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LOCAL AND GLOBAL BIFURCATION FOR SOME NONLINEARIZABLE EIGENVALUE PROBLEMS

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Abstract. In this paper we investigate the structure of the set of solutions for a wide class of nonlinear eigenvalue problems in Banach space with non differentiable by Frechet nonlinearities. We give a generalization of the classical theorem on the global bifurcation of the eigenvalues of odd multiplicity and prove the existence of global continua bifurcating from intervals of the line of trivial solutions.

1. Introduction

Let E be an real Banach space with norm denoted by $||\cdot||$, and $L : D(L) \to E$ be linear operator with compact resolvent, i.e. $(L - \lambda I)^{-1}$ is compact for some (and hence for all) λ not belonging to the spectrum $\sigma(L)$ of L, and $D(L)$ is dense in$E$.

We consider the nonlinear eigenvalue problem

$$Lu = \lambda u + F(\lambda, u) + G(\lambda, u), \tag{1.1}$$

where $F : \mathbb{R} \times E \to E$ is a continuous operator mapping bounded sets onto bounded sets and $F(\lambda, 0) = 0$ for any $\lambda \in \mathbb{R}$, and $G : \mathbb{R} \times E \to E$ is a continuous operator satisfying the following condition: $G(\lambda, u) = o(||u||)$ as $u \to 0$ uniformly with respect to $\lambda \in \Lambda$, where Λ is a finite interval of the real axis $\mathbb{R}$. As norm in $\mathbb{R} \times E$, we take $||(\lambda, u)|| = \left\{|\lambda|^2 + ||u||^2\right\}^{1/2}$.

Note that, due to the assumption on L, every eigenvalue of L is necessarily isolated and of finite multiplicity, and the whole spectrum $\sigma(L)$ consists of only such points.

It is well known (see [4, Ch. 4]) that in the case when $F \equiv 0$ each point of the form $(\mu, 0)$, where μ is a eigenvalue of of odd algebraic multiplicity of operator L, is a bifurcation point of the nonlinear problem (1.1). In addition, this bifurcation point corresponds to a continuous branch of nontrivial solutions. In [9] shows that there exists a maximal connected set (continua) of nontrivial solutions of (1.1) bifurcating from $(\mu, 0)$ which either is unbounded in $\mathbb{R} \times E$ or must also bifurcate from another point $(\tilde{\mu}, 0)$, where $\tilde{\mu} \neq \mu$ is a eigenvalue of L. This shows that bifurcation from eigenvalues of odd multiplicity is a global rather than a local phenomenon. He also obtains stronger results for bifurcation from a eigenvalue of

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multiplicity 1. These results are strengthened in [2], where it was established that this continuum decomposes into two subcontinua which either are unbounded in $\mathbb{R} \times E$ or are not intersect outside of the point $(\mu, 0)$.

In the present paper, we give generalization of the result in [8] to the case where F is not necessarily differentiable, but is merely assumed to be sublinear in a neighborhood of $(\lambda, 0)$, $\lambda \in \mathbb{R}$.

Because of the presence of the term F problem (1.1) do not in general have a linearization about $(\lambda, 0)$. For this reason, the set of bifurcation points for these problems with respect to the line of trivial solutions need not be discrete. Therefore, for investigating the question of bifurcation of nontrivial solutions of problem (1.1) it is natural, as in the cases of bifurcations of a solutions of problems considered in [1, 6, 7, 8], to study bifurcation from intervals.

2. Local bifurcation of solutions of problem (1.1)

Let $B_\varepsilon(\lambda)$ and B_ε denote respectively open balls in $\mathbb{R} \times E$ and E of radius ε centered at $(\lambda, 0)$ and 0.

