

MULTI-SUBLINEAR MAXIMAL OPERATOR AND MULTILINEAR SINGULAR INTEGRAL OPERATORS ON GENERALIZED MORREY SPACES

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Abstract. In this paper the authors study the boundedness of multi-sublinear maximal operator M_m and multilinear singular integral operators T_m on product generalized Morrey spaces $\mathcal{M}_{p_1, \varphi_1}(\mathbb{R}^n) \times \dots \times \mathcal{M}_{p_m, \varphi_m}$. We find the sufficient conditions on $(\varphi_1, \dots, \varphi_m, \varphi)$ which ensures the boundedness of the operators M_m and T_m from $\mathcal{M}_{p_1, \varphi_1}(\mathbb{R}^n) \times \dots \times \mathcal{M}_{p_m, \varphi_m}$ to $M_{p, \varphi}$. In all cases, the conditions for the boundedness of M_m are given in terms of suprema type inequalities on $(\varphi_1, \dots, \varphi_m, \varphi)$ and the conditions for the boundedness of T_m are given in terms of Zygmund-type integral inequalities on $(\varphi_1, \dots, \varphi_m, \varphi)$, which do not assume any assumption on monotonicity of $\varphi_1, \dots, \varphi_m, \varphi$ in r .

1. Introduction

Multilinear Calderón-Zygmund theory is a natural generalization of the linear case. The initial work on the class of multilinear Calderon-Zygmund operators was done by Coifman and Meyer in [4] and was later systematically studied by Grafakos and Torres in [8]-[10].

Let \mathbb{R}^n be the n -dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \dots \times \mathbb{R}^n$ be the m -fold product space ($m \in \mathbb{N}$). For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r , and by ${}^c B(x, r)$ denote its complement. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$. We denote by \vec{f} the m -tuple (f_1, f_2, \dots, f_m) , $\vec{y} = (y_1, \dots, y_n)$ and $d\vec{y} = dy_1 \cdots dy_n$.

The multilinear theory has been well developed in the past twenty years. In 1992, Grafakos [6] first study the following multilinear integrals, defined by

$$I_\alpha^m(\vec{f})(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} f_1(x - \theta_1 y) \dots f_m(x - \theta_m y) dy,$$

where $\theta_i (i = 1, \dots, m)$ are fixed distinct and nonzero real numbers and $0 < \beta < n$. Grafakos proved that the operator I_α^m is bounded from $L_{p_1}(\mathbb{R}^n) \times \dots \times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $0 < 1/q = 1/p_1 + \dots + 1/p_m - \beta/n < 1$, which can be regarded as an extension result for the classical fractional integral on Lebesgue

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spaces. In [14, 15] was proved a certain O'Neil type inequality for dilated multilinear convolution operators, including permutations of functions. This inequality was used to extend Grafakos result [6] to more general multi-linear operators of potential type and the relevant maximal operators.

Let $\vec{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \dots \times L_{p_m}^{loc}(\mathbb{R}^n)$. The multi-sublinear maximal operator M_m is defined by

$$M_m(\vec{f})(x) = \sup_{r>0} \prod_{i=1}^m \frac{1}{|B(x,r)|} \int_{B(x,r)} |f_i(y_i)| dy_i.$$

In 2002 Grafakos and Torres [7]-[10] studied the multilinear Calderón-Zygmund operator which can be written for $x \notin \bigcap_{j=1}^m \text{supp } f_j$ as

$$K_m(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} K(x, y_1, \dots, y_m) f_1(y_1) \dots f_m(y_m) dy_1 dy_2 \dots dy_m,$$

where $K(x, y_1, \dots, y_m)$ is the kernel function defined of the diagonal $x = y_1 = \dots = y_m$ in $(\mathbb{R}^n)^{m+1}$ satisfying

$$|K(y_0, y_1, \dots, y_m)| \leq c_1 \left(\sum_{k,l=0}^m |y_k - y_l| \right)^{-nm}, \quad (1.1)$$

and whenever $2|y_j - y'_j| \leq \frac{1}{2} \max_{0 \leq k \leq m} |y_j - y_k|$,

$$|K(y_0, \dots, y_j, \dots, y_m) - K(y_0, \dots, y'_j, \dots, y_m)| \leq \frac{c_1 |y_j - y'_j|^\epsilon}{\left(\sum_{k,l=0}^m |y_k - y_l| \right)^{nm+\epsilon}},$$

for some $\epsilon > 0$ and all $0 \leq j \leq m$. Grafakos and Torres [7] proved that the operator $K_m(\vec{f})$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \dots \times L_{p_m}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ for $p_i > 1$ ($i = 1, \dots, m$) and $1/p = 1/p_1 + \dots + 1/p_m$, and bounded from $L_1(\mathbb{R}^n) \times \dots \times L_1(\mathbb{R}^n)$ to $L_{\frac{1}{m}, \infty}(\mathbb{R}^n)$.

