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SOME SPECTRAL PROPERTIES OF THE BOUNDARY VALUE PROBLEM WITH SPECTRAL PARAMETER IN THE BOUNDARY CONDITIONS

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Abstract. In this paper we consider the boundary value problem for second-order differential operator with spectral parameter occurs in the boundary conditions. We study the structure of root subspaces and location of eigenvalues on the real axis of this problem.

1. Introduction

In this paper we consider the boundary value problem arising in a mathematical model of torsional vibrations of a rod with pulleys at the ends. The well-known mathematical model describing small torsional vibrations of a rod consists of the wave equation for the rod rotation angle and the corresponding boundary conditions. If there are pulleys at both ends of the rod, then the boundary conditions simulating the forces contain second time derivatives (see [14]). By solving the corresponding mathematical problem by separation of variables, we obtain the spectral problem

$$-y''(x) = \lambda y(x), \;\; 0 < x < 1, \tag{1.1}$$

$$y'(0) = -a_0 \lambda y(0), \tag{1.2}$$

$$y'(1) = (a_1 \lambda + b_1) y(1), \tag{1.3}$$

where $\lambda \in \mathbb{C}$ is a spectral parameter, a_0, a_1, b_1 are real constants, and $a_0 \neq 0$, $a_1 \neq 0$.

The structure of root subspaces and location of eigenvalues on the real axis of problem (1.1)-(1.3) were studied by Kapustin [7] for the case where $a_0 > 0$, $a_1 > 0$, $b_1 = 0$ and by Aliev [1, 3] for the cases where $a_0 > 0$, $a_1 < 0$, $b_1 = 0$ and $a_0 < 0$, $a_1 < 0$, $b_1 = 0$. In these papers, studied also basis properties in the space $L_p(0,1)$, $1 < p < \infty$, of the system of root functions, where obtained necessary and sufficient conditions for the basicity of subsystems of root functions of problem (1.1)-(1.3) in the space $L_p(0,1)$, $1 < p < \infty$. In [8] studied the eigenvalue problem for a second order differential equation with spectral parameter in the boundary conditions in the more general case, where investigate oscillation

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properties of eigenfunctions and obtained the sufficient condition for basicity of subsystem of eigenfunctions in the space $L_p(0,1)$, $1 < p < \infty$.

In this paper we study the structure of the root subspaces and location of eigenvalues on the real axis of this of problem (1.1)- (1.3) in the case $a_0 < 0$, $a_1 < 0$, $b_1 \neq 0$.

2. Operator interpretation of problem (1.1)-(1.3) and some properties of solutions of problem (1.1)- (1.2)

Let $H = L_2(0,1) \oplus \mathbb{C}^2$ be a Hilbert space with a scalar product

$$(\hat{u}, \hat{v})_H = (\{u(x),\, m,\, n\},\, \{v(x), s, t\})_H = (u, v)_{L_2} + |a_0|^{-1}\, m\bar{s} + |a_1|^{-1} n\bar{t} \quad (2.1)$$

where $(\cdot\,,\cdot)_{L_2}$ is an inner product in $L_2(0,1)$.

We define the operator in H by

$$L\hat{y} = L\{\, y(x), m, n\} = \{-y''(x),\, y'(0),\, y'(1) - b_1 y(1)\}$$

with the domain

$$D(L) = \{\, \hat{y} \in H \mid y(x),\, y'(x) \in AC[0,1],\, m = -a_0 y(0),\, n = a_1 y(1)\}$$

which is dense in H [11, 13]. With this framework it is easily seen that the eigenvalue problem (1.1)-(1.3) is equivalent to eigenvalue problem

$$L\hat{y} = \lambda \hat{y}, \ \ \hat{y} \in D(L).$$

We now introduce the the operator $J : H \to H$ by

$$J\{\, y, m, n\} = \{\, y, -m, -n\}.$$

Theorem 2.1 . *The operator J is a unitary and symmetric on H. Its spectrum consist of two eigenvalues : -1 with multiplicity 2, and $+1$ with infinite multiplicity.*

Proof. The simple calculations to show that the operator J is unitary and symmetric are easy. Note further that any vector $\hat{y}$ of the form $\hat{y} = \{\, 0, m, n\}$ satisfies the relation $J\hat{y} = -\hat{y}$ and that there is two-dimensional subspace of such $\hat{y}$, while all vectors $\hat{y}$ of the form $\hat{y} = \{\, y, 0, 0\}$ satisfy the relation $J\hat{y} = \hat{y}$ and these generate an infinite-dimensional subspace. Those two subspaces provide an orthogonal decomposition of H. The theorem is proved.

