Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 40, Special Issue, 2014, Pages 23–29

ASYMPTOTIC BEHAVIOR OF EIGENVALUES OF A BOUNDARY VALUE PROBLEM FOR A SECOND ORDER ELLIPTIC DIFFERENTIAL-OPERATOR EQUATION

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In memory of M. G. Gasymov on his 75th birthday

Abstract. In this paper, in the Hilbert space H we obtain asymptotic formulas for eigenvalues of a boundary value problems for differential-operator equations in the case when both boundary conditions contain a spectral parameter.

1. Introduction

Boundary value problems for second order differential-operator equations in the case when one and the same spectral parameter enters in both the equation and boundary conditions were studied in different aspects in a lot of papers (see for instance [1-7, 9, 10]).

Note that in the papers [1-7, 9, 10] the spectral parameter enters linearly in both the equation and boundary conditions.

In this paper we study asymptotic behavior of eigenvalues of boundary value problems for a second order elliptic differential-operator equation in the case when one and the same spectral parameter quadratically enters into the equation and linearly into one of boundary conditions.

Thus, in the present paper in a separable Hilbert space H, we consider the following boundary value problem on $[0, 1]$ for a second order elliptic differential-operator equation:

$$-u''(x) + Au(x) = \lambda^2 u(x), x \in (0, 1), \tag{1.1}$$

$$\begin{aligned} u'(0) - \lambda u(0) = 0, \\ u'(1) + \lambda u(1) = 0, \end{aligned} \tag{1.2}$$

where λ is a spectral parameter, A is a linear unbounded, self-adjoint, positive-definite operator in H and A^{-1} is completely continuous in H. It is proved that the eigenvalues of boundary valued problem (1.1), (1.2) are real. In what follows we find asymptotic formulas for these eigenvalues.

2010 *Mathematics Subject Classification.* 35J25, 35P20.

Key words and phrases. differential-operator equation, elliptic equations, spectral parameter, eigenvalues, Regge problem, asymptotic formula.

In the case when in (1.1) $A = Q(x)$, $Q(x) \in W_p^1(0,1)$, $1 < p < \infty$ and $Q(1) \neq 0$, and in (1.2) instead of the second condition $u(0) = 0$, then the obtained boundary value problem (1.1), (1.2) for second order ordinary differential equations that is called the Regge problem was studied in S. Yakubov and Ya.Yakubov's monograph [12, Section 3.5.4] (see also S.Yakubov's paper [11]), where the discreteness of the spectrum and double completeness of the root vectors of the considered boundary value problems in the direct sum $W_2^1((0,1); u(0) = 0) \oplus L_2(0,1)$ is studied.

Note that boundary value problems for second order ordinary differential equations with a spectral parameter in the equation and in boundary conditions were studied in a lot of papers. In particular, in the paper [8] the spectral parameter exists in the equation as λ^2, and in the boundary condition as λ, and as λ^2, where asymptotic behavior of the eigenvalues and eigenfunctions of the considered boundary value problems are studied.

2. Asymptotic formula for eigenvalues of (1.1), (1.2)

At first we prove the following lemma.

Lemma 2.1. *The eigenvalues of boundary value problem (1.1), (1.2) are real.*

Proof. Denote the eigen elements of the operator A, corresponding to the eigenvalues μ_k, by φ_k, where $k = 1, 2, 3, \ldots$. It is known that $\{\varphi_k\}$ forms a complete orthonormalized basis in H. Then, taking into account spectral expansion, for the Fourier coefficients $u_k(x) = (u(x), \varphi_k)_H$ we get the following spectral problem

$$-u_k''(x) + \mu_k u_k(x) = \lambda^2 u_k(x), \quad x \in (0,1), \tag{2.1}$$

$$u_k'(0) - \lambda u_k(0) = 0,$$
$$u_k'(1) + \lambda u_k(1) = 0, \tag{2.2}$$

Thus, the study of eigenvalues of boundary value problem (1.1), (1.2) is reduced to studying eigenvalues of boundary value problems (2.1), (2.2). The spectrum of boundary value problems (2.1), (2.2) consists of those λ, $\lambda^2 \neq \mu_k$, for which even if for one k the problem (2.1), (2.2) has a non-trivial solution. $\lambda = \pm\sqrt{\mu_k}$ may not be an eigenvalue of problem (2.1), (2.2) for sufficiently large k, since if $\lambda = \pm\sqrt{\mu_k}$, then problem (2.1), (2.2) has only a trivial solution.