In this work, we prove the boundedness of the multi-sublinear maximal operator M_m and multilinear singular integral operators T_m from product generalized Morrey space $M_{p_1, \varphi_1} \times \dots \times M_{p_m, \varphi_m}$ to $M_{p, \varphi}$, if $1 < p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \dots + 1/p_m$, and from the space $M_{p_1, \varphi_1} \times \dots \times M_{p_m, \varphi_m}$ to the weak space $WM_{1, \varphi}$, if $1 \leq p_1, \dots, p_m < \infty$, $1/p = 1/p_1 + \dots + 1/p_m$ and at least one exponent p_i equals one.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Generalized Morrey spaces

In the study of local properties of solutions to of partial differential equations, together with weighted Lebesgue spaces, Morrey spaces $M_{p, \lambda}(\mathbb{R}^n)$ play an important role, see [5], [17]. Introduced by C. Morrey [19] in 1938, they are defined by

the norm

$$\|f\|_{M_{p,\lambda}} := \sup_{x, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

where $0 \leq \lambda < n$, $1 \leq p < \infty$.

We also denote by $WM_{p,\lambda}$ the weak Morrey space of all functions $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\lambda}} = \sup_{x \in \mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))} < \infty,$$

where WL_p denotes the weak L_p -space.

We find it convenient to define the generalized Morrey spaces in the form as follows.

Definition 2.1. Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \leq p < \infty$. We denote by $M_{p,\varphi} \equiv M_{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{L_p(B(x,r))}.$$

Also by $WM_{p,\varphi} \equiv WM_{p,\varphi}(\mathbb{R}^n)$ we denote the weak generalized Morrey space of all functions $f \in WL_p^{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x, r)^{-1} |B(x, r)|^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the spaces $M_{p,\lambda}$ and $WM_{p,\lambda}$ under the choice $\varphi(x, r) = r^{\frac{\lambda-n}{p}}$:

$$M_{p,\lambda} = M_{p,\varphi} \Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}},$$

$$WM_{p,\lambda} = WM_{p,\varphi} \Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

In [20], the following condition was imposed on $\varphi(x, r)$:

$$c^{-1}\varphi(x, r) \leq \varphi(x, t) \leq c\varphi(x, r) \tag{2.1}$$

whenever $r \leq t \leq 2r$, where $c(\geq 1)$ does not depend on t, r and $x \in \mathbb{R}^n$, jointly with the condition:

$$\int_r^\infty \varphi(x, t)^p \frac{dt}{t} \leq C \varphi(x, r)^p, \tag{2.2}$$

for the maximal or singular operator, where $C(> 0)$ does not depend on r and $x \in \mathbb{R}^n$.

In [20] the following statements were proved.

Theorem 2.1. [20] *Let $1 \leq p < \infty$ and $\varphi(x, r)$ satisfies the conditions (2.1)-(2.2). Then for $p > 1$ the operators M and T are bounded in $M_{p,\varphi}(\mathbb{R}^n)$ and for $p = 1$ M and T are bounded from $M_{1,\varphi}(\mathbb{R}^n)$ to $WM_{1,\varphi}(\mathbb{R}^n)$.*

The following statement, containing results obtained in [18], [20] was proved in [11] (see also [12, 13, 21]).

Theorem 2.2. *Let $1 \leq p < \infty$ and (φ_1, φ) satisfies the condition*

$$\int_r^\infty \varphi_1(x, t) \frac{dt}{t} \leq C \varphi(x, r), \quad (2.3)$$

where C does not depend on x and r . Then the maximal operator M and Calderón-Zygmund operator T are bounded from M_{p, φ_1} to $M_{p, \varphi}$ for $p > 1$ and from M_{1, φ_1} to $WM_{1, \varphi}$.

The following statements, containing results Theorems 2.1 and 2.2 was proved in [1], see also [16].

Theorem 2.3. *Let $1 \leq p < \infty$ and (φ_1, φ) satisfy the condition*

$$\sup_{r < t < \infty} \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) s^{\frac{n}{p}}}{t^{\frac{n}{p}}} \leq C \varphi(x, r), \quad (2.4)$$

where C does not depend on x and r . Let the operator M is bounded from M_{p, φ_1} to $M_{p, \varphi}$ for $p > 1$ and from M_{1, φ_1} to $WM_{1, \varphi}$.

Theorem 2.4. *Let $1 \leq p < \infty$ and (φ_1, φ) satisfy the condition*

$$\int_r^\infty \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_1(x, s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \leq C \varphi(x, r), \quad (2.5)$$

where C does not depend on x and r . Let the operator T is bounded from M_{p, φ_1} to $M_{p, \varphi}$ for $p > 1$ and from M_{1, φ_1} to $WM_{1, \varphi}$.

