(x,t) \le \left(\frac{C}{\theta}\int_K |f(x,t)| dm_\alpha(x,t)\right)^q, \qquad (4.1)$$

where C is independent of f.

2) Let $1 < p < \frac{2\alpha+4}{\beta}$, $f \in L_p(K)$, then $M_\beta f \in L_q(K)$ and

$$\left(\int_K (M_\beta f(x,t))^q dm_\alpha(x,t)\right)^{\frac{1}{q}} \le C\left(\int_K |f(x,t)|^p dm_\alpha(x,t)\right)^{\frac{1}{p}}. \qquad (4.2)$$

where C is independent of f.

3) Let $p = \frac{2\alpha+4}{\beta}$, $f \in L_p(K)$, then $M_\beta f \in L_\infty(K)$ and

$$\sup_{(x,t)\in\mathbb{K}} M_\beta f(x,t) \le C\left(\int_K |f(x,t)|^p dm_\alpha(x,t)\right)^{\frac{1}{p}}. \qquad (4.3)$$

where C is independent of f.

Now we define the Riesz potential on the Laguerre hypergroup by

$$I_\beta f(x,t) = \sup_{r>0}(m_\alpha B_r)^{\frac{\beta}{2\alpha+4}-1}\int_{B_r} T^{(\alpha)}_{(x,t)}|(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4} f(y,s)dm_\alpha(y,s),$$

where $0<\beta<2\alpha+4$

In [5] the following theorems is proved.

Theorem 4.3. *Let $0<\beta<2\alpha+4$, $\frac{1}{p}-\frac{1}{q}=\frac{\beta}{2\alpha+4}$, $1\le p<\frac{2\alpha+4}{\beta}$.*
1) If $f\in L_1(K)$, then for all $\theta>0$

$$\int\limits_{\{(x,t)\in\mathbb{K}:I_\beta f(x,t)>\theta\}} dm_\alpha(x,t) \le \left(\frac{C}{\theta}\int_{\mathbb{K}}|f(x,t)|dm_\alpha(x,t)\right)^q, \tag{4.4}$$

where C is independent of f.
2) Let $1<p<\frac{2\alpha+4}{\beta}$, $f\in L_p(\mathbb{K})$, then $I_\beta f\in L_q(\mathbb{K})$ and

$$\left(\int_K (I_\beta f(x,t))^q dm_\alpha(x,t)\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{K}}|f(x,t)|^p dm_\alpha(x,t)\right)^{\frac{1}{p}}. \tag{4.5}$$

where C is independent of f.

The following theorems was proved in [10].

Theorem 4.4. *Let $0<\beta<2\alpha+4$, $1\le p<\frac{\lambda}{\beta}$. Then for any locally summable function f exists the positive numbers C_1 and C_2, such that for every $r>0$ and $(x,t)\in\mathbb{K}$ the following inequality is valid:*

$$I_\beta|f|(x,t)\le C_1\ r^\beta(Mf)(x,t)+C_2\ r^{\beta-\frac{\lambda}{p}}(M_{\frac{\lambda}{p}}f)(x,t), \tag{4.6}$$

Theorem 4.5. *Let $0<\beta<\lambda$, $1<p<\frac{\lambda}{\beta}$, $1\le r\le\infty$, $\frac{1}{q}=\frac{1}{p}-\frac{\beta}{\lambda}+\frac{\beta p}{\lambda r}$. Then for any $f\in L_p(\mathbb{K})$ and $M_{\frac{\lambda}{p}}f\in L_r(\mathbb{K})$ the following estimation is valid:*

$$\|I_\beta f\|_{L_q(\mathbb{K})}\le C_4\|M_{\frac{\lambda}{p}}f\|_{L_r(\mathbb{K})}^{\frac{\beta p}{\lambda}}\|f\|_{L_p(\mathbb{K})}^{1-\frac{\beta p}{\lambda}}. \tag{4.7}$$

where $C>0$ is independent of function f.

The following theorems is our main result in which we obtain pointwise and integral estimates for Riesz potentials in terms maximal and fractional maximal functions.

Theorem 4.6. *Let $0<\beta<2\alpha+4$, $1\le p<\frac{2\alpha+4}{\beta}$ and $f\in L_p(\mathbb{K})$. Then for any $(x,t)\in\mathbb{K}$ the following estimation is valid*

$$|I_\beta f(x,t)|\le(C_1+C_3)\|f\|_{L_p(\mathbb{K})}^{\frac{\beta p}{2\alpha+4}}\ (Mf(x,t))^{1-\frac{\beta p}{2\alpha+4}},$$

where $C_1=\frac{2^{2\alpha+4-\beta}}{1-2^{-\beta}}$ and $C_3=\left(\frac{\Omega_2}{p'(\frac{1}{p}-\frac{\beta}{2\alpha+2})}\right)^{\frac{1}{p'}}$.