Theorem 2.2 . *The operator JL is self-adjoint, bounded below and has compact resolvent in H.*

Proof. By using the formula for the integration by parts, we obtain

$$(JL\hat{y}, \hat{y})_H = \int_0^1 |y'(x)|^2 dx \; - b_1 |y(1)|^2,$$

where $\hat{y} \in D(L)$. Hence, the operator is symmetric. The equation

$$(JL - \lambda I)u = \hat{f}, \ \ \hat{f} = \{\, f, \tau, \varkappa\} \in H$$

can be rewritten in the form

$$-y''(x) - \lambda y(x) = f(x),\ 0 < x < 1,$$
$$y'(0) - \lambda a_0 y(0) = \tau,$$
$$y'(1) - (-a_1\lambda + b_1)y(1) = \varkappa.$$

This problem is obviously solvable for all λ that are not eigenvalues of the corresponding homogeneous problem. But the homogeneous problem

$$-y''(x) = \lambda y(x),\ 0 < x < 1,$$
$$y'(0) = \lambda a_0 y(0),$$
$$y'(1) = (-a_1\lambda + b_1)y(1),$$

is regular in the sense [13]; in particular, it has discrete spectrum, i.e. has compact resolvent in H. Hence the operator JL is symmetric and discrete. Therefore, it is self-adjoint. By [8] eigenvalues of the homogeneous problem form an infinite increasing sequence, so that the operator JL is bounded from below in H. The proof of Theorem 2.1 is complete.

By Theorem 2.1 the operator $J : H \to H$ generates the Pontryagin space $\Pi_2 = L_2(0,1) \oplus \mathbb{C}^2$ with inner product ($J-$ metric) [5, Ch.1]

$$(\hat{u}, \hat{v})_{\Pi_2} = (\{u(x),\, m,\, n\},\, \{v(x), s, t\})_{\Pi_2} = (u, v)_{L_2} +\, a_0^{-1}\, m\bar{s} + a_1^{-1} n\bar{t}. \tag{2.2}$$

From Theorem 2.2 imply
Corollary 2.1. *L is a self-adjoint operator on Π_2.*

Let λ be an eigenvalue of L of algebraic multiplicity ν. We set $\rho(\lambda)$ to be equal to ν if $\mathrm{Im}\lambda \neq 0$ and to the integer part $[\nu/2]$ if $\mathrm{Im}\lambda = 0$.

Theorem 2.3 [9]. *The eigenvalues of the operator L are arranged symmetrically around the real axis, and $\sum\limits_{k=1}^{n} \rho(\lambda_k) \leq 2$ for any system $\{\lambda_k\}_{k=1}^{n} (n \leq +\infty)$ of eigenvalues with nonnegative imaginary parts.*

It follows from Theorem 2.3 that problem (1.1)-(1.3) may have either at most two pair of complex conjugate non-real eigenvalues, or have at most two real multiple eigenvalues whose sum of the algebraic multiplicities not exceeding 5.

The solution of equation (1.1) satisfying the initial conditions $y(0,\lambda) = -1$ and $y'(0,\lambda) = a\lambda$ is

$$y(x,\lambda) = a_0\sqrt{\lambda}\sin\sqrt{\lambda}\,x - \cos\sqrt{\lambda}\,x. \tag{2.3}$$

The eigenvalues of the boundary value problem

$$-y''(x) = \lambda y(x),\ 0 < x < 1,$$
$$a_0\lambda y(0) + y'(0) = 0,\ y(1) = 0,$$

are real and simple, they form an unboundedly increasing sequence

$$\mu_1 < 0 < \mu_2 < \ldots < \mu_k < \ldots,$$

while the eigenvalues of the boundary value problem

$$-y''(x) = \lambda y(x),\ 0 < x < 1,$$

$$a_0\lambda y(0)+y'(0)=0,\ y'(1)=0,$$

are real and simple, except the case $a_0=-1$, where $\lambda=0$ is a double eigenvalue, and they form an unboundedly increasing sequence $\{\nu_k\}_{k=1}^{\infty}$. Moreover,

$$\mu_1<\nu_1<0=\nu_2<\mu_2<\nu_3<\mu_3<...<\nu_k<\mu_k<...,\ \text{ if } a>-1,$$

$$\mu_1<0=\nu_1=\nu_2<\mu_2<\nu_3<\mu_3<...<\nu_k<\mu_k<...,\ \text{ if } a=-1,$$

$$\mu_1<0=\nu_1<\nu_2<\mu_2<\nu_3<\mu_3<...<\nu_k<\mu_k<...,\ \text{ if } a<-1.$$