Let λ be an eigenvalue of boundary value problem (2.1), (2.2) and $u_k(x,\lambda)$ be a corresponding eigenfunction. Multiply the both sides of equality (2.1) by the function $\overline{u_k(x,\lambda)}$ and integrate the obtained identity with respect to x from 0 to 1:

$$-\int_0^1 u_k''(x,\lambda)\,\overline{u_k(x,\lambda)}dx + \mu_k \int_0^1 |u_k(x,\lambda)|^2 dx = \lambda^2 \int_0^1 |u_k(x,\lambda)|^2 dx. \tag{2.3}$$

Using the formula of integration by parts and boundary conditions (2.2), we get

$$\int_0^1 u_k''(x,\lambda)\,\overline{u_k(x,\lambda)}dx = \overline{u_k(x,\lambda)}u_k'(x,\lambda)\Big|_0^1 - \int_0^1 u_k'(x,\lambda)\,\overline{u_k'(x,\lambda)}dx =$$

$$\overline{u_k(1,\lambda)}u_k'(1,\lambda) - \overline{u_k(0,\lambda)}u_k'(0,\lambda) - \int_0^1 |u_k'(x,\lambda)|^2 dx =$$

$$-\lambda \left|u_k(1,\lambda)\right|^2 - \lambda \left|u_k(0,\lambda)\right|^2 - \int_0^1 \left|u_k'(x,\lambda)\right|^2 dx.$$

Hence and from (2.3) follows

$$\lambda^2 \int_0^1 \left|u_k(x,\lambda)\right|^2 dx - \lambda \left(\left|u_k(1,\lambda)\right|^2 + \left|u_k(0,\lambda)\right|^2 \right) -$$

$$\mu_k \int_0^1 \left|u_k(x,\lambda)\right|^2 dx - \int_0^1 \left|u_k'(x,\lambda)\right|^2 dx = 0 \tag{2.4}$$

Denote

$$a_k(\lambda) = \int_0^1 \left|u_k(x,\lambda)\right|^2 dx, \quad b_k(\lambda) = -\left(\left|u_k(1,\lambda)\right|^2 + \left|u_k(0,\lambda)\right|^2 \right),$$

$$c_k(\lambda) = -\mu_k \int_0^1 \left|u_k(x,\lambda)\right|^2 dx - \int_0^1 \left|u_k'(x,\lambda)\right|^2 dx.$$

Then we can rewrite equation (2.4) in the form

$$a_k(\lambda)\lambda^2 + b_k(\lambda)\lambda + c_k(\lambda) = 0. \tag{2.5}$$

Since for each k, $a_k(\lambda) > 0$, $b_k(\lambda) \le 0$, $c_k(\lambda) < 0$, then $b_k^2(\lambda) - 4a_k(\lambda) \cdot c_k(\lambda) > 0$.

Consequently, for each k equation (2.5) has only real roots λ. The lemma is proved.

Theorem 2.1. *Let A be a self-adjoint positive-definite operator in the Hilbert space H and A^{-1} be a compact operator in H. Then boundary value problem (1.1), (1.2) has the following series of eigenvalues:*

$$\lambda_k^{(1)} \sim -\sqrt{\frac{\mu_k}{2}}, \quad k \to \infty,$$

and, for each fixed natural k,

$$\lambda_{n,k}^{(2)} \sim \sqrt{\mu_k + \left(2n - \frac{1}{2}\right)^2 \pi^2}, \quad \lambda_{n,k}^{(3)} \sim -\sqrt{\mu_k + \left(2n - \frac{3}{2}\right)^2 \pi^2},$$

$$\lambda_{n,k}^{(4)} \sim \sqrt{\mu_k + \left(2n - \frac{3}{2}\right)^2 \pi^2}, \quad \lambda_{n,k}^{(5)} \sim -\sqrt{\mu_k + \left(2n - \frac{1}{2}\right)^2 \pi^2}, \quad n \to \infty,$$

where $\mu_k \to +\infty$ are the eigenvalues of the operator A.