Proof. Let r be an arbitrary positive real number. We write the integral as the sum of two integrals:

$$\begin{aligned} I_\beta|f|(x,t) &= \int_{B_r} |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4} T^{(\alpha)}_{(y,s)} |f(x,t)|\, dm_\alpha(y,s) \\ &+ \int_{\complement_{B_r}} |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4} T^{(\alpha)}_{(y,s)} |f(x,t)|\, dm_\alpha(y,s) \\ &:= J_{1,r}(x,t) + J_{2,r}(x,t). \end{aligned}$$

Firstly we estimate $J_{1,r}(x,t)$. Summarizing on all $k \geq 0$, we have

$$\begin{aligned} J_{1,r}(x,t) &\leq \int_{B_r} |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4} T^{(\alpha)}_{(y,s)} |f(x,t)| dm_\alpha(y,s) \\ &= \sum_{k=0}^{\infty} \int_{B_{2^{-k}r} \setminus B_{2^{-k-1}r}} |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4} T^{(\alpha)}_{(y,s)} |f(x,t)| dm_\alpha(y,s) \\ &\leq \sum_{k=0}^{\infty} \left(2^{-k-1} r\right)^{\beta-2\alpha-4} \int_{B_{2^{-k}r}} T^{(\alpha)}_{(y,s)} |f(x,t)| dm_\alpha(y,s) \\ &= 2^{2\alpha+4-\beta} \sum_{k=0}^{\infty} 2^{-k\beta} (2^{-k} r)^{-2\alpha-4} \int_{B_{2^{-k}r}} T^{(\alpha)}_{(y,s)} |f(x,t)| dm_\alpha(y,s) \\ &\leq 2^{2\alpha+4-\beta} r^\beta Mf(x,t) \sum_{k=0}^{\infty} 2^{-k\beta} \leq C_1 r^\beta Mf(x,t), \end{aligned} \tag{4.8}$$

where $C_1 = \frac{2^{2\alpha+4-\beta}}{1-2^{-\beta}}$.

Evaluation $J_{2,r}(x,t)$ obtained in Hölder inequality

$$\begin{aligned} |J_{2,r}(x,t)| &\leq \left(\int\limits_{\mathbb{K}\setminus B_r} \left(T^{(\alpha)}_{(x,t)} |f(y,s)| \right)^p dm_\alpha(y,s) \right)^{\frac{1}{p}} \\ &\times \left(\int\limits_{\mathbb{K}\setminus B_r} |(y,s)|_{\mathbb{K}}^{(\beta-2\alpha-4)p'} \; dm_\alpha(y,s) \right)^{\frac{1}{p'}} \end{aligned}$$

$$\begin{aligned} &\leq \left\| T^{(\alpha)}_{(x,t)} f \right\|_{L_p(\mathbb{K})} \left(\int\limits_{\mathbb{K}\setminus B_r} |(y,s)|_{\mathbb{K}}^{(\beta-2\alpha-4)p'} \; dm_\alpha(y,s) \right)^{\frac{1}{p'}} \\ &\leq \|f\|_{L_p(\mathbb{K})} \left(\int\limits_{\mathbb{K}\setminus B_r} |(y,s)|_{\mathbb{K}}^{(\beta-2\alpha-4)p'} \; dm_\alpha(y,s) \right)^{\frac{1}{p'}}. \end{aligned}$$

To pass coordinates sphere we have

$$\left(\int\limits_{\mathbb{K}\backslash B_r} |(y,s)|_{\mathbb{K}}^{(\beta-2\alpha-4)p'} \, dm_\alpha(y,s) \right)^{\frac{1}{p'}}$$

$$= \left(\int\limits_{\Sigma} \int\limits_{r}^{\infty} t^{2\alpha+3+(\beta-2\alpha-4)p'} \, dtd\xi' \right)^{\frac{1}{p'}} = C_3 r^{\beta-\frac{2\alpha+4}{p}}.$$

Consequently,

$$|J_{2,r}(x,t)| \leq C_3 r^{\beta-\frac{2\alpha+4}{p}} \|f\|_{L_p(\mathbb{K})}. \tag{4.9}$$