Remark 2.1. It is obvious that if condition (2.3) holds, then condition (3.5) holds too. In general, condition (3.5) does not imply condition (2.3). For example, the functions

$$\varphi_1(r) = \frac{1}{\chi_{(1, \infty)}(r) r^{\frac{n}{p}-\beta}}, \quad \varphi_2(r) = r^{-\frac{n}{p}} (1 + r^\beta), \quad 0 < \beta < \frac{n}{p}$$

satisfy condition (3.5) but do not satisfy condition (2.3) (see [16]).

3. The multi-sublinear maximal operator in the product spaces

$$\mathcal{M}_{p_1, \varphi_1}(\mathbb{R}^n) \times \dots \times \mathcal{M}_{p_m, \varphi_m}(\mathbb{R}^n)$$

Let v be a weight. We denote by $L_{\infty, v}(0, \infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty, v}(0, \infty)} = \sup_{t > 0} v(t) |g(t)|$$

and $L_\infty(0, \infty) \equiv L_{\infty, 1}(0, \infty)$. Let $\mathfrak{M}(0, \infty)$ be the set of all Lebesgue-measurable functions on $(0, \infty)$ and $\mathfrak{M}^+(0, \infty)$ its subset of all nonnegative functions on $(0, \infty)$. We denote by $\mathfrak{M}^+(0, \infty; \uparrow)$ the cone of all functions in $\mathfrak{M}^+(0, \infty)$ which are non-decreasing on $(0, \infty)$ and

$$\mathcal{A} = \left\{ \varphi \in \mathfrak{M}^+(0, \infty; \uparrow) : \lim_{t \rightarrow 0^+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $(0, \infty)$. We define the supremal operator \bar{S}_u on $g \in \mathfrak{M}(0, \infty)$ by

$$(\bar{S}_u g)(t) := \|u g\|_{L_\infty(t, \infty)}, \quad t \in (0, \infty).$$

The following theorem was proved in [2].

Theorem 3.1. *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t, \infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on $(0, \infty)$. Then the operator \bar{S}_u is bounded from $L_{\infty, v_1}(0, \infty)$ to $L_{\infty, v_2}(0, \infty)$ on the cone \mathcal{A} if and only if*

$$\left\| v_2 \bar{S}_u \left(\|v_1\|_{L_\infty(\cdot, \infty)}^{-1} \right) \right\|_{L_\infty(0, \infty)} < \infty. \tag{3.1}$$

In this section, we will prove the boundedness of multi-sublinear maximal operators on product generalized Morrey space, first we prove the following theorem.

Theorem 3.2. *Let $1 \leq p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \dots + 1/p_m$. Then, for $1 < p_1, \dots, p_m < \infty$ the inequality*

$$\|M_m(\vec{f})\|_{L_p(B(x_0, r))} \lesssim r^{\frac{n}{p}} \prod_{i=1}^m \sup_{t > 2r} t^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0, t))} \tag{3.2}$$

holds for any ball $B(x_0, r)$ and for all $\vec{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \dots \times L_{p_m}^{loc}(\mathbb{R}^n)$.

Moreover, if at least one exponent p_i equals one, the inequality

$$\|M_m(\vec{f})\|_{WL_p(B(x_0, r))} \lesssim r^{\frac{n}{p}} \prod_{i=1}^m \sup_{t > 2r} t^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0, t))} \tag{3.3}$$

holds for any ball $B(x_0, r)$ and for all $\vec{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \dots \times L_{p_m}^{loc}(\mathbb{R}^n)$.

Proof. $1 < p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \dots + 1/p_m$. For arbitrary $x_0 \in \mathbb{R}^n$, set $B = B(x_0, r)$ for the ball centered at x_0 and of radius r , $2B = B(x_0, 2r)$. We represent f as

$$f_j = f_j^0 + f_j^\infty, \quad f_j^0 = f_j \chi_{2B}, \quad f_j^\infty = f_j \chi_{\mathbb{R}^n \setminus 2B}, \quad j = 1, \dots, m. \tag{3.4}$$

