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b-FRAMES, b-ATOMIC DECOMPOSITIONS, BANACH g-FRAMES AND THEIR PERTURBATIONS UNDER NOETHERIAN MAPS

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Abstract. The work is devoted to the generalization of concepts of frames and atomic decomposition in Hilbert and Banach spaces. The concept of b-frame and b-atomic decomposition with respect to some Banach space of vector sequences is introduced, here b is some bounded bilinear map. The perturbations of b-frames and b-atomic decompositions under Noetherian map are studied. The stability properties of g-Banach frames are also investigated.

1. Introduction

Frames for a Hilbert space were formally defined by R. J. Duffin and A. C. Schaeffer [12] to study some deep problems in nonharmonic Fourier series. The frames are generalization of orthonormal bases and frame theory plays a fundamental role in signal processing, image processing, date compression, sampling theory, and more, as well as being a fruitful area of research in abstract mathematics, such as in characterization of functional spaces. That is why there is a great interest of studying of frames. Basic information on the frames can be found for example, in [18, 9]. Banach space case of frames takes its origins from the work [13]. In [13] the concept of a frame for a Banach space and atomic decomposition was introduced. One of the main problems in theory of frames is to study the stability of it. The Paley-Wiener type theorems in case of both Hilbert and Banach spaces were proved, for example, in [3, 6, 10, 7, 8, 14, 15]. The frame properties of trigonometric system with degeneration, when the coefficient of degeneration does not satisfy the Machenhoupt condition were studied in [4, 16].

One of the generalizations of frames for Hilbert spaces is the concept of g-frame, introduced in [17]. There are some other generalizations of frames for Hilbert and Banach spaces. In [2] the concept of p-frame in Banach spaces is introduced. The concept p-frame for Banach space was introduced in [11]. pg-frames, g-Banach frames were introduced as a generalization of p-frames and the stability properties of them were investigated in [1].

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In this paper it is considered the bilinear map b, by means of which the concept of b-frame for Hilbert and Banach spaces, as well as b-atomic decomposition in Banach space with respect to another Banach space of sequences of vectors is given and the perturbation of b-frames and b-atomic decomposition under a Noetherian operator, as well as the stability properties are studied. The stability of g-frames is also studied in this work. Before the Noetherian perturbations of frames were considered in [5].

2. Necessary concepts and denotations

Let X, Y, Z be Banach spaces with norms $\|\|_X$, $\|\|_Y$ and $\|\|_Z$, respectively, $b(x,y) : X \times Y \to Z$ is a bilinear form, which satisfies

$$\exists M > 0 : \|b(x,y)\|_Z \leq M \|x\|_X \|y\|_Y, \forall x \in X, \forall y \in Y. \tag{2.1}$$

Let $Y_0 \subset Y$. Denote by $L_b(Y_0)$ the set of all finite sums of the form $\sum_k b(x_k, y_k)$, where $x_k \in X$, $y_k \in Y_0$.

Definition 2.1. [4]. The system $\{y_n\}_{n\in N} \subset Y$ is called b-complete in Z, if $\overline{L_b(\{y_n\}_{n\in N})} = Z$, where $\overline{L_b(\{y_n\}_{n\in N})}$ is a closure of the set $L_b(\{y_n\}_{n\in N})$. The system $\{y_n\}_{n\in N} \subset Y$ is called b-basis in Z, if each $z \in Z$ can uniquely be represented as $z = \sum_{n=1}^{\infty} b(x_n, y_n)$, $x_n \in X$.

Let $\tilde{X}$ be a Banach space of sequences of vectors of X with coordinatewise linear operations, such that the operator $P_k : X \to \tilde{X}$, $P_k(x) = \{\delta_{ik}x\}_{i\in N}$ is bounded (briefly KB-space). If $\lim_{n\to\infty} \left\|\{x_k\}_{k\in N} - \sum_{k=1}^{n} \{\delta_{ik}x_k\}_{i\in N}\right\|_{\tilde{X}} = 0$ for each $\{x_k\}_{k\in N} \in \tilde{X}$, then $\tilde{X}$ is called a CB-space.

Obviously, if $\tilde{X}$ is a CB-space, then $(\tilde{X})^*$ is isometrically isomorph with the space

$$\tilde{X}^* = \left\{\{t_k\}_{k\in N} \subset X^* : t_k = \tilde{t}P_k, \tilde{t} \in \left(\tilde{X}\right)^*\right\}$$

with the norm $\left\|\{t_k\}_{k\in N}\right\|_{\tilde{X}^*} = \left\|\tilde{t}\right\|_{(\tilde{X})^*}$. Each $\tilde{t} \in (\tilde{X})^*$ is given by $\tilde{t}(\{x_k\}_{k\in N}) = \sum_{k=1}^{\infty} t_k(x_k)$. Furthermore, $(\tilde{X})^*$ is identified with $\tilde{X}^*$.