Thus from (4.8) and (4.9), we have

$$|I_\beta f|(x,t) \leq C_1 r^\beta Mf(x,t) + C_3 r^{\beta-\frac{2\alpha+4}{p}} \|f\|_{L_p(\mathbb{K})}.$$

Taking

$$r = \left[(Mf(x,t))^{-1} \|f\|_{L_p(\mathbb{K})} \right]^{\frac{p}{2\alpha+4}}$$

we obtain

$$|I_\beta f|(x,t) \leq (C_1 + C_3) \|f\|_{L_p(\mathbb{K})}^{\frac{\beta p}{2\alpha+4}} \; (Mf(x,t))^{1-\frac{\beta p}{2\alpha+4}}.$$

The Theorem 4.6 is proven.

□

Theorem 4.7. *Let* $0 < \beta < 2\alpha + 4$. *Then for any measurable functions* $f \geq 0$ *and* $0 < \theta < 1$ *for every* $(x,t) \in \mathbb{K}$ *we obtained following estimation*

$$I_{\beta\theta} f(x,t) \leq (C_4 + 1) \left(I_\beta f(x,t) \right)^\theta \left(Mf(x,t) \right)^{1-\theta}, \tag{4.10}$$

$$I_{\beta\theta} f(x,t) \leq (C_4 + C_5) \left(M_\beta f(x,t) \right)^\theta \left(Mf(x,t) \right)^{1-\theta}, \tag{4.11}$$

where

$$C_4 = \frac{2^{2\alpha+4}}{2^{\beta\theta}-1} \, \Omega_2,$$

$$C_5 = \frac{2^{2\alpha+4-\beta}}{1-2^{\beta\theta-\beta}} \, \Omega_2^{1-\frac{\beta}{2\alpha+4}}.$$

Proof. We obtained

$$I_{\beta\theta} f(x,t) = \int\limits_{\mathbb{K}} T_{(x,t)}^{(\alpha)} f(y,s) |(y,s)|_{\mathbb{K}}^{\beta\theta-2\alpha-4} dm_\alpha(y,s)$$

$$= \left(\int\limits_{B_r} + \int\limits_{\mathbb{K}\backslash B_r} \right) T_{(x,t)}^{(\alpha)} f(y,s) |(y,s)|_{\mathbb{K}}^{\beta\theta-2\alpha-4} dm_\alpha(y,s)$$

$$= I_1(x,t) + I_2(x,t).$$

First proof (4.10). We estimate $I_2(x,t)$. For $\beta > 0$, $0 < \theta < 1$ we get $\beta\theta - \beta < 0$ and $|(y,s)|_{\mathbb{K}}^{\beta\theta-\beta} \leq r^{\beta\theta-\beta}$ for every $(y,s) \in \mathbb{K}\backslash B_r$. Therefore

$$
\begin{aligned}
I_2(x,t) &= \int\limits_{\mathbb{K}\setminus B_r} T^{(\alpha)}_{(x,t)} f(y,s) |(y,s)|_{\mathbb{K}}^{\beta\theta-2\alpha-4} dm_\alpha(y,s) \\
&\le r^{\beta\theta-\beta} \int\limits_{\mathbb{K}\setminus B_r} T^{(\alpha)}_{(x,t)} f(y,s) |(y,s)|_{\mathbb{K}}^{\beta-2\alpha-4} dm_\alpha(y,s) \\
&\le r^{\beta\theta-\beta} I_\beta f(x,t).
\end{aligned} \tag{4.12}
$$

From inequalities (4.8) and (4.12) we get

$$I_1(x,t) \le C_4 r^{\beta\theta} M f(x,t), \tag{4.13}$$

$$I_2(x,t) \le r^{\beta\theta-\beta} I_\beta f(x,t). \tag{4.14}$$

Therefore from condition (4.13) and (4.14), we have

$$I_{\beta\theta} f(x,t) \le C_4 r^{\beta\theta}\; Mf(x,t) + r^{\beta\theta-\beta}\; I_\beta f(x,t).$$

Taking $r = \left[(Mf(x,t))^{-1} I_\beta f(x,t) \right]^{\frac{1}{\beta}}$ we have

$$I_{\beta\theta} f(x,t) \le (C_4+1) \left(I_\beta f(x,t) \right)^\theta \; \left(Mf(x,t) \right)^{1-\theta}.$$

