Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Number 2, 2014, Pages 107–114

ON APPROXIMATE SOLUTION OF MIXED BOUNDARY VALUE PROBLEM FOR LAPLACE EQUATION

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Abstract. In the paper the collocation method to boundary integral equation of the mixed boundary value problem for the Laplace equation is justified.

1. Introduction

Numerical problems of mechanics and physics reduce to various boundary integral equations [BIE]. It is known that integral equations in the closed form are solved only in very rare cases. Therefore, development of approximate methods for solving BIE with appropriate theoretical grounding acquires paramount value. It is known that one of the methods of solving the mixed boundary value problem for the Laplace equation is its reduction to BIE. Recall that the mixed boundary value problem for the Laplace equation is as follows: to find the function $u \in C^2(\mathbb{R}^3 \setminus \bar{D}) \cap C(\mathbb{R}^3 \setminus D)$ possessing a normal derivative in the sense of uniform convergence, satisfying the Laplace equation $\Delta u = 0$ in $\mathbb{R}^3 \setminus \bar{D}$, the Sommerfeld condition at infinity and the boundary condition $\frac{\partial u(x)}{\partial \vec{n}(x)} + \lambda u(x) = f(x)$ on S, where $D \subset \mathbb{R}^3$ is a bounded domain with the boundary S, $\vec{n}(x)$ is a unit external normal at the point $x \in S, \lambda$ is a given number, f is a given continuous function on S. Note that in the papers [2,3,5], the sequences of approximate solutions of BIE of exterior boundary value problems of Dirichlet and Neumann for a wave equation were constructed, while in the papers [1,8], the grounding of the collocation method of BIE was given just for the same boundary value problems. However, the grounding of the collocation method for BIE of the mixed boundary value problem for the Laplace equation has not been given yet, and this paper is devoted to this problem.

2. Main results

Let S be a Lyapunov surface with the index $\alpha \in (0,1]$, $\Phi(x,y) = 1/(4\pi\,|x-y|)$ be a fundamental solution of the Laplace equation,

$$v_1(x,\rho) = 2\int_S \frac{\partial \Phi(x,y)}{\partial \vec{n'}(y)}\rho(y)dS_y,$$

2010 *Mathematics Subject Classification.* 45E05, 31B10.
Key words and phrases. Collocation method, mixed boundary value problem, Laplace equation, cubic formula, surface integral.

$$\upsilon_2(x,\rho)=2\int_S \Phi(x,y)\rho(y)dS_y$$

and $\psi(x)=\upsilon_{20}(x,\rho)$ be a simple layer potential of density ρ for the Laplace equation, i.e.

$$\upsilon_{20}(x,\rho)=\int_S \frac{\rho(y)}{|x-y|}dS_y=4\pi\int_S \Phi(x,y)\rho(y)dS_y,$$

where $\vec{n'}(x)$ is a unit internal normal at the point $x\in S$, and $\rho\in C(S)$ ($C(S)$is a space of functions continuous on S and with the norm $\|\rho\|_\infty=\max_{x\in S}|\rho(x)|$).

In the paper [9] it was shown that the function $u(x)=\upsilon_2(x,\rho)+\mu\upsilon_1(x,\psi), x\in\mathbb{R}^3\setminus\bar{D}$, where μ is a complex number, moreover $Im\mu\neq 0$, is the solution of the mixed problem for the Laplace equation if the density ρ is the solution of the BIE

$$(\mu-1)\rho(x)+\frac{\partial\tilde{\upsilon}_2(x,\rho)}{\partial\vec{n}(x)}-\mu\frac{\partial\tilde{\upsilon}_{20}(x,\psi_1)}{\partial\vec{n}(x)}+$$
$$+\lambda\left[\tilde{\upsilon}_2(x,\rho)-\mu\upsilon_{20}(x,\rho)+\mu\tilde{\upsilon}_1(x,\psi)\right]=f(x),x\in S, \tag{2.1}$$

where the sign " $\sim$ " means the direct value, ψ_1 is the direct value of the normal derivative of a simple layer potential of density ρ for the Laplace equation on S.