Thus for $y \in B(x_0, r)$ we get

$$\begin{aligned} M_m(\vec{f})(y) &= \sup_{t > 0} \prod_{i=1}^m \left(\frac{1}{|B(y, t)|} \int_{B(y, t)} |f_i^0(z_i) + f_i^\infty(z_i)| dz_i \right) \\ &\leq \sup_{t > 0} \prod_{i=1}^m \left(\frac{1}{|B(y, t)|} \int_{B(y, t)} |f_i^0(z_i)| dz_i + \frac{1}{|B(y, t)|} \int_{B(y, t)} |f_i^\infty(z_i)| dz_i \right) \\ &\leq \sup_{t > 0} \left(\prod_{i=1}^m A_{B(y, t)} f_i^0 \right) + \sup_{t > 0} \left(\sum' A_{B(y, t)} f_1^{\beta_1} \dots A_{B(y, t)} f_m^{\beta_m} \right) \\ &= I_1(y) + I_2(y), \end{aligned}$$

where $\beta_1, \dots, \beta_m \in \{0, \infty\}$ and each term in the sum \sum' contains at least one $\beta_i = 1$, and where we denote

$$A_{B(y, t)} f_i^{\beta_i} = \frac{1}{|B(y, t)|} \int_{B(y, t)} |f_i^{\beta_i}(z_i)| dz_i.$$

By the boundedness of $\mathcal{M} : L_{p_1}(\mathbb{R}^n) \times \dots \times L_{p_m}(\mathbb{R}^n) \rightarrow L_p(\mathbb{R}^n)$ we have

$$\begin{aligned} \|I_1\|_{L_p(B(x_0, r))} &\leq \|\mathcal{M}(\vec{f}^0)\|_{L_p(B(x_0, r))} \\ &\leq C \prod_{i=1}^m \|f_i^0\|_{L_{p_i}(\mathbb{R}^n)} \\ &\leq C \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x_0, 2r))}. \\ &\leq C \prod_{i=1}^m r^{\frac{n}{p_i}} \sup_{t>2r} t^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0, t))}. \end{aligned}$$

To treat the term $I_2(y)$, we first consider the case $\beta_1 = \beta_2 = \dots = \beta_m = \infty$.

Let y be an arbitrary point from B . If $B(y, t) \cap {}^c(2B) \neq \emptyset$, then $t > r$. Indeed, if $z_i \in B(y, t) \cap {}^c(2B)$, then $t > |y - z_i| \geq |x - z_i| - |x - y| > 2r - r = r$ for $i = 1, \dots, m$.

On the other hand, $B(y, t) \cap {}^c(2B) \subset B(x_0, 2t)$. Indeed, $z_i \in B(y, t) \cap {}^c(2B)$, then we get $|x_0 - z_i| \leq |y - z_i| + |x_0 - y| < t + r < 2t$ for $i = 1, \dots, m$.

$$\begin{aligned} \sup_{t>0} A_{B(y, t)} f_1^\infty \dots A_{B(y, t)} f_m^\infty &= \prod_{i=1}^m \sup_{t>0} \frac{1}{|B(y, t)|} \int_{B(y, t) \cap {}^c B(x_0, 2r)} |f_i(z_i)| dz_i \\ &\leq 2^{nm} \prod_{i=1}^m \sup_{t>r} \frac{1}{|B(x_0, 2t)|} \int_{B(x_0, 2t)} |f_i(z_i)| dz_i \\ &\leq 2^{nm} \prod_{i=1}^m \sup_{t>2r} \frac{1}{|B(x_0, t)|} \int_{B(x_0, t)} |f_i(z_i)| dz_i \\ &\lesssim \prod_{i=1}^m \sup_{t>2r} t^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0, t))}. \end{aligned}$$

Therefore, for all $y \in B$ we have

$$\sup_{t>0} A_{B(y, t)} f_1^\infty \dots A_{B(y, t)} f_m^\infty \lesssim \prod_{i=1}^m \sup_{t>2r} t^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0, t))}.$$

Then

$$\begin{aligned} \left\| \sup_{t>0} A_{B(y, t)} f_1^\infty \dots A_{B(y, t)} f_m^\infty \right\|_{L_p(B)} &\lesssim r^{\frac{n}{p}} \prod_{i=1}^m \sup_{t>2r} t^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0, t))} \\ &= \prod_{i=1}^m r^{\frac{n}{p_i}} \sup_{t>2r} t^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0, t))}. \end{aligned}$$

For the case that $\beta_{j_1} = \dots = \beta_{j_l} = 0$ for some $\{j_1, \dots, j_l\} \subset \{1, \dots, m\}$ where $1 \leq l < m$, we only consider the case $\beta_1 = \infty$ since the other ones follow in analogous way. Note that

$$\sup_{t>0} A_{B(y, t)} f_1^\infty \dots A_{B(y, t)} f_m^\infty \lesssim r^{\frac{n}{p_1}} \sup_{t>2r} t^{-\frac{n}{p_1}} \|f_1\|_{L_{p_1}(B(x_0, t))} M f_2^0(x_0) \dots M f_m^0(x_0).$$