The function

$$F(\lambda)=\frac{y'(1,\lambda)}{y(1,\lambda)}$$

is defined in the set

$$K\equiv(\mathbb{C}\backslash\mathbb{R})\cup\left(\bigcup_{k=1}^{\infty}(\mu_{k-1},\mu_k)\right),$$

where is assumed $\mu_0=-\infty$. This is a meromorphic function of finite order, and μ_k and ν_k, $k\in\mathbb{N}$, are the zeros and poles of this function, respectively. Notice that, the eigenvalues (counted with multiplicities) of problem (1.1)-(1.3) are roots of the equation

$$F(\lambda)=a_1\lambda+b_1. \tag{2.4}$$

Lemma 2.2. *The following relations hold*

$$\frac{dF(\lambda)}{d\lambda}=-\frac{\int\limits_0^1 y^2(x,\lambda)dx+a_0}{y^2(1,\lambda)},\quad \lambda\in K, \tag{2.5}$$

$$\lim_{\lambda\to-\infty}F(\lambda)=+\infty\,. \tag{2.6}$$

Proof. The proof of the formula (2.5) follows from [3] (see also [4]). By (2.3) for $\lambda<0$ we have the relation

$$F(\lambda)=\frac{y'(1,\lambda)}{y(1,\lambda)}=\frac{a_0\lambda\cos\sqrt{\lambda}+\sqrt{\lambda}\sin\sqrt{\lambda}}{a_0\sqrt{\lambda}\sin\sqrt{\lambda}-\cos\sqrt{\lambda}}=$$

$$\frac{a_0\lambda\cos i\sqrt{|\lambda|}+\ i\sqrt{|\lambda|}\sin i\sqrt{|\lambda|}}{a_0\, i\sqrt{|\lambda|}\sin i\sqrt{|\lambda|}-\cos i\sqrt{|\lambda|}}=\frac{a_0\lambda\,\mathrm{ch}\sqrt{|\lambda|}-\ \sqrt{|\lambda|}\,\mathrm{sh}\sqrt{|\lambda|}}{-a_0\,\sqrt{|\lambda|}\,\mathrm{sh}\sqrt{|\lambda|}-\mathrm{ch}\sqrt{|\lambda|}}=$$

$$\sqrt{|\lambda|}\,\frac{a_0\sqrt{|\lambda|}\ \mathrm{ch}\sqrt{|\lambda|}+\ \mathrm{sh}\sqrt{|\lambda|}}{a_0\,\sqrt{|\lambda|}\ \mathrm{sh}\sqrt{|\lambda|}+\mathrm{ch}\sqrt{|\lambda|}},$$

which implies that

$$F(\lambda)=\sqrt{|\lambda|}\left(1+O\left(\frac{1}{\sqrt{|\lambda|}}\right)\right)\ \text{ for }\ \lambda\to-\infty. \tag{2.7}$$

From this asymptotic formula it follows the relation (2.6). The proof of Lemma 2.2 is complete.

It is seen from (2.4) that, if $a_0>0$, then the function $F(\lambda)$ is strictly decreasing in each interval (μ_{k-1},μ_k), $k\in\mathbb{N}$, and if $a_0<0$, then this formula gives no information about the behavior of this function in each interval (μ_{k-1},μ_k), $k\in\mathbb{N}$.

Theorem 2.4. *The following representation holds:*

$$F(\lambda)=\sum_{k=1}^{\infty}\frac{\lambda\, c_k}{\mu_k(\lambda-\mu_k)}, \tag{2.8}$$

where

$$c_k=\underset{\lambda=\mu_k}{res}\,F(\lambda)=\frac{y'_x(1,\lambda)}{y'_\lambda(1,\lambda)},\ k\in N, \tag{2.9}$$

$c_1<0,\ c_k>0,\ k\in\mathbb{N}\backslash\{1\}$.

Proof. According to the theorem of Mittag-Leffler [6; Ch.6, § 5] the meromorphic function $F(\lambda)$ of finite order with simple poles μ_k, $k\in\mathbb{N}$, admits the representation

$$F(\lambda)=F_1(\lambda)+\sum_{k=1}^{\infty}\left(\frac{\lambda}{\mu_k}\right)^{s_k}\frac{c_k}{\lambda-\mu_k}\,, \tag{2.10}$$

where $F_1(\lambda)$ is an entire function, the coefficients $c_k.\, k\in\mathbb{N}$ are defined by the formula (2.9), and integers s_k, $k\in\mathbb{N}$, are chosen so that series series on the right side of formula (2.10) is uniformly converges in any finite circle (after truncation of terms having poles in this circle).