It is known that equation (2.1) may be rewritten in the form (see [1]):

$$\rho+A\rho=\varphi, \tag{2.2}$$

where $\varphi=(\mu-1)^{-1}f, A=(\mu-1)^{-1}(2K-16\pi^2\mu T+2\lambda L-4\pi\mu\lambda L+8\pi\mu\lambda Q)$ is a linear compact operator,

$$(K\rho)(x)=\int_S \frac{\partial\Phi(x,y)}{\partial\vec{n}(x)}\rho(y)dS_y, x\in S;$$

$$(T\rho)(x)=\int_S \frac{\partial\Phi(x,y)}{\partial\vec{n}(x)}\left(\int_S \frac{\partial\Phi(y,t)}{\partial\vec{n}(y)}\rho(t)dS_t\right)dS_y, x\in S;$$

$$(L\rho)(x)=\int_S \Phi(x,y)\rho(y)dS_y, x\in S;$$

$$(Q\rho)(x)=\int_S \frac{\partial\Phi(x,y)}{\partial\vec{n}(y)}\left(\int_S \Phi(y,t)\rho(t)dS_t\right), x\in S.$$

From the known theorem [6, p. 52] it is easy to prove that $A: C(S)\to H_\beta(S)$, $\beta\in(0,\alpha)$, where $H_\beta(S)$ is a space of functions determined on the surface S and satisfying the Holder condition with the index β.

For grounding the collocation method, at first we construct a cubic formula for $(A\rho)(x), x\in S$. Take the sequence $\{h\}\subset\mathbb{R}$ of values of the discretization parameter h tending to zero, and partition S into elementary parts $S=\bigcup_{l=1}^{N(h)} S_l^h$, satisfying the following conditions:

(1) $\forall l\in\{1,2,...,N(h)\}$ S_l^h is closed and its set of points S internal with respect to $\mathring{S}_l^h$ is not empty, moreover $\operatorname{mes}\mathring{S}_l^h=\operatorname{mes}S_l^h$ and for $j\in\{1,2,...,N(h)\}$, $j\neq l, \mathring{S}_l^h\cap\mathring{S}_l^h=\oslash$;
(2) $\forall l\in\{1,2,...,N(h)\}$ S_l^h is a connected piece of surface S with continuous boundary;
(3) $\forall l\in\{1,2,...,N(h)\}$ $diam S_l^h\leq h$;

(4) $\forall l \in \{1, 2, ..., N(h)\}$ there exists a so called support point $x_l \in S_l^h$ such that
  (a) $r_l(h) \sim R_l(h) (r_l(h) \sim R_l(h) \Leftrightarrow C_1 \leq \frac{r_l(h)}{R_l(h)} \leq C_2$, where C_1 and C_2 are positive constants independent of h), here $r_l(h) = \min_{x \in \partial S_l^h} |x - x_l|$ and $R_l(h) = \max_{x \in \partial S_l^h} |x - x_l|$;
  (b) $R_l(h) \leq d/2$, where d is a radius of standard sphere (see [11])
  (c) $\forall l \in \{1, 2, ..., N(h)\}\, r_j(h) \sim r_l(h)$.
  Obviously, $r(h) \sim R(h)$, where $R(h) = \max_{l=\overline{1,N(h)}} R_l(h), r(h) = \min_{l=\overline{1,N(h)}} r_l(h)$.

Note that in the paper [7], the partition of a unit sphere into elementary parts is given.

Let $S_d(x)$ and $\Gamma_d(x)$ be the parts of the surface S and tangential plane $\Gamma(x)$ at the point $x \in S$ enclosed inside the sphere $B_d(x)$ of radius d centered at the point x. Furthermore, let $\tilde{y} \in \Gamma(x)$ be the projection of the point $y \in S$. Then

$$|x - \tilde{y}| \leq |x - y| \leq C_1(S) |x - \tilde{y}| \quad \text{and} \quad \operatorname{mes} S_d(x) \leq C_2(S) \cdot \operatorname{mes} \Gamma_d(x), \quad (2.3)$$

where $C_1(S)$ and $C_2(S)$ are positive constants dependent only on S (if S is a sphere, then $C_1(S) = \sqrt{2}$ and $C_1(S) = 2$). The following lemma is valid.