Then combine the estimates above we can easily get that

$$\begin{aligned}
 & \left\| \sup_{t>0} A_{B(y,t)} f_1^\infty A_{B(y,t)} f_2^0 \cdots A_{B(y,t)} f_m^0 \right\|_{L_p(B)} \\
 & \lesssim r^{\frac{n}{p_1}} \sup_{t>2r} t^{-\frac{n}{p_1}} \|f_1\|_{L_{p_1}(B(x_0,t))} \prod_{i=2}^m \|M f_i^0\|_{L_{p_i}(B)} \\
 & \leq r^{\frac{n}{p_1}} \sup_{t>2r} t^{-\frac{n}{p_1}} \|f_1\|_{L_{p_1}(B(x_0,t))} \prod_{i=2}^m \left(|B|^{\frac{1}{p_i}} \sup_{t>2r} t^{-\frac{n}{p_i}} \|f_i^0\|_{L_{p_i}(B(x_0,t))} \right) \\
 & \approx r^{\frac{n}{p_1}} \sup_{t>2r} t^{-\frac{n}{p_1}} \|f_1\|_{L_{p_1}(B(x_0,t))} \prod_{i=2}^m \|f_i\|_{L_{p_i}(B(x_0,2r))} \\
 & \lesssim \prod_{i=2}^m r^{\frac{n}{p_i}} \sup_{t>2r} t^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0,t))}.
 \end{aligned}$$

Hence we have obtained

$$\begin{aligned}
 \|M_m(\vec{f})\|_{L_p(B)} & \leq \|I_1\|_{L_p(B)} + \|I_2\|_{L_p(B)} \\
 & \lesssim \prod_{i=2}^m r^{\frac{n}{p_i}} \sup_{t>2r} t^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x_0,t))}.
 \end{aligned}$$

Thus we obtain (4.1).

For the case that at least one exponent p_i equals one, repeat the estimates above and note that $\vec{f} \rightarrow M_m(\vec{f})$ is boundedness from $L^{p_1}(\mathbb{R}^n) \times \cdots \times L^{p_m}(\mathbb{R}^n)$ to $L^{p,\infty}(\mathbb{R}^n)$, the proof of (4.2) can be treated similarly and we omit the details here. \square

Next we give the boundedness of multilinear maximal operator $\vec{f} \rightarrow M_m(\vec{f})$ on product generalized Morrey space.

Theorem 3.3. *Let $1 \leq p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \cdots + 1/p_m$, and suppose that (φ_1, φ_2) satisfies the condition*

$$\prod_{i=1}^m \sup_{r<t<\infty} \frac{\text{ess inf}_{t<s<\infty} \varphi_i(x, s) s^{\frac{n}{p_i}}}{t^{\frac{n}{p_i}}} \leq C \psi(x, r) r^{\frac{n}{p}}, \quad (3.5)$$

where C does not depend on x and r . Then, if all $p_i > 1$, it follows

$$\|M_m(\vec{f})\|_{\mathcal{M}_{p,\psi}} \leq C \|f_1\|_{\mathcal{M}_{p_1,\varphi_1}} \cdots \|f_m\|_{\mathcal{M}_{p_m,\varphi_m}},$$

and if at least one $p_i = 1$, it follows

$$\|M_m(\vec{f})\|_{W\mathcal{M}_{p,\psi}} \leq C \|f_1\|_{\mathcal{M}_{p_1,\varphi_1}} \cdots \|f_m\|_{\mathcal{M}_{p_m,\varphi_m}},$$

with the constant C independent of \vec{f} .

Proof. Let $1 \leq p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \dots + 1/p_m$ and $\vec{f} \in \mathcal{M}_{p_1, \varphi_1} \times \dots \times \mathcal{M}_{p_m, \varphi_m}$. By Theorems 3.1 and 3.2 we obtain

$$\begin{aligned} \|M_m(\vec{f})\|_{\mathcal{M}_p, \psi} &= \sup_{x \in \mathbb{R}^n, r > 0} \psi^{-1}(x, r) r^{-\frac{n}{p}} \|M_m(\vec{f})\|_{L_p(B(x, r))} \\ &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \prod_{i=1}^m \psi^{-\frac{1}{m}}(x, r) \sup_{t > 2r} t^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x, t))} \\ &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \prod_{i=1}^m \varphi_i^{-1}(x, r) r^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x, r))} \\ &\lesssim \sup_{x \in \mathbb{R}^n, r > 0} \prod_{i=1}^m \|f_i\|_{\mathcal{M}_{p_i, \varphi_i}} \cdots \|f_m\|_{\mathcal{M}_{p_m, \varphi_m}} \end{aligned}$$

by (3.5), which completes the proof for $1 < p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \dots + 1/p_m$.