Proof. The general solution of ordinary differential equation (2.1) has the form

$$u_k(x,\lambda) = C_1 e^{-x\sqrt{\mu_k - \lambda^2}} + C_2 e^{-(1-x)\sqrt{\mu_k - \lambda^2}}, \tag{2.6}$$

where C_i, $i = 1, 2$ are arbitrary constants.

Substituting (2.6) in (2.2), we get a system with respect to C_i, $i = 1, 2$, whose determinant is of the form

$$K(\lambda) = -\left(\lambda - \sqrt{\mu_k - \lambda^2}\right)^2 e^{-2\sqrt{\mu_k - \lambda^2}} + \left(\lambda + \sqrt{\mu_k - \lambda^2}\right)^2.$$

Consequently, the eigenvalues of boundary value problem (2.1), (2.2) consist of those real $\lambda \neq \pm\sqrt{\mu_k}$, that even if for one μ_k satisfy the equation

$$\left(\lambda + \sqrt{\mu_k - \lambda^2}\right) e^{2\sqrt{\mu_k-\lambda^2}} - \left(\lambda - \sqrt{\mu_k - \lambda^2}\right)^2 = 0. \tag{2.7}$$

Write equation (2.7) in the form of the system of equations

$$e^{\sqrt{\mu_k-\lambda^2}} \left(\lambda + \sqrt{\mu_k - \lambda^2}\right) - \left(\lambda - \sqrt{\mu_k - \lambda^2}\right) = 0, \tag{2.8}$$

$$e^{\sqrt{\mu_k-\lambda^2}} \left(\lambda + \sqrt{\mu_k - \lambda^2}\right) + \left(\lambda - \sqrt{\mu_k - \lambda^2}\right) = 0. \tag{2.9}$$

Thus, the eigenvalues of problem (1.1), (1.2) consist of those real $\lambda \neq \pm\sqrt{\mu_k}$, that even if for one μ_k satisfy at least one of the equations (2.8) or (2.9).

Rewrite equations (2.8) and (2.9) in the form:

$$\lambda sh\frac{1}{2}\sqrt{\mu_k - \lambda^2} + \sqrt{\mu_k - \lambda^2} ch\frac{1}{2}\sqrt{\mu_k - \lambda^2} = 0, \tag{2.10}$$

$$\lambda ch\frac{1}{2}\sqrt{\mu_k - \lambda^2} + \sqrt{\mu_k - \lambda^2} sh\frac{1}{2}\sqrt{\mu_k - \lambda^2} = 0. \tag{2.11}$$

Find the eigenvalues of λ for which $\lambda^2 < \mu_k$. Assume $\sqrt{\mu_k - \lambda^2} = y$ $(0 < y < \sqrt{\mu_k})$. Then $\lambda = \pm\sqrt{\mu_k - y^2}$. We study equation (2.10). As first in equation (2.10) take $\lambda = \sqrt{\mu_k - y^2}$. Then equation (2.10) takes the form

$$\sqrt{\mu_k - y^2} + sh\frac{y}{2} + ych\frac{y}{2} = 0, \qquad 0 < y < \sqrt{\mu_k}, \tag{2.12}$$

Consider the function $f_k(y) = \sqrt{\mu_k - y^2} sh\frac{y}{2} + ych\frac{y}{2}$, $y \in \left(0, \sqrt{\mu_k}\right)$. Obviously, for each fixed k and for all $y \in \left(0, \sqrt{\mu_k}\right)$, $f_k(y) > 0$. Therefore, equation (2.12) has no solutions on the interval $\left(0, \sqrt{\mu_k}\right)$ for any k.