Lemma 2.1. *(see [7]). There exist the constants $C_0' > 0$ and $C_1' > 0$ independent of h, for which for $\forall l, j \in \{1, 2, ..., N(h)\}, j \neq l$ and $\forall y \in S_j^h$ the following inequality is valid:*

$$C_0' |y - x_l| \leq |x_j - x_l| \leq C_1' |y - x_l| \quad (2.4)$$

Let

$$b_{lj} = |\operatorname{sgn}(l - j)| \frac{\partial \Phi(x_l, x_j)}{\partial \vec{n}(x_l)} \cdot \operatorname{mes} S_j^h$$
$$c_{lj} = |\operatorname{sgn}(l - j)| \, \Phi(x_l, x_j) \cdot \operatorname{mes} S_j^h,$$
$$e_{lj} = |\operatorname{sgn}(l - j)| \frac{\partial \Phi(x_l, x_j)}{\partial \vec{n}(x_j)} \cdot \operatorname{mes} S_j^h$$

and

$$a_{lj} = (\mu - 1)^{-1} \left(2b_{lj} - 16\pi^2 \mu \sum_{m=1}^{N(h)} b_{lm} b_{mj} + 2\lambda c_{lj} - 4\pi\mu\lambda c_{lj} + 8\pi\mu\lambda \cdot \sum_{m=1}^{N(h)} e_{lm} \cdot c_{mj} \right).$$

Prove the following theorem.

Theorem 2.1. *The expression*

$$(A^{N(h)} \rho)(x_l) = \sum_{j=1}^{N(h)} a_{lj} \cdot \rho(x_j), \quad (2.5)$$

at the points $x_l, l = \overline{1, N(h)}$ is a cubic formulas for $(A\rho)(x)$, moreover the following estimation is valid

$$\max_{l=\overline{1,N(h)}} \left| (A\rho)(x_l) - \left(A^{N(h)} \right) (x_l) \right| \leq M \left[\|\rho\|_\infty (R(h))^\beta + \omega(\rho, R(h)) \right],$$

where $\beta \in (0,\alpha)$ and $\omega(\rho, R(h))$ is a modulus of continuity of the function $\rho(x)$. Here and in the sequel, M denotes positive constants dependent only on S, k, λ, and μ different at various inequalities.

Proof. In the paper [4] it is proved that the expressions

$$\left(K^{N(h)}\rho\right)(x_l) = \sum_{j=1}^{N(h)} b_{lj}\rho(x_j)$$

and

$$\left(L^{N(h)}\rho\right)(x_l) = \sum_{j=1}^{N(h)} c_{lj}\rho(x_j)$$

at the points $x_l, l = \overline{1, N(h)}$ are cubic formulas for the integrals $(K\rho)(x)$ and $(L\rho)(x)$, respectively, and

$$\max_{l=\overline{1,N(h)}} \left|(K\rho)(x_l) - \left(K^{N(h)}\rho\right)(x_l)\right| \le M\left[\|\rho\|_\infty (R(h))^\alpha \left|lnR(h)\right| + \omega(\rho, R(h))\right],$$

$$\max_{l=\overline{1,N(h)}} \left|(L\rho)(x_l) - \left(L^{N(h)}\rho\right)(x_l)\right| \le M\left[\|\rho\|_\infty R(h) \left|lnR(h)\right| + \omega(\rho, R(h))\right].$$

Now lets construct a cubic formula for the integral $(T\rho)(x)$. The expression $\left(T^{N(h)}\rho\right)(x_l) = \sum_{j=1}^{N(h)}\left(\sum_{m=1}^{N(h)} b_{lm}b_{mj}\right)\rho(x_j)$ at the points $x_l, l = \overline{1, N(h)}$ is a cubic formula for the integral $(T\rho)(x)$. Estimate the error of this cubic formula. Obviously,