Let $\tilde{Y}$ be a KB-space on Y. Say that $\tilde{X}$ is normally subordinated to $\tilde{Y}$ if for sequences $\{x_n\} \subset X$, $\{y_n\} \subset Y$ such that $\|\{x_n\}\|_X \leq \|\{y_n\}\|_Y$ from $\{y_n\} \in \tilde{Y}$, it follows that $\{x_n\} \in \tilde{X}$ and $\|\{x_n\}\|_{\tilde{X}} \leq \|\{y_n\}\|_{\tilde{Y}}$.

The system $\{y_n\}_{n\in N} \subset Y$ is called ω_b-linear independent in Z with respect to $\tilde{X}$, if $\sum_{n=1}^{\infty} b(x_n, y_n) = 0$, $\{x_n\}_{n\in N} \in \tilde{X}$, implies $x_n = 0$ for any $n \in N$.

3. b-Frames in Hilbert Space

Let X, Z and W be Hilbert spaces, $(\cdot,\cdot)_X$, $(\cdot,\cdot)_Y$, $(\cdot,\cdot)_Z$ are scalar products, respectively. Consider a functional $f_{y,z}(x) = (b(x,y), z)$ for $y \in Y$ and $z \in Z$ in X. It is clear that $f_{y,z}$ is a linear continuous functional in X. Consequently, by the Riesz theorem on the general form of a continuous linear functional in Hilbert spaces there exists a unique element $x^*(y,z) \in X$ such that $f_{z,y}(x) = (x, x^*)_X$. We will denote $x^*(y,z)$ by $< z, y >$ throughout the paper.

Using (2.1), it is easy to get that

$$\|< z, y >\|_X \le M \|z\|_Z \|y\|_Y . \tag{3.1}$$

Let $l_2(X)$ be a Hilbert space of the sequences $\{x_n\}_{n\in N} \subset X$ with $\sum_{n=1}^{\infty} \|x_n\|^2 < +\infty$, endowed with inner product

$$(\tilde{x}', \tilde{x}'') = \sum_{n=1}^{\infty} (x_n', x_n'')_X , \tilde{x}' = \{x_n'\}_{n\in N} , \tilde{x}'' = \{x_n''\}_{n\in N} .$$

Definition 3.1. The system $\{y_n\}_{n\in N} \subset Y$ is called b-frame in Z with respect to $l_2(X)$, if there holds

1) $\{< z, y_n >\}_{n\in N} \in l_2(X), \forall z \in Z$;

2) there are numbers $A > 0$ and $B > 0$ such that: $A \|z\|_Z \le \left\|\{< z, y_n >\}_{n\in N}\right\|_{l_2(X)} \le B \|z\|_Z$, $\forall z \in Z$.

The numbers A and B are called bounds of b-frame $\{y_n\}_{n\in N}$. Thereafter, we will call b-frame in Z with respect to $l_2(X)$ just b-frame in Z.

Let W be a Hilbert space, Y_1 be a normed space and $b_1 : X \times Y_1 \to W$ be a bilinear map.

Theorem 3.1. *Let $\{y_n\}_{n\in N} \subset Y$ be a b-frame in Z with its bounds A and B , $T \in L(Z, W)$ be a Noetherian operator. If the system $\{\psi_n\}_{n\in N} \subset Y$ is such, that $T(b(x, y_n)) = b_1(x, \psi_n)$ for each $x \in X$, $n \in N$, then it also forms a b-frame for $\overline{L_{b_1}\left(\{\psi_n\}_{n\in N}\right)}$.*

Proof. From the b-completeness of $\{y_n\}_{n\in N}$ in Z we have $R_T = \overline{L_{b_1}\left(\{\psi_n\}_{n\in N}\right)}$. Indeed, from $L_{b_1}\left(\{\psi_n\}_{n\in N}\right) \subset R_T$ follows that $\overline{L_{b_1}\left(\{\psi_n\}_{n\in N}\right)} \subset R_T$. Take $\forall \omega \in R_T$. We have $\omega = R(z), z \in Z$. So, from b-completeness of system $\{y_n\}$ in Z, there exists $z_n \in L\left(\{y_n\}_{n\in N}\right)$ such that $z = \lim_{n\to\infty} z_n$. We have $\omega = T(z) = \lim_{n\to\infty} T(z_n) \in \overline{L_{b_1}\left(\{\psi_n\}_{n\in N}\right)}$, so $T(z_n) \in L_{b_1}\left(\{\psi_n\}_{n\in N}\right)$. Thus, $R_T = \overline{L_{b_1}\left(\{\psi_n\}_{n\in N}\right)}$. Let the operator $U : l_2(X) \to Z$ be defined as $U\left(\{x_n\}_{n\in N}\right) = \sum_{n=1}^{\infty} b(x_n, y_n)$. It is easy to show that U is a bounded linear operator and $\|U\| \le B$. The adjoint operator of U has the form $U^*(z) = \{< z, y_n >\}_{n\in N}$. The operator $S = UU^*$ is nonnegative, invertible and $S(z) = \sum_{n=1}^{\infty} b(< z, y_n >, y_n)$, $z \in Z$. It is clear that for arbitrary $z \in Z$