For $p_i = 1$ and $f_i \in \mathcal{M}_{1, \varphi_i}$ ($i = 1, \dots, m$), by the definition of $\mathcal{M}_{1, \varphi}$ and a similar argument as before we can get

$$\|M_m(\vec{f})\|_{W\mathcal{M}_{\frac{1}{m}, \psi}} \leq C \|f_1\|_{\mathcal{M}_{p_1, \varphi_1}} \cdots \|f_m\|_{\mathcal{M}_{p_m, \varphi_m}}.$$

The theorem has been proved. \square

Remark 3.1. Note that in the case $m = 1$ Theorems 3.2 and 3.3 were proved in [1] (see also [16]). Theorem 3.3 do not impose the pointwise doubling condition (2.1) and (2.2). In the case $\varphi_1(x, r) = \varphi_2(x, r) = \varphi(x, r)$ Theorem 3.3 containing the results Theorem 2.1.

4. The multilinear singular integral operators in the product spaces $\mathcal{M}_{p_1, \varphi_1}(\mathbb{R}^n) \times \dots \times \mathcal{M}_{p_m, \varphi_m}(\mathbb{R}^n)$

In this section we are going to use the following statement on the boundedness of the Hardy operator

$$(Hg)(t) := \frac{1}{t} \int_0^t g(r) dr, \quad 0 < t < \infty.$$

Theorem 4.1. ([3]) *The inequality*

$$\operatorname{ess\,sup}_{t>0} w(t) Hg(t) \leq c \operatorname{ess\,sup}_{t>0} v(t) g(t)$$

holds for all non-negative and non-increasing g on $(0, \infty)$ if and only if

$$A := \sup_{t>0} \frac{w(t)}{t} \int_0^t \frac{dr}{\operatorname{ess\,sup}_{0<s<r} v(s)} < \infty,$$

and $c \approx A$.

In this section, we will prove the boundedness of multilinear singular integral operators on product generalized Morrey space, first we prove the following theorem.

Theorem 4.2. *Let $1 \leq p_1, \dots, p_m < \infty$ and $1/p = 1/p_1 + \dots + 1/p_m$. Then, for $1 < p_1, \dots, p_m < \infty$ the inequality*

$$\|T_m(\vec{f})\|_{L_p(B(x_0, r))} \lesssim r^{\frac{n}{p}} \prod_{i=1}^m \int_{2r}^{\infty} t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x_0, t))} dt \quad (4.1)$$

holds for any ball $B(x_0, r)$ and for all $\vec{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \dots \times L_{p_m}^{loc}(\mathbb{R}^n)$.

Moreover, if at least one exponent p_i equals one, the inequality

$$\|T_m(\vec{f})\|_{W L_p(B(x_0, r))} \lesssim r^{\frac{n}{p}} \prod_{i=1}^m \int_{2r}^{\infty} t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x_0, t))} dt \quad (4.2)$$

holds for any ball $B(x_0, r)$ and for all $\vec{f} \in L_{p_1}^{loc}(\mathbb{R}^n) \times \dots \times L_{p_m}^{loc}(\mathbb{R}^n)$.

Proof. We just consider the case $p_i > 1$ for $i = 1, \dots, m$ and write $f_i = f_i^0 + f_i^\infty$. Then we split $T_m(\vec{f})$ as follows

$$T_m(\vec{f})(x) = T_m(f_1^0, \dots, f_m^0)(x) + \sum_{\beta_1, \dots, \beta_m} 'T_m(f_1^{\beta_1}, \dots, f_m^{\beta_m})(x),$$

where $\beta_1, \dots, \beta_m \in \{0, \infty\}$ and each term of \sum' contains at least $\beta_i \neq 0$. Then,

$$\begin{aligned} \|T_m(\vec{f})\|_{L_p(B(x, r))} &\leq \|T_m(f_1^0, \dots, f_m^0)\|_{L_p(B(x, r))} \\ &+ \left\| \sum_{\beta_1, \dots, \beta_m} 'T_m(f_1^{\beta_1}, \dots, f_m^{\beta_m}) \right\|_{L_p(B(x, r))} \leq I + II. \end{aligned}$$

For I , by the boundedness of T_m from product $L_{p_1}(\mathbb{R}^n) \times \dots \times L_{p_m}(\mathbb{R}^n)$ to $L_p(\mathbb{R}^n)$ with $1/p = 1/p_1 + \dots + 1/p_m$ for each $p_i > 1$ ($i = 1, \dots, m$), we have,

$$\|T_m(\vec{f}^0)\|_{L_p(B(x, r))} \leq \|T_m(\vec{f}^0)\|_{L_p(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i^0\|_{L_{p_i}(\mathbb{R}^n)} \lesssim \prod_{i=1}^m \|f_i\|_{L_{p_i}(B(x, 2r))}.$$

Taking into account that

$$\|f_i\|_{L_{p_i}(B(x, 2r))} \lesssim r^{\frac{n}{p_i}} \int_{2r}^{\infty} t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x, t))} dt, \quad i = 1, \dots, m$$

we get

$$\|T_m(\vec{f}^0)\|_{L_p(B(x, r))} \lesssim r^{\frac{n}{p}} \prod_{i=1}^m \int_{2r}^{\infty} t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x, t))} dt. \quad (4.3)$$

For II , first we consider the case $\beta_1 = \dots = \beta_m = \infty$.