According to [1] for $r > 0$ we set

$$k\,(F; r) = \sup_{||u|| \le r,\, \lambda \in \mathbb{R}} \frac{||F(\lambda, u)||}{||u||} \tag{2.1}$$

suppose that it is finite at least for small values of r. Since $k = k(F; r)$ is a nondecreasing function of parameter r, we also set

$$k_0(F) = \lim_{r \to 0} k(F; r). \tag{2.2}$$

For $r > 0$, we let $B_r = \{\, u \in D(L) \, : \, ||u|| \le r\}$; ∂B_r will denote the boundary of B_r. The coincidence degree of the pair of operators (L, N) with respect to B_r, denoted $d[(L, N), B_r]$, will be defined for any continuous operator $N : E \to E$ which mapping bounded sets onto bounded sets, provided $Lu \ne N(u)$ for $u \in \partial B_r$ (Gaines-Mawhin [3], Chapter iii). For convenience we shall introduce the following notation

$$d(L - N, B_r) = d[(L, N),\, B_r].$$

Moreover, for $\lambda \in \sigma(L)$ we set

$$c(\lambda) = \inf\{\, ||Lu - \lambda u|| \, : \, u \in D(L),\, ||u|| = 1\,\} = ||(L - \lambda I)^{-1}||^{-1}. \tag{2.3}$$

Throughout what follows we shall assume that μ is a eigenvalue of the operator L of odd multiplicity.

Theorem 2.1. *Suppose there exists* $\underline{\lambda}$, $\overline{\lambda}$ *such that:*

$$\underline{\lambda} < \mu < \overline{\lambda} \text{ and } \sigma(L) \cap \,[\underline{\lambda}, \overline{\lambda}] = \{\mu\}; \tag{2.4}$$

$$\min\{c(\underline{\lambda}), c(\overline{\lambda})\} > k_0(F), \tag{2.5}$$

where $c(\lambda)$ *and* $k_0(F)$ *are defined by* (2.3) *and* (2.2) *respectively. Then, for any sufficiently small* $r > 0$ *problem* (1.1) *has an solution* (λ_r, u_r) *with* $||u_r|| = r$ *and* $\lambda_r \in [\underline{\lambda}, \overline{\lambda}]$.

Proof . Assume the contrary, i.e. suppose that there exists $r_1 > 0$ such that the problem (1.1) has no solution $(\lambda, u) \in [\underline{\lambda}, \overline{\lambda}] \times (\partial B_r \cap D(L))$ for all $0 < r \le r_1$.

We define c_0 and δ_0 by the following equalities

$$c_0 = \min\{c(\underline{\lambda}), c(\overline{\lambda})\}, \ \ \delta_0 = \frac{c_0 - k_0(F)}{2}.$$

It follows by condition (2.5) and definition (2.2) that there exists $r_2 > 0$ such that

$$k(F, r) \ < \ c_0 - \delta_0 \tag{2.6}$$

for any $0 < r \leq r_2$.

Since $G(\lambda, u) = o(||u||)$ as $u \to 0$ uniformly with respect to $\lambda \in \Lambda$, then there exists $r_3 > 0$ such that for any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ and any $u \in D(L)$ with $||u|| \leq r_3$ we have

$$||G(\lambda, u)|| \ \leq \ \frac{\delta_0 ||u||}{2}. \tag{2.7}$$

Denote $r_0 = \min\{r_1, r_2, r_3\}$. Hence we have the following relation

$$Lu \neq \lambda u + F(\lambda, u) + G(\lambda, u)$$

for any $\lambda \in [\underline{\lambda}, \overline{\lambda}]$ and for any $u \in \partial B_r \cap D(L)$, where $r \in (0, r_0)$. Then, by homotopy invariance of the coincide degree we obtain

$$d(Lu - \underline{\lambda}\, u - F(\underline{\lambda}, u) - G(\underline{\lambda}, u),\ B_r) = d(Lu - \overline{\lambda}\, u - F(\overline{\lambda}, u) - G(\overline{\lambda}, u),\ B_r). \tag{2.8}$$