Since $\mu_1<\nu_1$, then by virtue of relation (2.6) we obtain $F(\lambda)>0$ for $\lambda\in(-\infty,\mu_1)$. Hence, we get

$$\lim_{\lambda\to\mu_1-0}F(\lambda)=+\infty. \tag{2.11}$$

From simplicity of the pole μ_1 it follows that

$$\lim_{\lambda\to\mu_1+0}F(\lambda)=-\infty. \tag{2.12}$$

Since $\nu_1,\nu_2\in(\mu_1,\mu_2)$, then

$$\begin{array}{l}F(\nu_1-0)<0,\ F(\nu_1)=0,\ F(\nu_1+0)>0,\\ F(\nu_2-0)>0,\ F(\nu_2)=0,\ F(\nu_2+0)<0,\end{array} \tag{2.13}$$

in the case $a_0\neq-1$,

$$F(\nu_1-0)<0,\ F(\nu_1)=0,\ F(\nu_1+0)<0(\nu_1=\nu_2=0), \tag{2.14}$$

in the case $a_0=-1$. Consequently, we have

$$\lim_{\lambda\to\mu_2-0}F(\lambda)=-\infty,\ \text{ and }\ \lim_{\lambda\to\mu_2+0}F(\lambda)=+\infty. \tag{2.15}$$

Further, since $\nu_k\in(\mu_{k-1},\mu_k),\ k\geq3$, is a simple zeros of the function $F(\lambda)$, we obtain the following equalities

$$F(\nu_k-0)>0,\ F(\nu_k+0)<0,$$

and

$$F(\mu_k-0)=-\infty,\ F(\mu_k+0)=+\infty \text{ for } k\geq3. \tag{2.16}$$

Without loss of generality, we can assume that $y(1,\lambda)>0$ for $\lambda\in(-\infty,\mu_1)$. Then, taking into account the above arguments, we obtain

$$y'_x(1,\mu_1)>0,\ \ y'_\lambda(1,\mu_1)<0;$$

$$y'_x(1,\mu_2)>0,\ y'_\lambda(1,\mu_2)>0;$$

and

$$(-1)^k y'_x(1,\mu_k) > 0, \quad (-1)^k y'_\lambda(1,\mu_k) > 0 \ \text{ for } \ k \geq 3.$$

Then, by (2.9) we have $c_1 < 0$ and $c_k > 0$ for $k \geq 2$.

Denote by $\Omega_k(\varepsilon) = \left\{\lambda : \left|\sqrt{\lambda} - \sqrt{\mu_k}\right| < \varepsilon\right\}$, where is some small number. It is easy to verify that the eigenvalues μ_k of the problem

$$-y''(x) = \lambda y(x), \ 0 < x < 1,$$

$$a\lambda y(0) + y'(0) = 0, \ y(1) = 0,$$

for large k have the asymptotic

$$\sqrt{\mu_k} = k\pi + O\left(\frac{1}{k}\right). \tag{2.17}$$

From this asymptotic, it follows that for $\varepsilon < 1$ the regions $\Omega_k(\varepsilon)$ asymptotically do not intersect and contain only one pole μ_k of the function $F(\lambda)$.

By (2.3), we see that outside of regions $\Omega_k(\varepsilon)$ the asymptotic formula

$$F(\lambda) = \frac{a\lambda\cos\sqrt{\lambda} + \sqrt{\lambda}\sin\sqrt{\lambda}}{a\sqrt{\lambda}\sin\sqrt{\lambda} - \cos\sqrt{\lambda}} = \sqrt{\lambda}\,\frac{\cos\sqrt{\lambda}}{\sin\sqrt{\lambda}}\left(1 + O\left(\frac{1}{\sqrt{|\lambda|}}\right)\right), \quad |\lambda| \to +\infty.$$

is valid. Following the corresponding reasoning (see [10, Ch.7, § 2, formula (27)]), we see that outside of regions $\Omega_k(\varepsilon)$ the estimation

$$|F(\lambda)| \leq M\sqrt{|\lambda|}, \ \ M = \text{const}, \tag{2.18}$$

holds; using it in (2.9) we get

$$c_k = \left|\frac{1}{2\pi i}\int\limits_{\partial\Omega_k(\varepsilon)} F(\lambda)\,d\,\lambda\right| = \frac{1}{\pi}\left|\int\limits_{|\nu-\sqrt{\mu_k}|=\varepsilon} \nu F(\nu^2)\,d\nu\right| \leq M\pi^2k^2. \tag{2.19}$$

By (2.19) and asymptotic formula (2.17) the series $\sum\limits_{k=1}^{\infty} c_k|\mu_k|^{-2}$ converges. Then, according to Theorem 2 in [6; Ch.6, § 5], in formula (2.10) we can assume $s_k = 1, \ k \in \mathbb{N}$.

Let $\{\Gamma_k\}_{k=1}^{\infty}$ be a sequence of the expanding circles which are not crossing regions $\Omega_k(\varepsilon)$. Then, according to Formula (9) in [12; Ch. 5, § 13], we have

$$\begin{aligned} F(\lambda) - \sum_{\mu_m \in \,\text{int}\Gamma_k} \tfrac{c_m}{\lambda-\mu_m} &= \int\limits_{\Gamma_k} \tfrac{F(\xi)}{\xi-\lambda}\,d\xi, \\ F(0) + \sum_{\mu_m \in \,\text{int}\Gamma_k} \tfrac{c_m}{\mu_m} &= \int\limits_{\Gamma_k} \tfrac{F(\xi)}{\xi}\,d\xi. \end{aligned} \tag{2.20}$$