Now to proof the inequality (4.11) we consider $I_2(x,t)$. Summarizing on all $j > 0$, we have

$$
\begin{aligned}
I_2(x,t) &\le \sum_{j=0}^\infty \int\limits_{B_{2^{j+1}r}\setminus B_{2^j r}} T^{(\alpha)}_{(x,t)} f(y,s) |(y,s)|_{\mathbb{K}}^{\beta\theta-2\alpha-4} dm_\alpha(y,s) \\
&\le \sum_{j=0}^\infty (2^j r)^{\beta\theta-2\alpha-4} \int\limits_{B_{2^{j+1}r}} T^{(\alpha)}_{(x,t)} f(y,s) dm_\alpha(y,s) \\
&\le 2^{2\alpha+4-\beta} \Omega_2^{1-\frac{\beta}{2\alpha+4}}\; r^{\beta\theta-\beta}\; M_\beta f(x,t) \sum_{j=0}^\infty 2^{(\beta\theta-\beta)j} \\
&\le C_5\; r^{\beta\theta-\beta}\; M_\beta f(x,t).
\end{aligned}
$$

Consequently,

$$I_2(x,t) \le C_5\; r^{\beta\theta-\beta}\; M_\beta f(x,t). \tag{4.15}$$

Thus from (4.13) and (4.15) we get

$$I_{\beta\theta} f(x,t) \le C_4\; r^{\beta\theta}\; Mf(x,t) + C_5\; r^{\beta\theta-\beta}\; M_\beta f(x,t).$$

Taking $r = \left[(Mf(x,t))^{-1}\; M_\beta f(x,t) \right]^{\frac{1}{\beta}}$, we have

$$I_{\beta\theta} f(x,t) \le (C_4+C_5)\; \left(M_\beta f(x,t) \right)^\theta \; \left(Mf(x,t) \right)^{1-\theta}.$$

Thus, the Theorem 4.7 was proved. □

Theorem 4.8. *Let* $0<\beta<2\alpha+4$, $f\in L_p(\mathbb{K})$, $1<p<\frac{2\alpha+4}{\beta}$. *Then*

$$\|I_{\beta\theta}f\|_{L_r(\mathbb{K})}\le (C_4+1)A_p^{1-\theta}\ \|I_\beta|f|\|^\theta_{L_q(\mathbb{K})}\ \|f\|^{1-\theta}_{L_p(\mathbb{K})} \tag{4.16}$$

and

$$\|I_{\beta\theta}f\|_{L_r(\mathbb{K})}\le (C_4+C_5)A_p^{1-\theta}\ \|M_\beta f\|^\theta_{L_q(\mathbb{K})}\ \|f\|^{1-\theta}_{L_p(\mathbb{K})}, \tag{4.17}$$

where $0<\theta<1$, $0<q\le\infty$, $\frac{1}{r}=\frac{\theta}{q}+\frac{1-\theta}{p}$, *and the constants* C_4, C_5 *are as in Theorem 4.7.*

Proof. Firstly we proof of the inequality (4.16). So that inequality (4.17) proved similar levels.

From condition (4.10) and Hölder inequalities we obtain

$$\begin{aligned}\|I_{\beta\theta}f\|_{L_r(\mathbb{K})}&\le (C_4+1)\|\,(I_\beta|f|)^\theta\,(Mf)^{1-\theta}\,\|_{L_r(\mathbb{K})}\\ &\le (C_4+1)\|\,(I_\beta|f|)^\theta\,\|_{L_{r\tau'}(\mathbb{K})}\|\,(Mf)^{1-\theta}\,\|_{L_{r\tau}(\mathbb{K})}.\end{aligned}$$

Denote

$$p=(1-\theta)r\tau,\ q=\theta r\tau',$$

where $\tau'=\frac{\tau}{\tau-1}$.

Then, it is clear that

$$\frac{1}{r\tau}=\frac{1-\theta}{p},\ \text{and}\ \frac{1}{r\tau'}=\frac{\theta}{q}.$$

We have

$$\|I_{\beta\theta}f\|_{L_r(\mathbb{K})}\le (C_4+1)\|I_\beta|f|\|^\theta_{L_q(\mathbb{K})}\|Mf\|^{1-\theta}_{L_p(\mathbb{K})}.$$

From the last inequalities and the Theorem 4.5 we get

$$\|I_{\beta\theta}f\|_{L_r(\mathbb{K})}\le (C_4+1)A_p^{1-\theta}\ \|\ I_\beta|f|\ \|^\theta_{L_q(\mathbb{K})}\ \|f\|^{1-\theta}_{L_p(\mathbb{K})}.$$

Therefore the proof of the Theorem 4.8 is completed. □

Remark 4.1. Note that, the theorems 4.4-4.7 were proved in [11].

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