$$(T\rho)(x_l) - \left(T^{N(h)}\rho\right)(x_l) = (T\rho)(x_l) - \sum_{j=1}^{N(h)}\left(b_{lj}\sum_{m=1}^{N(h)} b_{jm}\rho(x_m)\right) =$$
$$= \int_S \frac{\partial\Phi(x_l,y)}{\partial\vec{n}(x_l)}\left(\int_S \frac{\partial\Phi(y,t)}{\partial\vec{n}(y)}\rho(t)dS_t\right)dS_y -$$
$$-\sum_{\substack{j=1\\ j\neq l}}^{N(h)}\left(\frac{\partial\Phi(x_l,x_j)}{\partial\vec{n}(x_l)} \operatorname{mes} S_j^h \sum_{\substack{m=1\\ m\neq j}}^{N(h)} \frac{\partial\Phi(x_j,x_m)}{\partial\vec{n}(x_j)} \operatorname{mes} S_m^h \rho(x_m)\right) =$$
$$= \int_{S_l^h} \frac{\partial\Phi(x_l,y)}{\partial\vec{n}(x_l)}\left(\int_S \frac{\partial\Phi(y,t)}{\partial\vec{n}(y)}\rho(t)dS_t\right)dS_y +$$
$$+\sum_{\substack{j=1\\ j\neq l}}^{N(h)}\int_{S_j^h} \frac{\partial\Phi(x_l,y)}{\partial\vec{n}(x_l)}\left(\int_S \frac{\partial\Phi(y,t)}{\partial\vec{n}(y)}\rho(t)dS_t - \int_S \frac{\partial\Phi(x_j,t)}{\partial\vec{n}(x_j)}\rho(t)dS_t\right)dS_y +$$

$$+\sum_{\substack{j=1\\ j\neq l}}^{N(h)}\left[\int_{S_j^h}\left(\frac{\partial\Phi(x_l,y)}{\partial\vec{n}(x_l)}-\frac{\partial\Phi(x_l,x_j)}{\partial\vec{n}(x_l)}\right)\int_S\frac{\partial\Phi(x_j,t)}{\partial\vec{n}(x_j)}\rho(t)dS_t\right]dS_y+$$

$$+\sum_{\substack{j=1\\ j\neq l}}^{N(h)}\frac{\partial\Phi(x_l,x_j)}{\partial\vec{n}(x_l)}\left(\int_S\frac{\partial\Phi(x_j,t)}{\partial\vec{n}(x_j)}\rho(t)dS_t-\sum_{\substack{m=1\\ m\neq l}}^{N(h)}\frac{\partial\Phi(x_j,x_m)}{\partial\vec{n}(x_j)}\operatorname{mes}S_m^h\rho(x_m)\right)\operatorname{mes}S_j^h. \tag{2.6}$$

Denote the summands in equality (2.6) by $T_1(h), T_2(h), T_3(h)$ and $T_4(h)$, respectively. Since $\left|\frac{\partial\Phi(x,y)}{\partial\vec{n}(x)}\right|=\frac{M}{|x-y|^{2-\alpha}}$ we get

$$|T_1(h)|\leq\left|\int_S\frac{\partial\Phi(y,t)}{\partial\vec{n}(y)}\rho(t)dS_t\right|\int_{S_l^h}\left|\frac{\partial\Phi(x_l,y)}{\partial\vec{n}(x_l)}\right|dS_y\leq M\|\rho\|_\infty\left(R(h)\right)^\alpha.$$

In the paper [6] it was proved that $K: C(S)\to H_\beta(S), \beta\in(0,\alpha)$, this means that $\forall y\in S_j^h$

$$\left|\int_S\frac{\partial\Phi(y,t)}{\partial\vec{n}(y)}\rho(t)dS_t-\int_S\frac{\partial\Phi(x_j,y)}{\partial\vec{n}(x_j)}dS_t\right|\leq M\|\rho\|_\infty\left(R(h)\right)^\beta,$$

then we have:

$$|T_2(h)|\leq M\|\rho\|_\infty\left(R(h)\right)^\beta\int_S\left|\frac{\partial\Phi(x_l,y)}{\partial\vec{n}(x_l)}\right|dS_y\leq M\|\rho\|_\infty\left(R(h)\right)^\beta, \beta\in(0,\alpha).$$

Taking into attention (2.4), it is easy to prove that $\forall l,j\in\{1,2,...,N(h)\}, j\neq l$ and $\forall y\in S_j^h$

$$\left|\frac{\partial\Phi(x_l,y)}{\partial\vec{n}(x_l)}-\frac{\partial\Phi(x_l,x_j)}{\partial\vec{n}(x_l)}\right|\leq M\cdot\frac{|x_j-y|}{|x_l-y|^{3-\alpha}}.$$

Then for the expression $T_3(h)$ we get $|T_3(h)|\leq M\|\rho\|_\infty\left(R(h)\right)^\alpha$.