On the other hand, by (2.1), (2.3), (2.6) and (2.7) for any $t \in [0, 1]$ and any $u \in \partial B_r \cap D(L)$, we have

$$||Lu - \underline{\lambda}\, u - \ t\, F(\underline{\lambda},\, u) - t\, G(\underline{\lambda},\, u)|| \ \geq \ ||Lu - \underline{\lambda}\, u|| - \ t||F(\underline{\lambda},\, u)||$$

$$-t||G(\underline{\lambda},\, u)|| \ \geq c(\underline{\lambda})||u|| - ||F(\underline{\lambda},\, u)|| \ - ||G(\underline{\lambda},\, u)|| \ \geq \ c_0||u|| -$$

$$(c_0 - \delta_0)||u|| - \frac{\delta_0}{2}||u|| = \frac{\delta_0}{2}||u|| = \frac{\delta_0 r}{2} > 0.$$

It follows that for any $u \in D(L)$ with $0 < r \leq r_0$ and for any $t \in [0, 1]$ we get

$$Lu \neq \underline{\lambda}\, u + t\, F(\underline{\lambda},\, u) - t\, G(\underline{\lambda},\, u).$$

So, using the homotopy invariance of the coincide degree again, we obtain that

$$d(L - \underline{\lambda}\, I - \ F(\underline{\lambda},\, \cdot\,) - \ G(\underline{\lambda},\, \cdot\,), B_r) = d(L - \underline{\lambda}\, I, B_r) = i(\underline{\lambda}).$$

The some argument can be used for $\bar{\lambda}$. Hence, by (2.8) we have $i(\underline{\lambda}) = i(\overline{\lambda})$. Since μ is the only eigenvalue of L in $[\underline{\lambda}, \overline{\lambda}]$ and has odd multiplicity, then by Leray-Schauder formula [3, p. 501] $i(\underline{\lambda}) = -i(\overline{\lambda})$. This contradiction proves the Theorem 2.1.

We denote by $\mathbf{B}$ the set of bifurcation points of the problem (1.1) respect to the line of trivial solutions.

Corollary 2.1. $([\underline{\lambda}, \overline{\lambda}] \times \{0\}) \cap \mathbf{B} \neq \emptyset$.

Now suppose that E is a real Hilbert space H and $L : D(L) \subset H \to H$ is selfadjoint. Then we have

Theorem 2.2. *Let $\mu \in \sigma(L)$ is of odd multiplicity and suppose that its distance to the spectrum is greater than $2k_0(F)$, i.e.*

$$\operatorname{dist}(\,\mu,\, \sigma(L)\backslash\{\mu\}\,) \ > \ 2k_0(F). \tag{2.9}$$

Then, there exists are small r_0 and ε_0 such that for all $0 < r \leq r_0$, (1.1) has a solution (λ_r, u_r) with $||u_r|| = r$ and $\lambda \in [\mu - k(r) - \varepsilon_0,\, \mu + k(r) + \varepsilon_0]$, where $k(r) = k(F, r)$.

Proof . Let

$$c_1 = \frac{\mathrm{dist}\{\mu, \sigma(L)\backslash\{\mu\})}{2}, \quad \delta_1 = \frac{c_1 - k_0(F)}{2}.$$

By the definition (2.2) from (2.9) it follows that there exists $r_1 > 0$ such that for any $0 < r \leq r_1$ we have

$$k_0(F) < k(F; r) < c_1 - \frac{\delta_1}{2}. \tag{2.10}$$

We will choose the $\varepsilon_0 > 0$ so that the following inclusions are valid

$$\underline{J} = [\mu - k(r) - 2\varepsilon_0,\, \mu - k(r) - \varepsilon_0] \subset [\mu - k(r) - \delta_1/2,\, \mu - k(r)],$$

$$\overline{J} = [\mu + k(r) + \varepsilon_0,\, \mu + k(r) + 2\varepsilon_0] \subset [\mu + k(r),\, \mu + k(r) + \delta_1/2].$$