Now in equation (2.10) take $\lambda = -\sqrt{\mu_k - y^2}$. In this case equation (2.10) is equivalent to the equation

$$\sqrt{\mu_k - y^2} - ycth\frac{y}{2} = 0, \quad y \in (0, \sqrt{\mu_k}). \tag{2.13}$$

Consider the function $\varphi_k(y) = \sqrt{\mu_k - y^2} - ycth\frac{y}{2}$, $y \in \left(0, \sqrt{\mu_k}\right)$. The derivative $\varphi_k'(y) = -\dfrac{y}{\sqrt{\mu_k - y^2}} - \dfrac{shy - y}{2sh^2\frac{y}{2}} < 0$, for $y \in \left(0, \sqrt{\mu_k}\right)$, or $shy > y$ for $y > 0$. This means that $\varphi_k(y)$ monotonically decreases on $\left(0, \sqrt{\mu_k}\right)$ for each k.

Obviously, $\varphi_k\left(\sqrt{\dfrac{\mu_k}{2}}\right) = \sqrt{\dfrac{\mu_k}{2}}\left(1 - cth\dfrac{1}{2}\sqrt{\dfrac{\mu_k}{2}}\right) < 0$, for each k.

On the other hand, some simple calculations show that

$$\varphi_k\left(\sqrt{\frac{\mu_k}{2} - \frac{1}{\mu_k}}\right) = \sqrt{\frac{\mu_k}{2} - \frac{1}{\mu_k}}\left(\sqrt{1 + \frac{4}{\mu_k^2 - 2}} - \left(1 + \frac{2}{e^{\sqrt{\frac{\mu_k}{2} - \frac{1}{\mu_k}}} - 1}\right)\right) > 0$$

starting with some k, since, for any fixed number m, $\lim\limits_{x \to +\infty} \dfrac{e^x}{x^m} = +\infty$.

Therefore, the equation (2.13), starting with some k, has exactly one zero y_k and it belongs to the interval $\left(\sqrt{\frac{\mu_k}{2} - \frac{1}{\mu_k}}, \sqrt{\frac{\mu_k}{2}}\right)$, i.e., $y_k \sim \sqrt{\dfrac{\mu_k}{2}}$. Hence, for the first series of the eigenvalues $\lambda = -\sqrt{\mu_k - y^2}$ of the problem (2.1), (2.2), for $\lambda^2 < \mu_k$, we get the asymptotic formula

$$\lambda_k^{(1)} \sim -\sqrt{\frac{\mu_k}{2}}.$$

Now, let us study the eigenvalues λ of problem (2.1), (2.2), for which $\lambda^2 > \mu_k$. Assume $z = \sqrt{\lambda^2 - \mu_k}$, $(0 < z < \infty)$. Then $\sqrt{\mu_k - \lambda^2} = iz$, $sh(\sqrt{\mu_k - \lambda^2}) = sh(iz) = i\sin z$, $ch(\sqrt{\mu_k - \lambda^2}) = ch(iz) = \cos z$, $\lambda^2 = z^2 + \mu_k$, $\lambda = \pm\sqrt{z^2 + \mu_k}$.

At first in equation (2.10) we take $\lambda = \sqrt{z^2 + \mu_k}$. Then equation (2.10) takes the form

$$\sqrt{z^2 + \mu_k}\sin\frac{z}{2} + z\cos\frac{z}{2} = 0, \quad z \in (0, +\infty). \tag{2.14}$$

Obviously, $z \neq 2n\pi$, $n = 1, 2, \dots$. Then equation (2.14) is equivalent to the equation

$$1 + \frac{z}{\sqrt{z^2 + \mu_k}} ctg\frac{z}{2} = 0, \quad z \in (0, +\infty), \quad z \neq 2n\pi, \quad n = 1, 2, \dots. \tag{2.15}$$