$$(S(z), z)_Z = \|\{< z, y_n >\}\|^2_{l_2(X)}.$$

By definition 3.1, we have

$$A^2\|z\|_Z^2 \le (S(z), z)_Z \le \|S(z)\|_Z\|z\|_Z.$$

Hence

$$A^2\|z\|_Z \le \|S(z)\|_Z,$$

therefore $KerS = \{0\}$. It remains to show that $R_S = Z$. Let us assume the contrary. Put $\exists z_0 \in Z, z_0 \ne 0$, such that $z_0 \in R_S^{\perp}$. Then $0 = (z_0, S(z_0))_Z = \|\{< z_0, y_n >_{n\in N}\}\|^2_{l_2(X)}$, and hence by definition 3.1 we get $z_0 = 0$. This contradicts the assumption. Consequently, $R_S = Z$. Hence, for any $z \in Z$ we have the expansion

$$z = S(S^{-1}(z)) = \sum_{n=1}^{\infty} b(< S^{-1}(z), y_n >, y_n). \tag{3.2}$$

Define the operator $y_n^* : Z \to X$ as $y_n^*(z) =< S^{-1}(z), y_n >$. It is clear that $y_n^* \in L(Z, X)$. Indeed, the linearity of y_n^* is obvious and (3.1) implies

$$\|y_n^*(z)\|_X \leq M \left\|S^{-1}(z)\right\|_Z \|y_n\|_Y \leq M \left\|S^{-1}\right\| \|z\|_Z \|y_n\|_Y ,$$

i.e. $\|y_n^*\| \leq M \left\|S^{-1}\right\| \|y_n\|_Y$. So the expansion (3.2) is written as

$$z = \sum_{n=1}^{\infty} b(y_n^*(z), y_n), z \in Z. \tag{3.3}$$

Now we show that if $T(z) = w$, then $< w, \psi_n >=< T^*(w), y_n >$. For any $x \in X$ we have

$$(x, < w, \psi_n >)_X = (b_1(x, \psi_n), w)_W = (T(b(x, y_n)), w)_W =$$

$$= (b(x, y_n), T^*(w))_Z = (x, < T^*(w), y_n >)_X.$$

Then for any $w \in R_T$ we obtain $\{< w, \psi_n >\}_{n\in N} = \{< T^*(w), y_n >\}_{n\in N} \in l_2(X)$.

Now estimate $\left\|\{< w, \psi_n >\}_{n\in N}\right\|_{l_2(X)}$ for $w \in R_T$. We have

$$\left\|\{< w, \psi_n >\}_{n\in N}\right\|_{l_2(X)} = \left\|\{< T^*(w), y_n >\}_{n\in N}\right\|_{l_2(X)} \leq$$

$$\leq B \|T^*(w)\|_Z \leq B \|T^*\| \|w\|_W .$$

Since $Z = \ker T \oplus Z_1$, then one may consider $T_1 = T|_{Z_1}$. It is clear that T_1 maps Z_1 onto R_T. By the inverse mapping theorem T_1 is invertible. Take any $w \in R_T$. If $z = T_1^{-1}(w)$, then taking into account (3.3) we get

$$\|w\|_W^2 = (w, w) = (T_1(z), w) = (T_1(\sum_{n=1}^{\infty} b(y_n^*(z), y_n)), w) = (\sum_{n=1}^{\infty} b_1(y_n^*(z), \psi_n), w) =$$

$$= \sum_{n=1}^{\infty}(b_1(y_n^*(z), \psi_n), w) = \sum_{n=1}^{\infty}(y_n^*(z), < w, \psi_n >) = (\{y_n^*(z)\}, \{< w, \psi_n >\}) \leq$$

$$\|\{y_n^*(z)\}\|_{l_2(X)} \|\{< w, \psi_n >\}\|_{l_2(X)} = \left\|\{< S^{-1}(z), y_n >\}\right\|_{l_2(X)} \|\{< w, \psi_n >\}\|_{l_2(X)} \leq$$

$$\leq B \left\|S^{-1}(z)\right\|_Z \|\{< w, \psi_n >\}\|_{l_2(X)} \leq B \left\|S^{-1}\right\| \|z\|_Z \|\{< w, \psi_n >\}\|_{l_2(X)} \leq$$

$$\leq B \left\|S^{-1}\right\| \left\|T_1^{-1}\right\| \|w\|_W \|\{< w, \psi_n >\}\|_{l_2(X)} .$$

Hence, $\|w\|_W \leq B\left\|S^{-1}\right\|\left\|T_1^{-1}\right\|\|\{< w, \psi_n >\}\|_{l_2(X)}$.

