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ESTIMATES FOR FRACTIONAL INTEGRALS AND FRACTIONAL MAXIMAL OPERATORS ON COMMUTATIVE HYPERGROUPS

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Abstract. Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ, Haar measure λ, upper Ahlfors N-regular on an identity. We consider fractional integrals, fractional maximal operators and Hardy-Littlewood maximal operator on $(K,*)$ and give pointwise and integral estimates for fractional integrals in terms of fractional maximal operators and Hardy-Littlewood maximal operator.

1. Introduction and preliminaries

The objects of the present paper are fractional integrals, fractional maximal operators and Hardy-Littlewood maximal operators on commutative hypergroups. These operators were introduced in [8] and [9], as a convolution of two functions on commutative hypergroups. We give pointwise and integral estimates for the fractional integrals in terms of fractional maximal operators and Hardy-Littlewood maximal operator. The obtained results are the extensions of the analogous results for the classical fractional integrals, fractional maximal operators and Hardy-Littlewood maximal operators(for example see [1]). Also the results of the paper include the analogous results for Bessel hypergroups in [6], for Laguerre hypergroups in [13] and for Dunkl hypergroups in [5], [7] and [12].

Let K be a set. A function $\rho : K \times K \to [0,\infty)$ is called quasi-metric if:

(1) $\rho(x,y) = 0 \Leftrightarrow x = y$;
(2) $\rho(x,y) = \rho(y,x)$;
(3) there is a constant $c \geq 1$ such that for every $x, y, z \in X$

$$\rho(x,y) \leq c(\rho(x,z) + \rho(z,y)).$$

In the theory of locally compact groups there arise certain spaces which, though not groups, have some of the structure of groups. Often, the structure can be expressed in terms of an abstract convolution of measures on the space.

A hypergroup $(K,*)$ consists of a locally compact Hausdorff space K together with a bilinear, associative, weakly continuous convolution on the Banach space of all bounded regular Borel measures on K with the following properties:

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1. For all $x, y \in K$, the convolution of the point measures $\delta_x * \delta_y$ is a probability measure with compact support.
2. The mapping: $K \times K \to \mathcal{C}(K)$, $(x, y) \mapsto supp(\delta_x * \delta_y)$ is continuous with respect to the Michael topology on the space $\mathcal{C}(K)$ of all nonvoid compact subsets of K, where this topology is generated by the sets

$$U_{V,W} = \{L \in \mathcal{C}(K) : L \cap V \neq \varnothing, L \subset W\}$$

with V, W open in K.
3. There is an identity $e \in K$ with $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in K$.
4. There is a continuous involution $\sim$ on K such that

$$(\delta_x * \delta_y)^{\sim} = \delta_{y^{\sim}} * \delta_{x^{\sim}}$$

and $e \in supp(\delta_x * \delta_y) \Leftrightarrow x = y^{\sim}$ for $x, y \in K$ (see [10], [11], [14], [2]).

A hypergroup K is called commutative if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$. It is well known that every commutative hypergroup K possesses a Haar measure which will be denoted by λ (see [14]). That is, for every Borel measurable function f on K,

$$\int_K f(\delta_x * \delta_y) d\lambda(y) = \int_K f(y) d\lambda(y) \ \ (x \in K).$$

Define the generalized translation operators T^x, $x \in K$, by

$$T^x f(y) = \int_K f d(\delta_x * \delta_y)$$

for all $y \in K$. If K is a commutative hypergroup, then $T^x f(y) = T^y f(x)$ and the convolution of two functions is defined by

$$f * g(x) = \int_K T^x f(y) g(y^{\sim}) d\lambda(y).$$

Let $(K, *)$ be a commutative hypergroup, with quasi-metric ρ, Haar measure λ and all balls $B(x, r) = \{y \in K : \rho(x, y) < r\}$ be λ-measurable and $N \in (0, \infty)$. We will say Haar measure λ is upper Ahlfors N-regular on an identity, if there exists a constant $C > 0$, not depending $r > 0$, such that

$$\lambda B(e, r) \leq C r^N. \tag{1.1}$$

Let $p > 0$. By $L^p(K, \lambda)$ denote a class of all λ-measurable functions $f : K \to (-\infty, +\infty)$ with $\|f\|_{L^p(K,\lambda)} = \left(\int_K |f(x)|^p d\lambda(x)\right)^{\frac{1}{p}} < \infty$.

The notation $\chi_A(x)$ denotes the characteristic function of set A.