Consider the function $F_k(z) = 1 + \frac{z}{\sqrt{z^2 + \mu_k}} ctg\frac{z}{2}$, $z \in (0, +\infty)$, $z \neq 2n\pi$, $n = 1, 2, \dots$. Since at each interval $(2n\pi, 2(n+1)\pi)$, $n = 0, 1, 2, \dots$, the function $F_k(z)$ runs the values from $-\infty$ to $+\infty$, and its derivative

$$F_k'(z) = \left(1 + \frac{z}{\sqrt{z^2 + \mu_k}} ctg\frac{z}{2}\right)' = \frac{\mu_k(\sin z - z) - z^3}{2(\mu_k + z^2)^{3/2}\sin^2\frac{z}{2}} < 0,$$

then therein, for each k, the function $F_k(z)$ has only one zero $z_{n,k}$: $2(n-1)\pi < z_{n,k} < 2n\pi$, $n = 1, 2, \dots$. Find the asymptotic formulae for $z_{n,k}$, for each k, when $n \to +\infty$.

From (2.15) we have

$$ctg\frac{z}{2} = -\frac{\sqrt{z^2 + \mu_k}}{z}, \; z \in (0, +\infty), \; z \neq 2n\pi, \; n = 1, 2, \dots.$$

Denote $q_k(z) = -\frac{\sqrt{z^2 + \mu_k}}{z}$, $z \in (0, +\infty)$. Obviously, for each k, $q_k(z) < 0$, $q_k'(z) = \frac{\mu_k}{z^2\sqrt{z^2 + \mu_k}} > 0$, and $q_k''(z) = -\mu_k\frac{2\mu_k + 3z^2}{z^3(z^2 + \mu_k)^{3/2}} < 0$, i.e. $q_k(z)$ for each k, is a negative, increasing, concave up function. Moreover, $\lim_{z\to+\infty} q_k(z) = -1$, i.e. the straight line $z = -1$ is a horizontal asymptote of the function $q_k(z)$ for each k and $\lim_{z\to 0+} q_k(z) = -\infty$.

On the other hand, the points $z_{n,k}$, for each k, are the abscissas of the intersection points of $q_k(z)$ and the branches of the functions $ctgz$, $z > 0$. Then $z_{n,k}$, for each k, approach the abscissas of the intersection points of the branches of the function $ctgz$ and the straight line $z = -1$ when natural $n \to +\infty$, i.e., $z_{n,k}$, for each k, are the approximate solutions of the equation $ctgz = -1$ on $(2(n-1)\pi, 2n\pi)$. So for each k

$$z_{n,k} \sim 2arctg(-1) + 2(n-1)\pi = 2\frac{3\pi}{4} + 2(n-1)\pi = \left(2n - \frac{1}{2}\right)\pi, \;\; n \to \infty.$$

Since $\lambda = \sqrt{z^2 + \mu_k}$, we get that, for each fixed k,

$$\lambda_{n,k}^{(2)} \sim \sqrt{\mu_k + \left(2n - \frac{1}{2}\right)^2 \pi^2}.$$

Now, take $\lambda = -\sqrt{z^2 + \mu_k}$ in the equation (2.10). Then equation (2.10) takes the form

$$\sqrt{z^2 + \mu_k} \sin \frac{z}{2} - z \cos \frac{z}{2} = 0, \ z \in (0, +\infty). \tag{2.16}$$

Obviously, $z \neq 2n\pi$, $n = 1, 2, \dots$. Then equation (2.16) is equivalent to the equation

$$1 - \frac{z}{\sqrt{z^2 + \mu_k}} ctg \frac{z}{2} = 0, \ z \in (0, +\infty), \ z \neq 2n\pi, \ n = 1, 2, \dots. \tag{2.17}$$

In the same way as equation (2.17), investigate equation (2.17) and show that the last series of the eigenvalues of the boundary value problem (2.1), (2.2) has the following asymptotic formula, for each fixed k,

$$\lambda_{n,k}^{(3)} \sim -\sqrt{\mu_k + \left(2n - \frac{3}{2}\right)^2 \pi^2}, \quad n \to \infty.$$

The equation (2.11) is studied in the same way. The theorem is proved.

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Received: September 01, 2014; Accepted: October 02, 2014