□

4. b-atomic decomposition in Banach spaces

Let X, Z and W be Banach spaces, $\tilde{X}$ be a KB space over X. Consider $\{y_n\}_{n\in N} \subset Y$ and $\{y_n^*\}_{n\in N} \subset L(Z,X)$. For fixed $f \in Z^*$ and $y \in Y$ the expression $f(b(x,y))$ defines a linear continuous functional of x. Denote this functional by $< f, y >$. It is easy to show that

$$\|< f, y >\|_X \leq M \|f\| \|y\|_Y . \tag{4.1}$$

Definition 4.1. The pair $(\{y_n^*\}, \{y_n\})$ is called b-atomic decomposition of Z with respect to $\tilde{X}$, if

1)$\{y_n^*(z)\}_{n\in N} \in \tilde{X}, \forall z \in Z$;

2)$\exists A > 0, B > 0 : A \|z\|_Z \leq \left\|\{y_n^*(z)\}_{n\in N}\right\|_{\tilde{X}} \leq B \|z\|_Z , \forall z \in Z$;

3)$z = \sum_{n=1}^{\infty} b(y_n^*(z), y_n), \forall z \in Z.$

The constants A and B are called b-frame boundaries of the b-atomic decomposition.

b-atomic decomposition of Z with respect to $\tilde{X}$ is called $b_{\tilde{X}}$-atomic decomposition.

Theorem 4.1. *Let $(\{y_n^*\}, \{y_n\})$ be a $b_{\tilde{X}}$-atomic decomposition of Z, with boundaries A and B, $T \in L(Z,W)$ be a Noetherian operator. Let the system $\{\psi_n\}_{n\in N} \subset Y$ is such that $T(b(x,y_n)) = b_1(x,\psi_n)$ for all $x \in X$, $n \in N$. Then there exist $\{\psi_n^*\}_{n\in N} \subset L(W, X)$ such that $(\{\psi_n^*\}, \{\psi_n\})$ is a $b_{\tilde{X}}$-atomic decomposition of $\overline{L_b\left(\{\psi_n\}_{n\in N}\right)}$.*

Proof. Decompose Z as $Z = \ker T \oplus Z_1$ and let $T_1 = T|_{Z_1}$. The operator T_1 acts from Z_1 onto R_T continuously, then it is invertible. It is easy to show that $R_T = \overline{L_b\left(\{\psi_n\}_{n\in N}\right)}$. Define the operator $\psi_n^* : R_T \to X$ as $\psi_n^*(w) = y_n^*(T_1^{-1}(w))$, $w \in R_T$. We get $\{\psi_n^*(w)\}_{n\in N} \in \tilde{X}$, for any $w \in R_T$, since $\{\psi_n^*(w)\}_{n\in N} = \{y_n^*(z)\}_{n\in N} \in \tilde{X}$, here $z = T_1^{-1}(w)$. For any $w \in R_T$, we have

$$\|\{\psi_n^*(w)\}\|_{\tilde{X}} = \left\|\{y_n^*(T_1^{-1}(w))\}\right\|_{\tilde{X}} \leq B \left\|T_1^{-1}(w)\right\|_Z \leq B \left\|T_1^{-1}\right\| \|w\|_W ,$$

$$\|\{\psi_n^*(w)\}\|_{\tilde{X}} = \left\|\{y_n^*(T_1^{-1}(w))\}\right\|_{\tilde{X}} \geq A \left\|T_1^{-1}(w)\right\|_Z \geq A \|T_1\|^{-1} \|w\|_W ,$$

and

$$w = T(T_1^{-1}(w)) = T(\sum_{n=1}^{\infty} b(y_n^*(T_1^{-1}(w)), y_n)) = \sum_{n=1}^{\infty} b_1(\psi_n^*(w), \psi_n).$$

□

Theorem 4.2. *Let $\tilde{Y}$ be a KB-space of sequences of vectors of Y, $\tilde{X}$ be a CB-space such that $\tilde{X}^*$ be normally subordinated to $\tilde{Y}$. Suppose that the systems $\{y_n\}\subset Y$ and $\{y_n^*\}_{n\in N}\subset K(Z,X)$ are such that $(\{y_n^*\}, \{y_n\})$ is a $b_{\tilde{X}}$-atomic decomposition of Z with its boundaries A and B. Let the system $\{\psi_n\}_{n\in N} \subset Y$ be ω_b-linear independent with respect to $\tilde{X}$ such that $\{y_n - \psi_n\}_{n\in N} \in \tilde{Y}$. Then there exists $\{\psi_n^*\}_{n\in N} \subset L(Z,X)$ such that $(\{\psi_n^*\}, \{\psi_n\})$ is a $b_{\tilde{X}}$-atomic decomposition of Z.*