Note that there exists $r_2 > 0$ such that for any $\lambda \in [\,\mu - c_1 + \delta_1/2\,,\, \mu + c_1 - \delta_1/2\,]$ and any $u \in D(L)$ with $||u|| \leq r_2$ we have

$$||G(\lambda, u)|| \leq \frac{\varepsilon_0 ||u||}{2}. \tag{2.11}$$

Denote $r_0 = \min\{r_1, r_2\}$. Let $0 < r \leq r_0$. Obviously, if $\lambda \in \underline{J} \cup \overline{J}$, then

$$\mathrm{dist}\{\lambda\,,\, \sigma(L)\} > k(r). \tag{2.12}$$

Now take $\underline{\lambda} \in \underline{J}$ and $\overline{\lambda} \in \overline{J}$. Since by (2.10), $[\mu - k(r), \mu + k(r)] \cap \sigma(L) = \{\mu\}$, the inequality (2.12) shows that $\underline{\lambda}$, $\overline{\lambda}$ satisfy the condition (2.4) from Theorem 2.1 as well as the inequality (2.6) because, in connection with selfadjointness of L, $c(\lambda) = \mathrm{dist}(\lambda, \sigma(L))$. Consequently, arguing as above, we obtain the existence of a solution (λ_r, u_r) of problem (1.1) with $\lambda_r \in [\underline{\lambda}, \overline{\lambda}]$ and $||u_r|| = r$. On the other hand, if $\lambda \in [\,\underline{J}, \overline{J}\,]$ and $0 < ||u|| \leq r$, then by (2.10)-(2.12) we have

$$||Lu - \lambda u - F(\lambda, u) - G(\lambda, u)|| \geq ||Lu - \lambda u|| - ||F(\lambda, u)|| -$$

$$||G(\lambda, u)|| \geq c(\lambda)||u|| - k(r)||u|| - \frac{\varepsilon_0}{2}||u|| = (dist(\lambda, \sigma(L)) -$$

$$k(r) - \frac{\varepsilon_0}{2}\Big)\,||u|| \geq \Big(k(r) + \varepsilon_0 - k(r) - \frac{\varepsilon_0}{2}\Big)\,||u|| = \frac{\varepsilon_0}{2}\,||u||.$$

Hence, it follows that the problem (1.1) has no nontrivial solution in $[\,\underline{\lambda}\,,\, \overline{\lambda}\,] \times B_r$.

Corollary 2.2. $(\mathbf{B} \cap [\mu - c_1,\, \mu + c_1]) \subset [\mu - k_0(F),\, \mu + k_0(F)]$.

3. Global bifurcation of solutions of problem (1.1)

The closure of the set of nontrivial solutions of (1.1) will be denoted by $\Im$. By a subcontinuum of A we mean a subset of A which is closed and connected in $\mathbb{R} \times E$.

Let denote by $\tilde{\mathrm{Y}}_\mu$ the union of all components of $\Im$ bifurcating from the points $(\lambda, 0) \in \mathbf{B} \cap I_\mu$, where $I_\mu = [\mu - k_0(F), \mu + k_0(F)$. Let $\mathrm{Y}_\mu = \tilde{\mathrm{Y}}_\mu \cup (I_\mu \times 0)$. Note that the set Y_μ is connected in $\mathbb{R} \times E$, but $\tilde{\mathrm{Y}}_\mu$ my not be connected.

We give a global bifurcation result which generalizes Theorem 2.3 [9]

Theorem 3.1. *Let $\mu \in \sigma(L)$ is of odd multiplicity and assume that executed the condition (2.9). Then the connected component Υ_μ of the set $\Im$ either (i) is unbounded in $\mathbb{R} \times E$, or (ii) contains the set $I_\nu \times \{0\}$, where $\mu \neq \nu \in \sigma(L)$.*