By (2.20), we get

$$F(\lambda) - F(0) = \sum_{\mu_m \in \,\text{int}\Gamma_k} \frac{\lambda c_m}{\mu_m(\lambda-\mu_m)} = \int\limits_{\Gamma_k} \frac{\lambda F(\xi)}{\xi(\xi-\lambda)}\,d\xi. \tag{2.21}$$

By (2.18) the right side of (2.21) tends to zero as $k \to +\infty$. Then, passing to the limit in (2.21), we obtain

$$F(\lambda) = F(0) + \sum_{k=1}^{\infty} \frac{\lambda c_k}{\mu_k(\lambda - \mu_k)},$$

which implies (2.8), since $F(0) = 0$. Theorem 2.4 is proved.

Corollary 2.2. *The function $F(\lambda)$ is convex upward in the interval (μ_1, μ_2).*
Proof. Formula (2.8) implies

$$\frac{d^2F(\lambda)}{d\lambda^2} = 2\sum_{k=1}^{\infty} \frac{c_k}{(\lambda - \mu_k)^3},$$

it follows that

$$\frac{d^2F(\lambda)}{d\lambda^2} > 0, \text{ if } \lambda \in (\mu_1, \mu_2),$$

which means that the function $F(\lambda)$ is convex upward in the interval (μ_1, μ_2).

3. The structure of root subspaces and location of eigenvalues on the real axis of problem (1.1)-(1.3)

Lemma 3.1. *If $b_1 < 0$, then the problem (1.1)-(1.3) does not have nonreal eigenvalues.*
Proof. Let $\mu \in \mathbb{C}\backslash\mathbb{R}$ be an eigenvalue of problem (1.1)-(1.3). Then $\bar{\mu}$ is also an eigenvalue of this problem, since the coefficients a_0, a_1 and b_1 are real; moreover $y(x, \bar{\mu}) = \overline{y(x, \mu)}$. Multiplying the both parts of equation (1.1) by the function $\overline{y(x, \mu)}$ and integrating the obtained equality by parts in the range from 0 to 1, and also taking into account (1.2)-(1.3) we get

$$\begin{gathered}\int_0^1 |y'(x, \mu)|^2\, dx \; - b_1|y(1, \mu)|^2 = \\ \mu\left\{\int_0^1 |y(x, \mu|^2 + \; a_0|y(0, \mu)|^2 + a_1|y(1, \mu)|^2\right\}.\end{gathered} \tag{3.1}$$

On the other hand by virtue of (1.1), we have

$$-y''(x, \mu)\overline{y(x, \mu)} \; + \overline{y''(x, \mu)}\, y(x, \mu) = (\mu - \bar{\mu})|y(x, \mu)|^2.$$

Integrating this relation from 0 to 1, using the formula for the integration by parts, and taking into account conditions (1.2)-(1.3), we obtain

$$-(\mu - \bar{\mu})\{a_1|y(1, \mu)|^2 + \; a_0|y(0, \mu)|^2 = (\mu - \bar{\mu}) \int_0^1 |y(x, \mu)|^2\, dx,$$

which implies that

$$\int_0^1 |y(x, \mu)|^2\, dx + a_0|y(0, \mu)|^2 + \; a_1|y(1, \mu)|^2 = 0. \tag{3.2}$$

In view the relation (3.2), from (3.1) we get

$$\int_0^1 |y'(x,\mu)|^2\,dx - b_1|y(1,\mu)|^2 = 0,$$

which contradicts condition $b_1 < 0$. The proof of Lemma 3.1 is complete.

Lemma 3.2. *If $b_1 \neq 0$, then zero is not an eigenvalue of the problem* (1.1)-(1.3).
Proof. If zero is an eigenvalue of problem (1.1)-(1.3), by virtue of (2.3) we have $y(x,0) = -1$ whence taking into account the condition (1.3) we obtain $0 = -b_1$. Lemma 3.2 is proved.

Lemma 3.3. *If $b_1 < 0$, then the eigenvalues of the boundary value problem* (1.1)-(1.3) *are simple.*
Proof. If $\tilde{\lambda}$ is multiple root of the equation (2.4), then by (2.5) we obtain

$$\int_0^1 y^2(x,\tilde{\lambda})\,dx + a_0 + a_1 y^2(1,\tilde{\lambda}) = 0. \tag{3.3}$$

Multiplying the both parts of equation (1.1) by the function $y(x,\tilde{\lambda})$ and integrating the obtained equality by parts in the range from 0 to 1, and also taking into account the boundary conditions (1.2)-(1.3) we have

$$\int_0^1 y'^2(x,\tilde{\lambda})\,dx - b_1 y^2(1,\tilde{\lambda}) = \tilde{\lambda}\left[\int_0^1 y^2(x,\tilde{\lambda})\,dx + a_0 + a_1 y^2(1,\tilde{\lambda})\right]. \tag{3.4}$$

By (3.3), from (3.4) we get

$$\int_0^1 y'^2(x,\tilde{\lambda})\,dx - b_1 y^2(1,\tilde{\lambda}) = 0,$$

which is impossible in view of condition $b_1 < 0$. The proof of Lemma 3.3 is complete.