Proof. By the Han-Banach Theorem for any $m, p \in N$ and $z \in Z$ there exists $f_{m,p} \in Z^*$ with $\|f_{m,p}\| = 1$ such that

$$\left\| \sum_{n=m+1}^{m+p} b(y_n^*(z), y_n - \psi_n) \right\|_Z = f_{m,p}\left(\sum_{n=m+1}^{m+p} b(y_n^*(z), y_n - \psi_n) \right).$$

By (4.1) we obtain

$$\left\| \sum_{n=m+1}^{m+p} b(y_n^*(z), y_n - \psi_n) \right\|_Z =$$

$$= \left| \sum_{n=m+1}^{m+p} f_{m,p}(b(y_n^*(z), y_n - \psi_n)) \right| = \left| \sum_{n=m+1}^{m+p} < f_{m,p}, y_n - \psi_n > (y_n^*(z)) \right| =$$

$$= \left| \left(\left\{ < f_{m,p}, \sum_{n=m+1}^{m+p} \{\delta_{in}(y_n - \psi_n)\}_{i\in N} > \right\}, \{y_n^*(z)\}_{n\in N} \right) \right| \leq$$

$$\leq \left\| \left\{ < f_{m,p}, \sum_{n=m+1}^{m+p} \{\delta_{in}(y_n - \psi_n)\}_{i\in N} > \right\} \right\|_{\tilde{X}^*} \times$$

$$\times \left\| \{y_n^*(z)\}_{n\in N} \right\|_{\tilde{X}} \leq M \|f_{m,p}\| \left\| \left\{ \sum_{n=m+1}^{m+p} \{\delta_{in}(y_n - \psi_n)\}_{i\in N} \right\} \right\|_{\tilde{Y}} \left\| \{y_n^*(z)\}_{n\in N} \right\|_{\tilde{X}} \leq$$

$$\leq MB \left\| \left\{ \sum_{n=m+1}^{m+p} \{\delta_{in}(y_n - \psi_n)\}_{i\in N} \right\} \right\|_{\tilde{Y}} \|z\|_Z .$$

Hence, the series $\sum_{n=1}^{\infty} b(y_n^*(z), \psi_n)$ is convergent for any $z \in Z$. Consider the operator $T : Z \to Z$

$$T(z) = \sum_{n=1}^{\infty} b(y_n^*(z), y_n - \psi_n), z \in Z.$$

Obviously, T is compact operator. Then $F = I - T$ is Fredholm operator and $F(z) = \sum_{n=1}^{\infty} b(y_n^*(z), \psi_n)$. If $z \in KerF$, then $\sum_{n=1}^{\infty} b(y_n^*(z), \psi_n) = 0$. Since the system $\{\psi_n\}_{n\in N}$ is ω_b-linear independent with respect to $\tilde{X}$, we have $y_n^*(z) = 0$ for any $n \in N$. Then $z = 0$. From here we have that F is invertible. Let $\psi_n^* : Z \to X$ is defined by the formula $\psi_n^*(z) = y_n^*(F^{-1}(z))$, $z \in Z$. Then $\psi_n^* \in L(Z, X)$ and $\{\psi_n^*(z)\} = \left\{y_n^*(F^{-1}(z))\right\}_{n\in N} \in \tilde{X}$, $z \in Z$. For any $z \in Z$ we have

$$z = F(F^{-1}(z)) = \sum_{n=1}^{\infty} b(y_n^*(F^{-1}(z)), \psi_n) = \sum_{n=1}^{\infty} b(\psi_n^*(z), \psi_n)$$

and

$$\left\| \{\psi_n^*(z)\}_{n\in N} \right\|_{\tilde{X}} = \left\| \left\{y_n^*(F^{-1}(z))\right\}_{n\in N} \right\|_{\tilde{X}} \leq B \left\|F^{-1}(z)\right\|_Z \leq B \left\|F^{-1}\right\| \|z\|_Z ,$$

$$\left\| \{\psi_n^*(z)\}_{n\in N} \right\|_{\tilde{X}} = \left\| \left\{y_n^*(F^{-1}(z))\right\}_{n\in N} \right\|_{\tilde{X}} \geq A \left\|F^{-1}(z)\right\|_Z \geq A \|F\|^{-1} \|z\|_Z .$$

□

Theorem 4.3. *Let $\tilde{Y}$ be a KBspace of sequences of vectors of Y, $\tilde{X}$ be a CB space such that $\tilde{X}^*$ be normally subordinated to $\tilde{Y}$, the system $\{y_n\}\subset Y$ and $\{y_n^*\}_{n\in N}\subset L(Z,X)$ are such that $(\{y_n^*\},\{y_n\})$ is $b_{\tilde{X}}$-atomic decomposition of Z with its boundaries A and B, the system $\{\psi_n\}_{n\in N}\subset Y$ be so that $\left\|\{\varphi_n-\psi_n\}_{n\in N}\right\|_{\tilde{Y}}<\dfrac{1}{MB}$. Then there exists $\{\psi_n^*\}_{n\in N}\subset L(Z,X)$ such that $(\{\psi_n^*\},\{\psi_n\})$ is $b_{\tilde{X}}$-atomic decomposition of Z.*