Proof . Assume the contrary, i.e. let the assertion of Theorem 2.1 is not true. Then Υ_μ is a bounded subset of $\mathbb{R} \times E$ and, therefore, is compact due to the fact that the operator L has a compact resolvent. Thus, similarly to Lemma 1.2 from [8] we can prove that, there exists a bounded open set A and number $\tilde{\delta} < \delta_1/2$ such that

$$(I_\mu \times 0) \subset \mathrm{A}, \quad \partial \mathrm{A} \cap \Im = \emptyset \tag{3.1}$$

and A contains no trivial solutions $(\lambda, 0)$ of problem (1.1) for $\mathrm{dist}(\lambda,\, I_\mu) > \tilde{\delta}$. Obviously, if $0 < \mathrm{dist}(\lambda,\, I_\mu) \leq \tilde{\delta}$ then there exists $\rho(\lambda)$ such that $(\lambda, 0)$ is the only solution of (1.1) in $\{\lambda\} \times \bar{B}_{\rho(\lambda)}$. We define the

$$\rho(\lambda) = \rho(\mu + k_0(F) + \tilde{\delta}) \text{ for } \lambda > \mu + k_0(F) + \tilde{\delta}$$

and

$$\rho(\lambda) = \rho(\mu - k_0(F) - \tilde{\delta}) \text{ for } \lambda < \mu - k_0(F) - \tilde{\delta}.$$

Let

$$\mathrm{A}_\lambda = \{u \in E \mid (\lambda, u) \in \mathrm{A}\}.$$

Choosing $\rho(\mu \pm (k_0(F) + \tilde{\delta}))$ is sufficiently small, we can assume that

$$\bar{B}_{\rho(\lambda)} \cap \bar{\mathrm{A}}_\lambda = \emptyset \text{ if } |\lambda - \mu| \geq k_0(F) + \tilde{\delta}. \tag{3.2}$$

We define

$$\underline{\lambda} = \mu - k_0(F) - \varepsilon_2 \text{ and } \overline{\lambda} = \mu + k_0(F) + \varepsilon_2,$$

where $\tilde{\delta} < \varepsilon_2 < \delta_1/2$. Note that

$$\mathrm{dist}(\underline{\lambda}, \sigma(L)) = \mathrm{dist}(\overline{\lambda}, \sigma(L)) = k_0(F) + \varepsilon_2. \tag{3.3}$$

By (3.1) and (3.2) it is clear that the coincidence degree $d\,[(L, \lambda I + F(\lambda, \cdot) + G(\lambda, \cdot)), A_\lambda\,]$ of the couple $(L, \lambda I + F(\lambda, \cdot) + G(\lambda, \cdot))$ with respect to A_λ is well defined for all $\lambda \in [\,\underline{\lambda}, \overline{\lambda}\,]$. Then, by the homotopy invariance of the degree we have

$$d\,[(L, \lambda I + F(\lambda, \cdot) + G(\lambda, \cdot)), A_\lambda\,] = \text{const}, \text{ for } \lambda \in [\,\underline{\lambda}, \overline{\lambda}\,]. \tag{3.4}$$

Using similar arguments in the proof of Theorems 2.1 and 2.2 we obtain

$$d[(L, \underline{\lambda} I + F(\underline{\lambda}, \cdot) + G(\underline{\lambda}, \cdot)),\, A_{\underline{\lambda}}] = i(\underline{\lambda}), \tag{3.5}$$

$$d[(L, \overline{\lambda} I + F(\overline{\lambda}, \cdot) + G(\overline{\lambda}, \cdot)),\, A_{\overline{\lambda}}] = i(\overline{\lambda}). \tag{3.6}$$

Hence, in view of (3.5) and (3.6), by (3.4) we have

$$i(\underline{\lambda}) = i(\overline{\lambda}).$$

But on the other hand

$$i(\underline{\lambda}) = -i(\overline{\lambda}),$$

since μ is the only eigenvalue of odd multiplicity of the operator L in $[\underline{\lambda}, \overline{\lambda}]$. This contradiction completes the proof of Theorem 3.1.