For the sake of simplicity, the letter C always denotes a positive constant which may change from one step to the next.

2. Main results

Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ, Haar measure λ. For $0 \leq \beta < 1$ define fractional maximal operator

$$M_\beta f(x) = \sup_{r>0} \frac{1}{\lambda B(e,r)^{1-\beta}} \left(|f| * \chi_{B(e,r)}\right)(x).$$

If $\beta = 0$, then we have Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{\lambda B(e,r)} \left(|f| * \chi_{B(e,r)}\right)(x) \tag{2.1}$$

on commutative hypergroup $(K,*)$ equipped with the pseudo-metric ρ.
We have

Theorem 2.1. *If $f \in L^{\frac{1}{\beta}}(K,\lambda)$ then $M_\beta f \in L^\infty(K)$ and*

$$\sup_{x\in K} M_\beta f(x) \leq \left(\int_K |f(x)|^p d\lambda\right)^{\frac{1}{p}}.$$

Proof. Using Hölder's inequality we can write

$$\frac{1}{\lambda B(e,r)^{1-\beta}} \int_K T^x |f(y)| \chi_{B(e,r)}(y^\sim) d\lambda(y)$$

$$\leq \frac{1}{\lambda B(e,r)^{1-\beta}} \left(\int_K \left(T^x|f(y)|\right)^{\frac{1}{\beta}} d\lambda(y)\right)^{\beta} \times$$

$$\times \left(\int_K \left(\chi_{B(e,r)}(y^\sim)\right)^{\frac{1}{1-\beta}} d\lambda(y)\right)^{1-\beta}$$

$$\leq \left(\int_K |f(y)|^{\frac{1}{\beta}} d\lambda(y)\right)^{\beta}$$

and we have the required result. □

Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ, Haar measure λ, upper Ahlfors N-regular on an identity. Define fractional integral

$$I_\alpha f(x) = \left(\rho(e,\cdot)^{\alpha-N} * f\right)(x),\ 0 < \alpha < N \tag{2.2}$$

on commutative hypergroup $(K,*)$ equipped with the quasi-metric ρ.

Theorem 2.2. *Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ and Haar measure λ, upper Ahlfors N-regular on an identity and $0 < \eta < 1$, $0 \leq \alpha < N\eta$. Then*

$$I_\alpha |f(x)| \leq C\left(r^\alpha M_0 f(x) + r^{\alpha - N\eta} M_\eta f(x)\right). \tag{2.3}$$

Proof. Let $r > 0$ be arbitrary and split $I_\alpha|f(x)|$ in the standard way

$$I_\alpha|f(x)| = \int\limits_{B(e,r)} \rho(e,y)^{\alpha-N} T^x|f(y^\sim)|d\lambda(y)$$
$$+ \int\limits_{K\setminus B(e,r)} \rho(e,y)^{\alpha-N} T^x|f(y^\sim)|d\lambda(y)$$
$$= U_1(x,r) + U_2(x,r). \tag{2.4}$$

For $U_1(x,r)$ we have the estimate

$$U_1(x,r) = \int\limits_{B(e,r)} \rho(e,y)^{\alpha-N} T^x|f(y^\sim)|d\lambda(y).$$
$$= \sum_{k=1}^{\infty} \int\limits_{2^{-k}r\le\rho(e,y)<2^{-k+1}r} \rho(e,y)^{\alpha-N} T^x|f(y^\sim)|d\lambda(y)$$
$$\le \sum_{k=1}^{\infty} \left(2^{-k}r\right)^{\alpha-N} \int\limits_{\rho(e,y)<2^{-k+1}r} T^x|f(y^\sim)|d\lambda(y)$$
$$= \sum_{k=1}^{\infty} \left(2^{-k}r\right)^{\alpha-N} \lambda B(e, 2^{-k+1}r)\times$$
$$\times \frac{1}{\lambda B(e,2^{-k+1}r)} \int\limits_{B(e,2^{-k+1}r)} T^x|f(y^\sim)|d\lambda(y)$$
$$\le Cr^\alpha M_0 f(x).$$