Proof. By the same way in proof of Theorem 3.1 we get that the series $\sum_{n=1}^{\infty} b(\varphi_n^*(z),\psi_n)$ is convergent for any $z\in Z$ and

$$\left\|\sum_{n=1}^{\infty} b(\varphi_n^*(z),\varphi_n-\psi_n)\right\|_Z \le MB\left\|\{\varphi_n-\psi_n\}_{n\in N}\right\|_{\tilde{Y}}\|z\|_Z .$$

Let the operator $T:Z\to Z$ is given by the formula $T(z)=\sum_{n=1}^{\infty} b(\varphi_n^*(z),\varphi_n-\psi_n)$, $z\in Z$. Then $\|T\|\le MB\left\|\{\varphi_n-\psi_n\}_{n\in N}\right\|_{\tilde{Y}}$ and hence $\|T\|<1$. That is why the operator $F=I-T$ is invertible. The remainder part of proof coincides with that part of proof in Theorem 3.2.

□

Let $\tilde{y}=\{y_n\}\subset Y$ and $\tilde{y}^*=\{y_n^*\}_{n\in N}\subset L(Z,X)$. Denote

$$Z_b(\tilde{y},\tilde{y}^*)=\left\{\mathrm{w}\in Z:\exists z\in Z\Rightarrow w=\sum_{n=1}^{\infty} b(y_n^*(z),y_n)\right\}.$$

Lemma 4.1. *Let $\tilde{y}=\{y_n\}\subset Y$, $\tilde{y}^*=\{y_n^*\}_{n\in N}\subset K(Z,X)$, $(\{y_n^*\},\{y_n\})$ be $b_{\tilde{X}}$-atomic decomposition of Z with its $b_{\tilde{X}}$- frame bounds A and B. Let assume that the system $\tilde{\psi}=\{\psi_n\}_{n\in N}\subset Y$ is derived from the system $\{y_n\}_{n\in N}$ by changing of finite number of elements of $\{y_n\}_{n\in N}$. Then there exists $\{\psi_n^*\}_{n\in N}\subset L(W,X)$ such that $(\{\psi_n^*\},\{\psi_n\})$ is $b_{\tilde{X}}$-atomic decomposition of $Z_b(\tilde{\psi},\tilde{\psi}^*)$.*

Proof. Obviously, the series $\sum_{n=1}^{\infty} b(y_n^*(z),\psi_n)$ is convergent for any $z\in Z$. Let the number n_0 so that $\psi_n=y_n$ for any $n\ge n_0+1$. Consider the operator defined by

$$T(z)=\sum_{n=1}^{\infty} b(y_n^*(z),y_n-\psi_n)=\sum_{n=1}^{n_0} b(y_n^*(z),y_n-\psi_n), z\in Z.$$

It is clear that T is a compact operator. Let $F=I-T$. For any $\{\psi_n^*\}$ we have

$$F(z)=\sum_{n=1}^{\infty} b(y_n^*(z),\psi_n).$$

Hence, $R_F=Z_b(\tilde{\psi},\tilde{y}^*)$. Decompose Z as $Z=KerF\oplus Z_1$, and let $F_1=F|_{Z_1}$. The operator F_1 maps Z_1 onto R_F and invertible. Let $\psi_n^*(w)=y_n^*(F_1^{-1}(w))$, $w\in R_F$. The remainder part of proof coincides with that part of above theorems.

□

Theorem 4.4. *Let $\tilde{Y}$ be a KB space of sequences of vectors of Y, $\tilde{X}$ be a CB space such that $\tilde{X}^*$ be normally subordinated to $\tilde{Y}$. Let suppose that the systems $\{y_n\}\subset Y$ and $\{y_n^*\}_{n\in N}\subset K(Z,X)$ are such that $(\{y_n^*\},\{y_n\})$ is $b_{\tilde{X}}$- atomic decomposition of Z with bounds A and B, the system $\{\psi_n\}_{n\in N}\subset Y$ is such that*

$\{y_n - \psi_n\}_{n\in N} \in \tilde{Y}$. *Then there is* $\{\psi_n^*\}_{n\in N} \subset L(Z,X)$ *such that* $(\{\psi_n^*\}, \{\psi_n\})$ *is* $b_{\tilde{X}}$*- atomic decomposition of* $Z_b(\tilde{\psi}, \tilde{y}^*)$.