Now we give one application to a nonlinear eigenvalue problem for ordinary differential equations of second order.

Consider the following problem

$$-y'' + q(x)y = \lambda y + f(x, y, y', \lambda) + g(x, y, y', \lambda),\ 0 < x < \pi, \tag{3.7}$$

$$a_0 y(0) + b_0 y'(0) = 0, \tag{3.8}$$

$$a_1 y(\pi) + b_1 y'(\pi) = 0, \tag{3.9}$$

where $\lambda \in \mathbb{R}$, $q(x)$ is a real-valued continuous function on $[0, \pi]$, the function a_i, b_i are real numbers such that $|a_i| + |b_i| > 0,\ i = 0,\ 1,$ f and g are continuous functions on $[0, \pi] \times \mathbb{R}^3$, and g satisfied the conditions: there exists a finite number of

$$M = \inf_{\eta > 0} \sup_{\substack{x \in [0,\pi] \\ 0 < |u| + |s| < \eta,\, \lambda \in \mathbb{R}}} \left| \frac{f(x, y, s, \lambda)}{y} \right|; \tag{3.10}$$

and

$$g(x, y, s, \lambda) = o(|y| + |s|) \text{ near } (y, s) = (0, 0), \tag{3.11}$$

uniformly for $x \in [0, \pi]$ and in $\lambda \in \Lambda$, for every bounded interval $\Lambda \subset \mathbb{R}$.

Let $H = L_2(0, \pi)$. Define the operators $L : D(L) \subset H \to H,\ F : D(L) \to H$ and $G : D(L) \to H$ as follows:

$$D(L) = \left\{ y \in H \mid y \in W_2^1(0, \pi),\ -y'' + qy \in L_2(0, \pi),\ y \in \text{B.C.} \right\},$$

$$Ly = -y'',\quad F(\lambda, y) = f(x, y, y', \lambda),\quad G(\lambda, y) = g(x, y, y', \lambda)$$

where by B.C denotes the set of functions satisfying the boundary conditions (3.8)-(3.9). Then the problem (3.7)-(3.9) can be written as an operator equation in the form of (1.1), i.e.

$$Ly = \lambda y + F(\lambda, y) + G(\lambda, y).$$

It is known that (see [5]) L is a semi-bounded from below self-adjoint operator in H with compact resolvent. The corresponding linear problem

$$Ly = \lambda y$$

possesses infinitely many eigenvalues $\mu_1 < \mu_1 < \mu_2 < ... < \mu_k < ...$, all of which are simple, and $\lambda_k = k^2 + C + O\left(\frac{1}{k}\right)$, where C is a some constant [5, Ch 1].

By (3.10) for this problem we have $k_0(F) = M$, and by (3.10) we obtain

$$G(\lambda, y) = o\left(||y||\right) \text{ as } ||y|| \to 0 \text{ in } H$$

uniformly with respect to $\lambda \in \Lambda$. Accordingly, $I_k = I_{\mu_k} = [\mu_k - M, \mu_k + M]$ for our problem. From the asymptotic formula for the eigenvalues μ_k, $k \in \mathbb{N}$, it follows that there exists a natural number $\tilde{k}$ such that $\text{dist}(\mu_k, \sigma(L) \backslash \{\mu_k\}) > 2M$ for $k \geq \tilde{k}$. Then by Theorem 3.1 for each $k \geq \tilde{k}$ connected component $\Upsilon_k \equiv \Upsilon_{\mu_k}$ of solutions of the problem (3.7)-(3.9), containing $I_k \times \{0\}$ is either (i) unbounded in $\mathbb{R} \times H$, or (ii) contains the interval $I_{\mu_m} \times \{0\}$, where $\mu_k \neq \mu_m \in \sigma(L)$. Notice that, using the nodal properties of eigenfunction can be shown that alternative (ii) is not possible (see [10]). Hence, the component Υ_k for $k \geq \tilde{k}$ is unbounded in $\mathbb{R} \times H$.

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