Taking into account inequality (2.4) and error estimation of the constructed cubic formula for the integral $K(x)$, we have

$$|T_4(h)|\leq M\left[\|\rho\|_\infty\left(R(h)\right)^\beta+\omega(\rho,R(h))\right], \beta\in(0,\alpha).$$

As a result, summing the obtained estimations for the expressions $T_1(h), T_2(h)$, $T_3(h)$ and $T_4(h)$, we get

$$\max_{l=\overline{1,N(h)}}\left|(T\rho)(x_l)-\left(T^{N(h)}\rho\right)(x_l)\right|\leq M\cdot\left[\|\rho\|_\infty\left(R(h)\right)^\beta+\omega(\rho,R(h))\right], \beta\in(0,\alpha).$$

Behaving in the same way, we can prove that the expression

$$\left(Q^{N(h)}\rho\right)(x_l)=\sum_{j=1}^{N(h)}\left(\sum_{m=1}^{N(h)}e_{lm}\cdot c_{mj}\right)\rho(x_j)$$

at the points $x_l, l=\overline{1,N(h)}$ is a cubic formula for that integral $(Q\rho)(x)$, and

$$\max_{l=\overline{1,N(h)}} \left|(Q\rho)(x_l) - \left(Q^{N(h)}\rho\right)(x_l)\right| \leq M \left[\|\rho\|_\infty (R(h))^\beta + \omega(\rho, R(h))\right],$$

where $\beta \in (0,\alpha)$. This completes the proof of the theorem. □

For $z^{N(h)} \in \mathbb{C}^{N(h)}$ ($\mathbb{C}^{N(h)}$ is the space of $N(h)$-dimensional vectors

$$z^{N(h)} = \left(z_1^{N(h)}, z_2^{N(h)}, ..., z_{N(h)}^{N(h)}\right), \qquad z_l^{N(h)} \in \mathbb{C}, \quad l = \overline{1, N(h)}$$

with the norm

$$\left\|z^{N(h)}\right\| = \max_{l=\overline{1,N(h)}} \left|z_l^{N(h)}\right|)$$

we assume

$$A_l^{N(h)} z^{N(h)} = \sum_{j=1}^{N(h)} a_{lj} \cdot z_j^{N(h)}, \quad l = \overline{1, N(h)};$$

$$A^{N(h)} z^{N(h)} = \left(A_1^{N(h)} z^{N(h)}, A_2^{N(h)} z^{N(h)}, ..., A_{N(h)}^{N(h)} z^{N(h)}\right).$$

Using cubic formula (2.5), we replace BIE (2.2) by the system of algebraic equations with respect to $z_l^{N(h)}$-approximate values $\rho(x_l), l = \overline{1, N(h)}$ that we write in the form

$$z^{N(h)} + A^{N(h)} z^{N(h)} = \varphi^{N(h)}, \tag{2.7}$$

where $\varphi^{N(h)} = p^{N(h)}\varphi = \left(\varphi_1, \varphi_2, ..., \varphi_{N(h)}\right)$; $\varphi_l = \varphi(x_l), l = \overline{1, N(h)}$; $p^{N(h)} \in L\left(C(S), \mathbb{C}^{N(h)}\right)$ is a simple drift operator, and $A^{N(h)} : \mathbb{C}^{N(h)} \to \mathbb{C}^{N(h)}$ is a linear bounded operator.

We get the justification of the collocation method from the G.M. Vainicco theorem on convergence for linear operator equations (see [10]). For its formulation we give necessary definition and conjecture in the denotation of the paper [10].