Proof. Let $n_0 \in N$ be such that $\left\|\left\{\sum_{n=n_0}^{\infty} \{\delta_{in}(y_n - \psi_n)\}_{i\in N}\right\}\right\|_{\tilde{Y}} < \frac{1}{MB}$. Let the system $\{\theta_n\}\subset Y$ be defined as

$$\theta_n = \begin{cases} y_n, & n < n_0 \\ \psi_n, & n \geq n_0 \end{cases}.$$

Then $\left\|\{y_n - \theta_n\}_{n\in N}\right\|_{\tilde{Y}} < \frac{1}{MB}$, and by Theorem 4.3, there is $\tilde{\theta}^* = \{\theta_n^*\}_{n\in N} \subset L(Z,X)$ such that $(\{\theta_n^*\}, \{\theta_n\})$ is $b_{\tilde{X}}$-atomic decomposition of Z. The system $\{\psi_n\}_{n\in N}$ differs from $\{\theta_n\}$ by a finite number of elements, so using Lemma 4.1 we get that there is $\{\psi_n^*\}_{n\in N} \subset L(W,X)$ such that $(\{\psi_n^*\}, \{\psi_n\})$ is $b_{\tilde{X}}$-atomic decomposition of $Z_b(\tilde{\psi}, \tilde{\psi}^*)$.

□

5. Banach g-Frames and their stability

Let X , Z and W be Banach spaces, $\tilde{X}$ be a KB space over X.

Definition 5.1. ([1]). Let $\{y_n^*\}_{n\in N} \subset L(Z,X)$ and $S : \tilde{X} \to Z$. The pair $(\{\varphi_n^*\}, S)$ is called g-Banach frame in Z with respect to $\tilde{X}$, if

1) $\{y_n^*(z)\}_{n\in N} \in \tilde{X}$ for any $z \in Z$;
2) there are numbers $A > 0$ and $B > 0$ such that

$$A\|z\|_Z \leq \left\|\{y_n^*(z)\}_{n\in N}\right\|_{\tilde{X}} \leq B\|z\|_Z, \forall z \in Z;$$

3) The operator S is a linear and continuous in $\tilde{X}$ and $S\left(\{y_n^*(z)\}_{n\in N}\right) = z$, $\forall z \in Z$.

The constants A and B are called g-frame bounds for $(\{y_n^*\}, S)$, the operator S is called a reconstruction operator. In future, g-Banach frame for Z with respect to $\tilde{X}$ will be called $g_{\tilde{X}}$-Banach frame for Z.

Theorem 5.1. *Let* $(\{y_n^*\}, S)$ *be* $g_{\tilde{X}}$*-Banach frame for* Z*,* $T \in L(Z,W)$ *be a Noetherian operator and* $S_1 = TS$*. Then there exists* $\{\psi_n^*\}_{n\in N} \subset L(R_T, X)$ *such that* $(\{\psi_n^*\}, S_1)$ *forms* $g_{\tilde{X}}$*-Banach frame for* R_T*.*

Proof. Let $T_1 = T|_{Z_1}$, where $Z = KerT \oplus Z_1$. It is clear that $T_1 \in L(Z_1, R_T)$ and invertible. Denote $\psi_n^*(w) = y_n^*(T_1^{-1}(w))$. Then $\{\psi_n^*\}_{n\in N} \subset L(R_T, X)$ and $\{\psi_n^*(w)\}_{n\in N} \in \tilde{X}$, $w \in R_T$. For any $w \in R_T$ we have

$$w = T(T_1^{-1}(w)) = T(S(\{y_n^*(T_1^{-1}(w))\}_{n\in N}) =$$
$$= T(S(\{\psi_n^*(w)\}_{n\in N}) = S_1(\{\psi_n^*(w)\}_{n\in N}),$$

and

$$\left\|\{\psi_n^*(w)\}_{n\in N}\right\|_{\tilde{X}} = \left\|\{y_n^*(T_1^{-1}(w))\}_{n\in N}\right\|_{\tilde{X}} \leq B\left\|T_1^{-1}(w)\right\|_Z \leq B\left\|T_1^{-1}\right\|\|w\|_W,$$

$$\left\|\{\psi_n^*(w)\}_{n\in N}\right\|_{\tilde{X}} = \left\|\{y_n^*(T_1^{-1}(w))\}_{n\in N}\right\|_{\tilde{X}} \geq A\left\|T_1^{-1}(w)\right\|_Z \geq A\|T_1\|^{-1}\|w\|_W.$$

□

Theorem 5.2. *Let* $(\{y_n^*\}, S)$ *be* $g_{\tilde{X}}$*-Banach frame in* Z*, the operator* $U : Z \to \tilde{X}$ *is defined as* $U(z) = \{y_n^*(z)\}_{n\in N}$*,* $z \in Z$*, the operator* $S_1 \in L(\tilde{X}, Z)$ *is such that* $\|S - S_1\| < \|U\|^{-1}$*. Then there exists* $\{\psi_n^*\}_{n\in N} \subset L(W, X)$ *such that* $(\{\psi_n^*\}, S_1)$ *forms* $g_{\tilde{X}}$*-Banach frame for* Z*.*