Therefore

$$U_1(x,r) \le Cr^\alpha M_0 f(x), \tag{2.5}$$

where $C > 0$ does not depend f, x and r.
Estimating $U_2(x,r)$ we obtain:

$$U_2(x,r) = \int\limits_{K\setminus B(e,r)} \rho(e,y)^{\alpha-N} T^x|f(y^\sim)|d\lambda(y)$$
$$= \sum_{k=0}^{\infty} \int\limits_{2^k r\le\rho(e,y)<2^{k+1}r} \rho(e,y)^{\alpha-N} T^x|f(y^\sim)|d\lambda(y)$$
$$\le \sum_{k=0}^{\infty} \left(2^k r\right)^{\alpha-N} \left(\lambda B(e,2^{k+1}r)\right)^{1-\eta} \times$$
$$\times \frac{1}{(\lambda B(e,2^{k+1}r))^{\eta-1}} \int\limits_{\rho(e,y)<2^{k+1}r} T^x|f(y^\sim)|d\lambda(y)$$
$$\le C\sum_{k=0}^{\infty} \left(2^k r\right)^{\alpha-N} \left(2^{k+1}r\right)^{N-N\eta} M_\eta f(x)$$
$$\le Cr^{\alpha-N\eta} M_\eta f(x)$$

since $\alpha - N\eta < 0$ by assumption.
Thus (2.3) is proved. □

Theorem 2.3. *Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ and Haar measure λ, upper Ahlfors N-regular on an identity and $1 \le p < \frac{N}{\alpha}$. Then*

$$I_\alpha|f(x)| \le C\left(M_0 f(x)\right)^{1-\frac{\alpha p}{N}} \|f\|_{L^p(K,\lambda)}^{\frac{\alpha p}{N}} \tag{2.6}$$

$$I_{\alpha\theta}|f(x)| \le C\left(I_\alpha|f(x)|\right)^{\theta}\left(M_0 f(x)\right)^{1-\theta} \tag{2.7}$$

$$I_{\alpha\theta}|f(x)| \le C\left(M_\alpha f(x)\right)^{\theta}\left(M_0 f(x)\right)^{1-\theta} \tag{2.8}$$

Note that, for classical operators, the analogue of inequality (2.8) is known as the Welland inequality.

Proof. Split $I_\alpha|f(x)|$ again as in (2.4). By Hölder's inequality we have

$$U_2(x,r) \le \left(\int\limits_{K\setminus B(e,r)} |T^x f(y^\sim)|^p d\lambda(y)\right)^{\frac{1}{p}} \times$$

$$\times\left(\int\limits_{K\setminus B(e,r)} \rho(e,y)^{(\alpha-N)p'} d\lambda(y)\right)^{\frac{1}{p'}}.$$

Here

$$\left(\int\limits_{K\setminus B(e,r)} |T^x f(y^\sim)|^p d\lambda(y)\right)^{\frac{1}{p}}$$

$$\le \left(\int\limits_{K} |T^x f(y^\sim)|^p d\lambda(y)\right)^{\frac{1}{p}}$$

$$\le \left(\int\limits_{K} |T^x f^p(y^\sim)| d\lambda(y)\right)^{\frac{1}{p}}$$

$$= \left(\int\limits_{K} |f(y)|^p d\lambda(y)\right)^{\frac{1}{p}}$$

and

$$\left(\int\limits_{K\setminus B(e,r)} \rho(e,y)^{(\alpha-N)p'} d\lambda(y)\right)^{\frac{1}{p'}}$$

$$= \left(\sum_{k=0}^{\infty}\int\limits_{2^k r\le\rho(e,y)<2^{k+1}r} \rho(e,y)^{(\alpha-N)p'} d\lambda(y)\right)^{\frac{1}{p'}}$$

$$\leq \left(\sum_{k=0}^{\infty} \left(2^k r\right)^{(\alpha-N)p'} \int\limits_{\rho(e,y)<2^{k+1}r} d\lambda(y) \right)^{\frac{1}{p'}}$$

$$\leq C \left(\sum_{k=0}^{\infty} \left(2^k r\right)^{(\alpha-N)p'} \left(2^{k+1} r\right)^{N} \right)^{\frac{1}{p'}}$$

$$\leq C r^{\alpha - N + \frac{N}{p'}}$$

$$= C r^{\alpha - \frac{N}{p}}.$$

Therefore

$$U_2(x,r) \leq C r^{\alpha - \frac{N}{p}} \|f\|_{L^p(K,\lambda)} \tag{2.9}$$

From (2.5) and (2.9), we have

$$I_\alpha |f(x)| \leq C \left(r^\alpha M_0 f(x) + r^{\alpha - \frac{N}{p}} \|f\|_{L^p(K,\lambda)} \right)$$