When $|x - y_i| \leq r$, $|z - y_i| \geq 2r$, we have $\frac{1}{2}|z - y_i| \leq |x - y_i| \leq \frac{3}{2}|z - y_i|$, and so by the condition (1.1) we have

$$\begin{aligned} |T_m(\vec{f}^\infty)(z)| &\lesssim \int_{(c_{B(x, 2r)})^m} \frac{|f_1(y_1) \cdots f_m(y_m)|}{|(x - y_1, \dots, x - y_m)|^{mn}} d\vec{y} \\ &\lesssim \prod_{i=1}^m \int_{c_{B(x, 2r)}} \frac{|f_i(y_i)|}{|x - y_i|^n} dy_i \end{aligned}$$

and

$$\begin{aligned} \|T_m(\overrightarrow{f^\infty})\|_{L_p(B(x,r))} &\leq \prod_{i=1}^m \int_{\mathring{c}_{B(x,2r)}} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \|\chi_{B(x,r)}\|_{L_p(\mathbb{R}^n)} \\ &\lesssim r^{\frac{n}{p}} \prod_{i=1}^m \int_{\mathring{c}_{B(x,2r)}} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i. \end{aligned}$$

By Fubini's theorem we have

$$\begin{aligned} \int_{\mathring{c}_{B(x,2r)}} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i &\approx \int_{\mathring{c}_{B(x,2r)}} |f_i(y_i)| \int_{|x_0-y_i|}^{\infty} \frac{dt}{t^{n+1}} dy_i \\ &\approx \int_{2r}^{\infty} \int_{2r \leq |x_0-y_i| < t} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}} \\ &\lesssim \int_{2r}^{\infty} \int_{B(x_0,t)} |f_i(y_i)| dy_i \frac{dt}{t^{n+1}}. \end{aligned}$$

Applying Hölder's inequality, we get

$$\int_{\mathring{c}_{B(x,2r)}} \frac{|f_i(y_i)|}{|x-y_i|^n} dy_i \lesssim \int_{2r}^{\infty} t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x,t))} dt. \quad (4.4)$$

Moreover, for all $p_i \in [1, \infty)$, $i = 1, \dots, m$ the inequality

$$\|T_m(\overrightarrow{f^\infty})\|_{L_p(B(x,r))} \lesssim r^{\frac{n}{p}} \prod_{i=1}^m \int_{2r}^{\infty} t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x,t))} dt \quad (4.5)$$

is valid.

Next we consider the case that some $\alpha_i = 0$ and other $\alpha_j = \infty$. To this end we may assume that $\alpha_1 = \alpha_2 = \infty$ and $\alpha_3 = \dots = \alpha_m = 0$. Recall the condition (1.1) and the fact that $|x-y_i| \approx |z-y_i|$ for $z \in B(x,r)$ and $y_i \in \mathring{c}_{B(x,2r)}$, we have that

$$\begin{aligned} &T(f_1^\infty, f_2^\infty, f_3^0, \dots, f_m^0)(z) \\ &\lesssim \int_{\mathring{c}_{B(x,2r)} \times \mathring{c}_{B(x,2r)}} \frac{|f_1(y_1)| |f_2(y_2)|}{\{|x-y_1| + |x-y_2|\}^{mn}} dy_1 dy_2 \prod_{i=3}^m \int_{B(x,2r)} |f_i(y_i)| dy_i \\ &\lesssim \int_{\mathring{c}_{B(x,2r)}} \frac{|f_1(y_1)|}{|x-y_1|^n} dy_1 \int_{\mathring{c}_{B(x,2r)}} \frac{|f_2(y_2)|}{|x-y_2|^n} dy_2 \prod_{i=3}^m \int_{B(x,2r)} |f_i(y_i)| dy_i. \end{aligned}$$