Proof. Since

$$I - S_1U = SU - S_1U = (S - T)U.$$

Then by the condition the operator $F = S_1U$ is invertible. Define $\psi_n^* : Z \to X$ with the formula $\psi_n^*(z) = y_n^*(F^{-1}(z))$. Then, it is obvious that $\psi_n^* \in L(Z, X)$. Since $F^{-1}(z) \in X$, for any $z \in Z$, then $\{\psi_n^*(z)\} = \left\{y_n^*(F^{-1}(z))\right\}_{n\in N} \in \tilde{X}$. Furthermore, for any $z \in Z$ we have

$$\left\|\{\psi_n^*(z)\}_{n\in N}\right\|_{\tilde{X}} = \left\|\left\{y_n^*(F^{-1}(z))\right\}_{n\in N}\right\|_{\tilde{X}} \le B\left\|F^{-1}(z)\right\|_Z \le B\left\|F^{-1}\right\|\|z\|_Z,$$

and

$$\left\|\{\psi_n^*(z)\}_{n\in N}\right\|_{\tilde{X}} = \left\|\left\{y_n^*(F^{-1}(z))\right\}_{n\in N}\right\|_{\tilde{X}} \ge A\left\|F^{-1}(z)\right\|_Z \ge A\|F\|^{-1}\|z\|_Z.$$

Now, let's verify the last condition 3) in definition we have

$$z = F(F^{-1}(z)) = S_1U(F^{-1}(z)) = S_1\left(\left\{y_n^*(F^{-1}(z))\right\}_{n\in N}\right) =$$

$$= S_1\left(\{\psi_n^*(z)\}_{n\in N}\right), z \in Z.$$

□

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References

[1] M. R. Abdollahpour, M. H. Faroughi and A. Rahimi, *PG*-frames in Banach spaces, *Methods of functional Analysis and Topology*, **13(3)** (2007), 201–210.

[2] A. Aldroubi, Q. Sun and W. Tang, p-frames and shift invariant subspaces of L_p, *J. Fourier Anal. Appl.*, **7** (2001), 1–21.

[3] R. Balan, Stability theorems for Fourier frames and wavelet Riesz bases, *J. Fourier Anal. And Appl.* **3** (1997), 499–504.

[4] B. T. Bilalov and F. A. Guliyeva, On The Frame Properties of Degenerate System of Sines, *Journal of Function Spaces and Applications*, **2012** (2012), Article ID 184186, 12 pages, doi:10.1155/2012/184186.

[5] B. T. Bilalov and F. A. Guliyeva, Noetherian perturbation of frames, *Pensee Journal*, **75(12)** (2013), 425–431.

[6] P. G. Casazza and O. Christensen, Perturbation operators and applications to frame theory, *J. Fourier Anal. Appl.*, **35** (1997) 543–557.

[7] P. G. Casazza and O. Christensen, Approximation of the inverse frame operator and applications to Gabor frames, *J. Approx. Theory*, **103(2)** (2000), 338–356.

[8] P. G. Casazza, D. Han and D. R. Larson, Frames for Banach space, *Contemp. Math.*, **247** (1999), 149–182.

[9] O. Christensen, An Introduction to Frames and Riesz Bases, *Appl. Numer. Harmon. Anal.*, Birkhuser, Boston, MA, 2002.

[10] O. Christensen and C. Heil, Perturbations of Frames and Atomic Decompositions, *Math. Nachr.*, **185** (1997), 33–47.
[11] O. Christensen and D. Stoeva, p-frames in separable Banach spaces, *Adv. Comp. Math.*, **18** (2003), 117–126.
[12] R. J. Duffin and A. C. Schaeffer, A class of nonharmonic Fourier series, *Trans. Amer. Math. Soc.*, **72** (1952), 341–366.
[13] K. Gróchenig, Describing functions: atomic decomposition versus frames, *Monatsh. Math.*, **112(1)** (1991), 1–41.
[14] D. Han and D. R. Larson, Frames, bases and group representations, *Memoirs Amer. Math. Soc.*, **147(697)** (2000), 1–91.
[15] P. K. Jain, S. K. Kaushik and L. K. Vashisht, On perturbations of Banach frames, *Int. J. Wavelet multiresolut.Inf. Process*, **4(3)** (2006), 559–565.
[16] S. R. Sadigova and Z. V. Mamedova, Frames from Cosines with the Degenerate Coefficients, *American Journal of Applied Mathematics and Statistics*, **1(3)**(2013), 36–40.
[17] W. Sun, G-frames and G-Riesz bases, *J. Math. Anal. Appl.*, **322** (2006), 473–452.
[18] R. Young, An introduction to nonharmonic Fourier series, *New York*, 1980.

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