Minimum of the right-hand side is attained at $r = \left[\frac{\|f\|_{L^p(K,\lambda)}}{M_0 f(x)} \right]^{\frac{p}{N}}$. So

$$I_\alpha |f(x)| \leq C \left(M_0 f(x)\right)^{1 - \frac{\alpha p}{N}} \|f\|_{L^p(K,\lambda)}^{\frac{\alpha p}{N}}.$$

We now prove (2.7). Consider

$$I_{\alpha\theta} |f(x)| = \int\limits_{B(e,r)} \rho(e,y)^{\alpha\theta - N} T^x |f(y^\sim)| d\lambda(y)$$

$$+ \int\limits_{K \setminus B(e,r)} \rho(e,y)^{\alpha\theta - N} T^x |f(y^\sim)| d\lambda(y)$$

$$= u_1(x,r) + u_2(x,r)$$

For $u_2(x,r)$ we have the estimate

$$u_2(x,r) \leq r^{\alpha\theta - \alpha} \int\limits_{K \setminus B(e,r)} \rho(e,y)^{\alpha - N} T^x |f(y^\sim)| d\lambda(y)$$

$$\leq r^{\alpha\theta - \alpha} I_\alpha |f(x)|$$

Thus

$$u_2(x,r) \leq r^{\alpha\theta - \alpha} I_\alpha |f(x)| \tag{2.10}$$

From (2.5) and (2.10) we have

$$I_{\alpha\theta} |f(x)| \leq C r^{\alpha\theta} M_0 f(x) + r^{\alpha\theta - \alpha} I_\alpha |f(x)|.$$

Minimum of the right-hand side is attained at $r = \left[\frac{I_\alpha |f(x)|}{M_0 f(x)} \right]^{\frac{1}{\alpha}}$. So

$$I_{\alpha\theta} |f(x)| \leq C \left(I_\alpha |f(x)|\right)^\theta \left(M_0 f(x)\right)^{1-\theta}.$$

Turn to the proof of (2.7). For this purpuse we estimate $u_2(x,r)$ by the different way.

$$u_2(x,r)=\sum_{k=0}^{\infty}\int\limits_{2^k r\le\rho(e,y)<2^{k+1}r}\rho(e,y)^{\alpha\theta-N}T^x|f(y^{\sim})|d\lambda(y)$$

$$\le\sum_{k=0}^{\infty}\left(2^k r\right)^{\alpha\theta-N}\left(\lambda B(e,2^{k+1}r)\right)^{1-\frac{\alpha}{N}}\times$$

$$\times\frac{1}{(\lambda B(e,2^{k+1}r))^{\frac{\alpha}{N}-1}}\int\limits_{\rho(e,y)<2^{k+1}r}T^x|f(y^{\sim})|d\lambda(y)$$

$$\le Cr^{\alpha\theta-\alpha}M_{\frac{\alpha}{N}}f(x).$$

Therefore,

$$u_2(x,r)\le Cr^{\alpha\theta-\alpha}M_{\frac{\alpha}{N}}f(x). \tag{2.11}$$

Using (2.5) and (2.11) we have

$$I_{\alpha\theta}|f(x)|\le r^{\alpha\theta}M_0f(x)+Cr^{\alpha\theta-\alpha}M_{\frac{\alpha}{N}}f(x).$$

Minimum of the right-hand side is attained at $r=\left[\frac{M_\alpha f(x)}{M_0f(x)}\right]^{\frac{1}{\alpha}}$. So

$$I_{\alpha\theta}|f(x)|\le C\left(M_\alpha f(x)\right)^{\theta}\left(M_0f(x)\right)^{1-\theta}$$

Theorem 2.3 is proved. □

Theorem 2.4. *Let $(K,*)$ be a commutative hypergroup, with quasi-metric ρ and Haar measure λ, upper Ahlfors N-regular on an identity. Assume that $0<\alpha<N$, $0<p<\infty$ and Hardy-Littlewood maximal operator* (2.1) *is bounded on $L^p(K,\lambda)$.*
1) If $1<p<\frac{N}{\mu}$, $1\le b\le\infty$ and $\frac{\alpha p}{\mu b}=\frac{1}{q}-\frac{1}{p}+\frac{\alpha}{\mu}$ then

$$\|I_\alpha f\|_{L^q(K,\lambda)}\le C\|M_{\frac{\mu}{Np}}f\|_{L^b(K,\lambda)}^{\frac{\alpha p}{\mu}}\|f\|_{L^p(K,\lambda)}^{1-\frac{\alpha p}{\mu}} \tag{2.12}$$