Set $B_k = (\mu_{k-1}, \mu_k)$, $k = 1, 2, \ldots$, where $\mu_0 = -\infty$.

Lemma 3.4. *If $b_1 < 0$, then the equation* (2.4) *has a unique solution in each interval B_k, $k = 1, 3, 4, \ldots$.*
Proof. Let $\tilde{\lambda} \in B_k$, $k \in \mathbb{N}\backslash\{2\}$ is an eigenvalue of the problem (1.1)- (1.3). Then, by (3.4) we obtain

$$\int_0^1 y^2(x,\tilde{\lambda})\,dx + a_0 + a_1 y^2(1,\tilde{\lambda}) < 0, \text{ if } \tilde{\lambda} \in B_1,$$

and

$$\int_0^1 y^2(x,\tilde{\lambda})\,dx + a_0 + a_1 y^2(1,\tilde{\lambda}) > 0, \text{ if } \tilde{\lambda} \in B_k,\ k \in \mathbb{N}\backslash\{1,2\}.$$

By virtue of (2.5), from these relations follows that $\frac{d}{d\lambda}\left(F(\lambda)-(a_1\lambda+b_1)\right)|_{\lambda=\tilde{\lambda}}$ is positive, if $\tilde{\lambda}\in B_1$ and is negative, if $\tilde{\lambda}\in B_k,\ k\in\mathbb{N}\backslash\{1,2\}$. Thus, the function $F(\lambda)-(a_1\lambda+b_1)$ is takes a value zero only strictly increasing (decreasing) in the interval B_1 $(B_k,\ k\in\mathbb{N}\backslash\{1,2\})$. Consequently, equation (2.3) has a unique solution in each interval $B_k,\ k=1,\,3,\,4,\ \ldots$. The proof of Lemma 3.4 is complete.

Theorem 3.1. *In the case* $b_1<0$ *all eigenvalues of problem* (1.1)-(1.3) *are real and simple;* B_2 *contains two eigenvalues, and* $B_k,\ k=1,\,3,\,4,\,\ldots,$ *contain one eigenvalue. In the case* $b_1>0$ *one of the following assertions holds: (i) all eigenvalues of problem* (1.1)-(1.3) *are real; in this case,* B_2 *contains algebraically two eigenvalues (either two simple eigenvalues or one double eigenvalue), and* $B_k,\ k=1,\,3,\,4,\,\ldots,$ *contains one simple eigenvalue; (ii) all eigenvalues of problem* (1.1)-(1.3) *are real; in this case,* B_2 *contains no eigenvalues, while there exists a positive integer* m $(m\neq 2)$ *such that* B_m *contains algebraically three eigenvalues (either three simple eigenvalues, or one double eigenvalue and one simple eigenvalue, or one triple eigenvalue), and* $B_k,\ k=1,\,3,\,4,\,\ldots k\neq m,$ *contains one simple eigenvalue; (iii) problem* (1.1)- (1.3) *has one pair of nonreal complex conjugate eigenvalues; in this case,* B_2 *contains no eigenvalues, and* $B_k,\ k=1,\,3,\,4,\,\ldots,$ *contains one simple eigenvalue.*

Proof. Recall that the eigenvalues of problem (1.1)-(1.3) are the roots of the equation (2.4). It follows from Corollary 2.2 that $F(\lambda)$ is a convex upward function in the interval B_2. By virtue of the relations (2.12), (2.15) and

$$\lim_{\lambda\to\mu_1+0}F(\lambda)=-\infty,\quad \lim_{\lambda\to\mu_2-0}F(\lambda)=-\infty,$$

$$\max_{\lambda\in B_2}F(\lambda)\ >0 \text{ in the case } a_0\neq -1,\ \max_{\lambda\in B_2}F(\lambda)\ =0 \text{ in the case } a_0=-1,$$

(see (2.13) and (2.14)) for each given number a_1, there exists a number $\tilde{b}_1=b_{1,a_1}\geq 0$ ($\tilde{b}_1=0$ in the case $a_0=-1$), such that the line $a_1\lambda+b_1,\ \lambda\in\mathbb{R}$, is tangent to the graph of the function $F(\lambda)$ at some point $\tilde{\lambda}$ of the interval B_2. Consequently, in the interval B_2, equation (2.4) has two simple roots $\tilde{\lambda}_1<\tilde{\lambda}_2$ if $b_1<\tilde{b}_1$, one double root $\tilde{\lambda}_1=\tilde{\lambda}$ if $b_1=\tilde{b}_1$, and no root if $b_1>\tilde{b}_1$.