Definition 2.1. We call the system $P = \left\{p^{N(h)}\right\}$ of the operators $p^{N(h)} : C(S) \to \mathbb{C}^{N(h)}$ connecting for $C(S)$ and $\mathbb{C}^{N(h)}$ if

$$\|p^{N(h)}\varphi\| \to \|\varphi\|_\infty$$

as

$$h \to 0, \forall \varphi \in C(S)$$

$$\left\|p^{N(h)}(a\varphi + a'\varphi') - \left(ap^{N(h)}\varphi + a'p^{N(h)}\varphi'\right)\right\| \to 0$$

as

$$h \to 0, \forall \varphi, \varphi' \in C(S), a, a' \in \mathbb{C}.$$

Definition 2.2. The sequence $\left\{\varphi_{N(h)}\right\}$ of the elements $\varphi_{N(h)} \in \mathbb{C}^{N(h)}$ P-converges to $\varphi \in C(S)$ if

$$\|\varphi_{N(h)} - p^{N(h)}\|\varphi \to 0$$

as $h \to 0$. Herewith well write $\varphi_{N(h)} \xrightarrow{p} \varphi$.

Definition 2.3. The sequence $\left\{\varphi_{N(h)}\right\}$ of the elements $\varphi_{N(h)} \in \mathbb{C}^{N(h)}$ is P-compact if its any subsequence contains P-convergent subsequence.

Definition 2.4. The sequence of the operators $B^{N(h)} : \mathbb{C}^{N(h)} \to \mathbb{C}^{N(h)}$ PP-converges to the operator $B : C(S) \to C(S)$ if for any P-convergent sequence $\{\varphi_{N(h)}\}$ we have $\varphi_{N(h)} \xrightarrow{p} \varphi \Rightarrow B^{N(h)}\varphi_{N(h)} \xrightarrow{p} B\varphi$. Herewith well write $B^{N(h)} \xrightarrow{pp} B$.

Definition 2.5. The sequence of operators $B^{N(h)} \in L\left(\mathbb{C}^{N(h)}, \mathbb{C}^{N(h)}\right)$ regularly converges to the operator $B \in L(C(S), C(S))$ if $B^{N(h)} \xrightarrow{pp} B$ and the following regularity condition is fulfilled: $\varphi_{N(h)} \in \mathbb{C}^{N(h)}$, $\|\varphi_{N(h)}\| \le M$, $\{B^{N(h)}\varphi_{N(h)}\}$ P-compact $\Rightarrow \{\varphi_{N(h)}\}$ P-compact.

Theorem 2.2. *Let $B^{N(h)} \to B$ regularly, $B^{N(h)}(N(h) \ge N_0)$ be Fredholm with a zero index, $KerB = \{0\}$ and $\psi_{N(h)} \xrightarrow{p} \psi$, $\psi \in C(S)$. Then the equation $B\varphi = \psi$ has a unique solution $\tilde{\varphi} \in C(S)$, the equation $B^{N(h)}\varphi_{N(h)} = \psi_N(h)$ $(N(h) \ge N_0)$ has a unique solution $\tilde{\varphi}_{N(h)} \in \mathbb{C}^{N(h)}$, and $\tilde{\varphi}_{N(h)} \xrightarrow{p} \tilde{\varphi}$ with the estimation*

$$c_1\|B^{N(h)}p^{N(h)}\tilde{\varphi} - \psi_{N(h)}\| \le \|\tilde{\varphi}_N(h) - p^{N(h)}\tilde{\varphi}\| \le c_2\|B^{N(h)}p^{N(h)}\tilde{\varphi} - \psi_{N(h)}\|,$$

where

$$c_1 = 1/\sup_{N(h)\ge N_0}\|B^{N(h)}\| > 0,$$

$$c_2 = \sup_{N(h)\ge N_0}\|\left(B^{N(h)}\right)^{-1}\| < +\infty.$$

Now we formulate the main result of this paper.

Theorem 2.3. *Equations (2.2) and (2.7) have unique solutions $\rho_* \in C(S)$ and $z_*^{N(h)} \in \mathbb{C}^{N(h)}(N(h) \ge N_0)$ and $\|z_*^{N(h)} - p^{N(h)}\rho_*\| \to 0$ as $h \to 0$ with the estimation $\|z_*^{N(h)} - p^{N(h)}\rho_*\| \le M \cdot \left[(R(h))^{\beta} + \omega(\varphi, R(h))\right]$ where $\beta \in (0, \alpha)$.*