for all $f\in L^p(K,\lambda)$ and all $M_{\frac{\mu}{Np}}\in L^b(K,\lambda)$, where C is a constant independent of f.
2)

$$\|I_{\alpha\theta}f\|_{L^b(K,\lambda)}\le C\|I_\alpha f\|_{L^q(K,\lambda)}^{\theta}\|f\|_{L^p(K,\lambda)}^{1-\theta}, \tag{2.13}$$

$$\|I_{\alpha\theta}f\|_{L^b(K,\lambda)}\le C\|M_\alpha f\|_{L^q(K,\lambda)}^{\theta}\|f\|_{L^p(K,\lambda)}^{1-\theta}. \tag{2.14}$$

where $0<\theta<1$, $0<q\le\infty$, $\frac{1}{b}=\frac{\theta}{q}+\frac{1-\theta}{p}$.

Proof. Taking $r=r(x)=\left[\frac{M_\eta f(x)}{M_0f(x)}\right]^{\frac{1}{N\eta}}$ in (2.3), we have

$$|I_\alpha f(x)|\le C\left(M_\eta f(x)\right)^{\frac{\alpha}{N\eta}}\left(M_0f(x)\right)^{1-\frac{\alpha}{N\eta}}.$$

If $\eta=\frac{\mu}{Np}$, then

$$|I_\alpha f(x)|\le C\left(M_{\frac{\mu}{Np}}f(x)\right)^{\frac{\alpha p}{\mu}}\left(M_0f(x)\right)^{1-\frac{\alpha p}{\mu}} \tag{2.15}$$

Using the inequality (2.15) and applying Hölder's inequality we can write

$$\int_K |I_\alpha f(x)|^q d\lambda(x) \le C \int_K \left(M_{\frac{\mu}{Np}} f(x)\right)^{\frac{q\alpha p}{\mu}} (M_0 f(x))^{q-\frac{q\alpha p}{\mu}} d\lambda(x)$$

$$\le C \| \left(M_{\frac{\mu}{Np}} f\right)^{\frac{q\alpha p}{\mu}} \|_{L^{\zeta'}(K,\lambda)} \| (M_0 f)^{q-\frac{q\alpha p}{\mu}} \|_{L^{\zeta}(K,\lambda)},$$

where $\zeta = \frac{\mu p}{\mu q - \alpha p q}$, $\zeta' = \frac{\zeta}{\zeta - 1} = \frac{\mu b}{\alpha p q}$, $\frac{\alpha p}{\mu b} = \frac{1}{q} - \frac{1}{p} + \frac{\alpha}{\mu}$. Therefore

$$\|I_\alpha f\|_{L^q(K,\lambda)} \le C \|M_{\frac{\mu}{Np}} f\|_{L^b(K,\lambda)}^{\frac{\alpha p}{\mu}} \|M_0 f\|_{L^p(K,\lambda)}^{1-\frac{\alpha p}{\mu}}$$

$$\le C \|M_{\frac{\mu}{Np}} f\|_{L^b(K,\lambda)}^{\frac{\alpha p}{\mu}} \|f\|_{L^p(K,\lambda)}^{1-\frac{\alpha p}{\mu}}.$$

Now we prove (2.13). Consider $\|I_{\alpha\theta} f\|_{L^b(K,\lambda)}$. From (2.7) and Hölder's inequality we obtain

$$\|I_{\alpha\theta} f\|_{L^b(K,\lambda)} \le C \| (I_\alpha f)^\theta (M_0 f)^{1-\theta} \|_{L^b(K,\lambda)}$$

$$\le C \| (I_\alpha f)^\theta \|_{L^{b\tau'}(K,\lambda)} \| (M_0 f)^{1-\theta} \|_{L^{b\tau}(K,\lambda)}.$$

where $\tau' = \frac{\tau}{\tau - 1}$. Put $p = (1-\theta) b\tau$, $q = \theta b \tau'$. Then $\frac{1}{b\tau} = \frac{1-\theta}{p}$ and $\frac{1}{b\tau'} = \frac{\theta}{q}$. From the last inequality we obtain

$$\|I_{\alpha\theta} f\|_{L^b(K,\lambda)} \le C \|I_\alpha f\|_{L^q(K,\lambda)}^\theta \|M_0 f\|_{L^p(K,\lambda)}^{1-\theta}$$

$$\le C \|I_\alpha f\|_{L^q(K,\lambda)}^\theta \|f\|_{L^p(K,\lambda)}^{1-\theta}$$

The inequality (2.14) can be proved analogously. □

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