By virtue of the relations (2.7), (2.11), (2.15) and (2.16) the equation (2.4) has at least one solution in each interval $B_k,\ k=1,\,3,\,4,\,\ldots$.

Thus the assertion of the theorem in the case $b_1<0$ follows from this reasoning in view of Lemmas 3.1-3.4.

Let the number $b_1=b_1^*<0$ is fixed. Take a sufficiently large number $k_1\in\mathbb{N}$ such that

$$a_1R_{k_1}+\ b_1<0,\ \ |F(\lambda)-(a_1\lambda+b_1^*)|\ >\ |b-b_1^*|,\ \lambda\in S_{R_{k_1}},\tag{3.5}$$

where $R_{k_1}=\nu_{k_1}+1+\delta_1$, δ_1 is a small number, $S_{R_{k_1}}=\{\,z\in\mathbb{C}\,:\,|z|\,=\,R_{k_1}\}$. Then, by (4.7) from [2], we get

$$\Delta_{S_{R_{k_1}}}\arg\left(F(\lambda)-(a_1\lambda+b_1)\right)=\Delta_{S_{R_{k_1}}}\arg\left(F(\lambda)-(a_1\lambda+b_1^*)\right),\tag{3.6}$$

where

$$\Delta_{S_{R_{k_1}}}\arg f(z)=\int\limits_{S_{R_{k_1}}}\frac{f'(z)}{f(z)}\,dz$$

(see [12, Ch. 4, § 10]). By the principle of argument [12, Ch.4, § 10], we have

$$\frac{1}{2\pi}\Delta_{S_{R_{k_1}}} \arg\left(F(\lambda)-(a_1\lambda+b_1^*)\right)=\sum_{\lambda_k^*\in \mathrm{B}_{R_{k_1}}}\varkappa(\lambda_k^*)-\sum_{\mu_k\in \mathrm{B}_{R_{k_1}}}\varkappa(\mu_k) \tag{3.7}$$

where $\mathrm{B}_{R_{k_1}}=\operatorname{int} S_{R_{k_1}}$, $\varkappa(\lambda_k^*)$ and $\varkappa(\mu_k)$ are the multiplicities of the zero λ_k^* and the pole μ_k of the function $F(\lambda)-(a_1\lambda+b_1^*)$, respectively. Obviously,

$$\sum_{\lambda_k^*\in \mathrm{B}_{R_{k_1}}}\varkappa(\lambda_k^*)=k_1+2 \quad \text{and} \quad \sum_{\mu_k\in \mathrm{B}_{R_{k_1}}}\varkappa(\mu_k)=k_1.$$

Consequently, from (3.7), we obtain

$$\frac{1}{2\pi}\Delta_{S_{R_{k_1}}} \arg\left(F(\lambda)-(a_1\lambda+b_1^*)\right)=2,$$

which by (3.6) we have

$$\frac{1}{2\pi}\Delta_{S_{R_{k_1}}} \arg\left(F(\lambda)-(a_1\lambda+b_1)\right)=2. \tag{3.8}$$

By again using the principle of argument, from (3.6), we obtain the relation

$$\sum_{\lambda_k\in \mathrm{B}_{R_{k_1}}}\varkappa(\lambda_k)-\sum_{\mu_k\in \mathrm{B}_{R_{k_1}}}\varkappa(\mu_k)=2$$

which implies that

$$\sum_{\lambda_k\in B_{R_{k_1}}}\varkappa(\lambda_k)=k_1+2. \tag{3.9}$$

By using the above argument, from (3.9), we obtain the relations

$$\sum_{\lambda_k\in \mathrm{B}_{R_k}}\varkappa(\lambda_k)=k+2,\ k=k_1,\ k_1+1,\ \ldots. \tag{3.10}$$

Let $0<b_1\leq\tilde{b}_1$. If $0<b_1<\tilde{b}_1$, then equation (2.4) has two simple roots in the interval B_2, and if $b_1=\tilde{b}_1$, then this equation has one double root in the interval B_2. Furthermore, the equation (2.4) has at least one root in each interval B_k, $k=1,\ 3,\ 4,\ \ldots$. Then, by formula (3.10) this equation has exactly one simple root in each interval B_k, $k=1,\ 3,\ 4,\ \ldots$.

Now let $b_1>\tilde{b}_1$. In this case the equation (2.4) has no root in the interval B_2, while has at least one root in each interval B_k, $k=1,\ 3,\ 4,\ \ldots$.