By the inequality (4.4) and use the Hölder's inequality for integrals,

$$\begin{aligned}
 & \|T(f_1^\infty, f_2^\infty, f_3^0, \dots, f_m^0)\|_{L_p(B(x,r))} \\
 & \lesssim r^{-(m-2)n+\frac{n}{p}} \int_{\mathfrak{C}_{B(x,2r)}} \frac{|f_1(y_1)|}{|x-y_1|^m} dy_1 \int_{\mathfrak{C}_{B(x,2r)}} \frac{|f_2(y_2)|}{|x-y_2|^n} dy_2 \prod_{i=3}^m \int_{B(x,2r)} |f_i(y_i)| dy_i \\
 & \lesssim r^{\frac{n}{p}} \int_{\mathfrak{C}_{B(x,2r)}} \frac{|f_1(y_1)|}{|x-y_1|^n} dy_1 \int_{\mathfrak{C}_{B(x,2r)}} \frac{|f_2(y_2)|}{|x-y_2|^n} dy_2 \prod_{i=3}^m r^{-\frac{n}{p_i}} \int_{B(x,2r)} |f_i(y_i)| dy_i \\
 & \leq r^{\frac{n}{p}} \prod_{i=3}^m \int_r^\infty t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x,t))} dt.
 \end{aligned}$$

For the proof of the inequality (4.2), by a similar argument as in the proof of (4.1) and pay attention to the fact that $\vec{f} \rightarrow T_m(\vec{f})$ is bounded from $L_{p_1}(\mathbb{R}^n) \times \dots \times L_{p_m}(\mathbb{R}^n)$ to $WL_p(\mathbb{R}^n)$, we can similarly prove (4.2) and we omit the details here. \square

Now we give the boundedness of multilinear singular integral operators on product generalized Morrey space.

Theorem 4.3. *Let $1 \leq p_1, \dots, p_m < \infty$ with $1/p = 1/p_1 + \dots + 1/p_m$ and $(\varphi_1, \dots, \varphi_m, \varphi)$ satisfies the condition*

$$\prod_{i=1}^m \int_r^\infty \frac{\operatorname{ess\,inf}_{t < s < \infty} \varphi_i(x, s) s^{\frac{n}{p_i}}}{t^{\frac{n}{p_i}+1}} dt \lesssim \varphi(x, r). \quad (4.6)$$

Then the operator T_m is bounded from product space $M_{p_1, \varphi_1}(\mathbb{R}^n) \times \dots \times M_{p_m, \varphi_m}(\mathbb{R}^n)$ to $M_{p, \varphi}(\mathbb{R}^n)$ for $p_i > 1$, $i = 1, \dots, m$, and from product space $M_{p_1, \varphi_1}(\mathbb{R}^n) \times \dots \times M_{p_m, \varphi_m}(\mathbb{R}^n)$ to $WM_{p, \varphi}(\mathbb{R}^n)$ for $p_i \geq 1$, $i = 1, \dots, m$.

Proof. Let $1 < p_1, \dots, p_m < \infty$ and $\vec{f} \in M_{p_1, \varphi_1}(\mathbb{R}^n) \times \dots \times M_{p_m, \varphi_m}(\mathbb{R}^n)$. By Theorems 4.1 and 4.2 we have

$$\begin{aligned}
 \|T_m(\vec{f})\|_{M_{p, \varphi}} & \lesssim \sup_{x \in \mathbb{R}^n, r > 0} \varphi(x, r)^{-1} \prod_{i=1}^m \int_r^\infty t^{-\frac{n}{p_i}-1} \|f_i\|_{L_{p_i}(B(x,t))} dt \\
 & \approx \sup_{x \in \mathbb{R}^n, r > 0} \prod_{i=1}^m \varphi(x, r)^{-\frac{1}{m}} \int_0^r t^{-\frac{n}{p_i}} \|f_i\|_{L_{p_i}(B(x, t^{-\frac{p_i}{n}}))} dt \\
 & = \sup_{x \in \mathbb{R}^n, r > 0} \prod_{i=1}^m \varphi(x, r^{-\frac{p_i}{n}})^{-\frac{1}{m}} \int_0^r \|f_i\|_{L_{p_i}(B(x, t^{-\frac{p_i}{n}}))} dt \\
 & \lesssim \prod_{i=1}^m \sup_{x \in \mathbb{R}^n, r > 0} \varphi_i(x, r^{-\frac{p_i}{n}})^{-1} r \|f_i\|_{L_{p_i}(B(x, r^{-\frac{p_i}{n}}))} \\
 & = \prod_{i=1}^m \|f_i\|_{M_{p_i, \varphi_i}}.
 \end{aligned}$$

When $p_i = 1$, $i = 1, \dots, m$, the proof is similar and we omit the details here. \square

Remark 4.1. As shown in [16], the condition (3.5) is weaker than (4.6): the latter implies the former, in particular, the functions

$$\varphi_i(r) = \frac{1}{\chi_{(1,\infty)}(r)r^{\frac{n}{p_i}-\alpha_i}}, \quad 0 < \alpha_i < \frac{n}{p_i}, \quad i = 1, \dots, m, \quad \psi(r) = r^{-\frac{n}{p}}(1+r^\alpha), \quad 0 < \alpha < \frac{n}{p}$$

satisfy condition (3.5) but do not satisfy condition (4.6).

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