Proof. Applying Theorem 2.2, we get that equations (2.2) and (2.7) have unique solutions $\rho_* \in C(S)$ and $z_*^{N(h)} \in \mathbb{C}^{N(h)}(N(h) \ge N_0)$, moreover

$$c_1 \cdot \delta_{N(h)} \le \|z_*^{N(h)} - p^{N(h)}\rho_*\| \le c_2 \cdot \delta_{N(h)},$$

where

$$c_1 = 1/\sup_{N(h)\ge N_0}\|I^{N(h)} + A^{N(h)}\| > 0,$$

$$c_2 = \sup_{N(h)\ge N_0}\left|\left(I^{N(h)} + A^{N(h)}\right)^{-1}\right| < +\infty,$$

$$\delta_{N(h)} = \max_{l=\overline{1,N(h)}}\left|A_l^{N(h)}(p^{N(h)}\rho_*) - (A\rho_*)(x_l)\right|.$$

Since $\rho_* = (I + A)^{-1}\varphi$, then $\|\rho_*\| \le \|(I + A)^{-1}\| \cdot \|\varphi\|_\infty$. Furthermore, taking into account $\omega(A\rho_*, R(h)) \le M \cdot \|\rho_*\| \cdot (R(h))^{\beta}$, we have:

$$\omega(\rho_*, R(h)) = \omega(\varphi - A\rho_*, R(h)) \le \omega(\varphi, R(h)) + \omega(A\rho_*, R(h)) \le$$

$$\le \omega(\varphi, R(h)) + M \cdot (R(h))^{\beta}, \beta \in (0, \alpha).$$

As a result, from the obtained estimations and from Theorem 2.1 we get $\delta_{N(h)} \le$ $\le M \cdot \left[(R(h))^{\beta} + \omega(\varphi, R(h))\right]$, where $\beta \in (0, \alpha)$. □

Note that in particular for $\lambda = 0$ we get exterior Neumann boundary value problem. The justification of the collocation method for BIE corresponding to the exterior Neumann boundary value problem is given in the paper [1]. But in the paper [1], for constructing a cubic formula in addition to support points the additional points on the boundary of elementary parts were taken. One of the advantages of this paper is that these additional points were not used in it and this in its turn shortens much the finding of approximate values of corresponding integrals.

References

[1] Abdullayev F.A., Khalilov E.H. Justification of the collocation method for a class of boundary integral equations. Differentsialniye Uravneniya, 2004, vol. 40, No1, p. 82-86. (Russian)

[2] Ahner J.F. The exterior Dirichlet problem for the Helmholtz equation.-J. Math. Anal. Appl., 1975, v. 52, p. 415-429.

[3] Ahner J.F., Kleinman ER.E. The exterior Neumann problem for the Helmholtz equation.-Arch. Rational Mech. Anal., 1973, v. 52, p. 26-43.

[4] Khalilov E.H. Cubic formula for class of weakly singular surface integrals.-Proceedings of IMM of NAS of Azerbaijan, 2013, v. XXXIX(XLVII), p. 69-76.

[5] Kleinman R.E., Wendland W. On Neumann's method for the exterior Neumann problem for the Helmholtz equation.-J. Math. Anal. Appl., 1977, v. 57, p. 170-202.

[6] Kolton D., Cress P. Methods of integral equations in scattering theory. M., Mir, 1987, 311 p. (Russian)

[7] Kustov Yu.A., Musayev B.I. Cubic formula for two-dimensional integral and its applications. M., 1981. Dep. in VINITI, No 4281-81, 60 p. (Russian)

[8] Musayev B.I., Khalilov E.H. On approximate solution of a class of boundary integral equations by the collocation method. Trudy IMM AN Azerb. Rep. 1998, vol. IX (XVII) p. 28-84. (Russian)

[9] Panich O.I. To solvability of exterior boundary value problems for wave equation and Maxwell equation. Uspekhi mat. Nauk, 1965, vol. 20, No 1, p. 221-226. (Russian)

[10] Vainicco G.M. Regular convergence of operators and approximate solution of equations. Itogi nauki i tekhniki. Mat. analiz. Vol. 16, p. 5-53. (Russian)

[11] Vladimirov V.S. Equations of mathematical physics. M., Nauka, 1976, 527 p. (Russian)

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Received: October 1, 2014; Accepted: December 12, 2014