Let λ and μ, $\lambda\neq\mu$, be eigenvalues of the operator L. Since such an operator is J-self-adjoint in Π_2, it follows that the eigenvectors

$\hat{y}(\lambda)=\{y(x,\lambda),-a_0y(0,\lambda),\ a_1y(1,\lambda)\}$ and $\hat{y}(\mu)=\{y(x,\mu),-a_0y(0,\mu),\ a_1y(1,\mu)\}$

corresponding to eigenvalues λ and μ are J-orthogonal in Π_2; consequently, by (2.2), we obtain

$$\int_0^1 y(x,\lambda)\overline{y(x,\mu)}\,dx=-a_0y(0,\lambda)\overline{y(0,\mu)}-a_1y(1,\lambda)\overline{y(1,\mu)}\,. \tag{3.11}$$

On the other hand, multiplying the both parts of equation (1.1) by the function $\overline{y(x,\mu)}$ and integrating the obtained equality by parts in the range from 0 to 1, and also taking into account the boundary conditions (1.2)-(1.3) we have

$$\begin{gathered}\int\limits_0^1 y'(x,\lambda)\overline{y'(x,\mu)}\,dx - b_1 y(1,\lambda)\overline{y(1,\mu)} = \\ \lambda\left[\int\limits_0^1 y(x,\lambda)\overline{y(x,\mu)}\,dx + a_0 y(0,\lambda)\overline{y(0,\mu)} + a_1 y(1,\lambda)\overline{y(1,\mu)}\right]\end{gathered} \tag{3.12}$$

By (3.11), from (3.12) we obtain

$$\int\limits_0^1 \frac{y'(x,\lambda)}{y(1,\lambda)}\overline{\left(\frac{y'(x,\mu)}{y(1,\mu)}\right)}\,dx = b_1. \tag{3.13}$$

Consequently, we have

$$\int\limits_0^1 \frac{y'(x,\lambda)}{y(1,\lambda)}\frac{y'(x,\mu)}{y(1,\mu)}\,dx = b_1. \tag{3.14}$$

By adding the relations (3.13) and (3.14), we get

$$2\int\limits_0^1 \frac{y'(x,\lambda)}{y(1,\lambda)}\,\mathrm{Re}\frac{y'(x,\mu)}{y(1,\mu)}dx = 2b_1. \tag{3.15}$$

If $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{C}\backslash\mathbb{R}$, then it follows from (3.12) that

$$\begin{gathered}\int\limits_0^1 y'^2(x,\lambda)\,dx\; - b_1 y^2(1,\lambda) = \\ \lambda\left[\int\limits_0^1 y^2(x,\lambda)\,dx\; + a_0 y^2(0,\lambda) + b_1 y^2(1,\lambda)\right],\end{gathered} \tag{3.16}$$

and

$$\int\limits_0^1 \left|\frac{y'(x,\mu)}{y(1,\mu)}\right|^2 dx = b_1. \tag{3.17}$$

Note that, if $F'(\lambda) \leq a_1$ at $\lambda < 0$ or $F'(\lambda) \geq a_1$ at $\lambda > 0$, then taking into account the relation (2.5), from (3.16) we have

$$\int\limits_0^1 \left(\frac{y'(x,\lambda)}{y(1,\lambda)}\right)^2 dx \leq b_1. \tag{3.18}$$

By virtue of relations (3.15), (3.17) and (3.18), we obtain

$$\int\limits_0^1 \left\{\left(\frac{y'(x,\lambda)}{y(1,\lambda)} - \mathrm{Re}\frac{y'(x,\mu}{y(1,\mu)}\right)^2 + \mathrm{Im}^{\,2}\frac{y'(x,\mu}{y(1,\mu)}\right\} dx \leq 0,$$

with contradicts condition $\mu \in \mathbb{C}\backslash\mathbb{R}$. Hence, if $(\operatorname{sgn}\lambda)(F'(\lambda) - a_1) \geq 0$, then problem (1.1)- (1.3) does not have nonreal eigenvalues.

Further, if $\lambda, \mu \in \mathbb{R}$, $\lambda \neq \mu$ and $(\operatorname{sgn} \lambda)(F'(\lambda) - a_1) \geq 0$, $(\operatorname{sgn} \mu)(F'(\mu) - a_1) \geq 0$, then by following the corresponding argument above, we obtain

$$\int_0^1 \left(\frac{y'(x,\lambda)}{y(1,\lambda)} - \frac{y'(x,\mu}{y(1,\mu)} \right)^2 dx \leq 0,$$

which is impossible in view of condition $\lambda \neq \mu$.

Therefore, if $\lambda, \mu \in \mathbb{R}$, $\lambda \neq \mu$ be eigenvalues of problem (1.1)- (1.3) and $(\operatorname{sgn} \lambda)(F'(\lambda) - a_1) \geq 0$, then $(\operatorname{sgn} \mu)(F'(\mu) - a_1) < 0$.

Next, the proof of assertions (ii) and (iii) of second parts of theorem can be proved in accordance with the scheme of the proof of Theorem 4.1 in [2] with use of the formula (3.10) and the above reasoning. The proof of Theorem 3.1 is complete.

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