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INTEGRAL REPRESENTATIONS FOR A SOLUTIONS FOR THE DIFFUSION DIFFERENTIAL EQUATION

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In memory of M. G. Gasymov on his 75th birthday

Abstract. We construct useful new integral representations for the fundamental solutions of the quadratic pencil of the Sturm-Liouville equation with piecewise-constant leading coefficient and discontinuous conditions inside an interval. We also study some significant properties of the kernels of these integral representations for the solutions.

1. Introduction

We consider the differential equation

$$-y'' + [q(x) + 2\lambda p(x)]\, y = \lambda^2 \rho(x) y, \qquad x \in [0, a) \cup (a, \pi], \tag{1}$$

with the boundary conditions

$$U(y) := y'(0) = 0,\ V(y) := y(\pi) = 0$$

and with the jump conditions

$$y(a + 0) = \beta y(a - 0), \quad y'(a + 0) = \beta^{-1} y'(a - 0),$$

where λ is the spectral parameter, $\beta \neq 1$ is real numbers, $y = y(x, \lambda)$ is an unknown function, $q(x) \in L_2(0, \pi)$, $p(x) \in W_2^1(0, \pi)$ are real-valued functions, and $\rho(x)$ is the following piecewise-constant function with discontinuity at the point $a \in (0, \pi)$ such that $a > \dfrac{\alpha\pi}{\alpha + 1}$:

$$\rho(x) = \begin{cases} 1, & 0 \le x \le a, \\ \alpha^2 & a \le x \le \pi, \end{cases} \qquad 0 < \alpha \neq 1. \tag{2}$$

Sturm-Liouville equations with potentials depending on the spectral parameter arise in various problems of mathematics and physics(see [11, 12, 28, 36, 51] for details). It is well known that in the case $\rho(x) = 1$, $\beta = 1$ the equation (1) appears for modelling of some problems connected with the scattering of waves and particles in physics [26]. In this classical case Jaulent and Jean [23, 24] have constructed the integral representations of Jost solutions and using them treated the inverse scattering problem by Marchenko method (see [35] and [7]). Note

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that this method which is an effective device in the theory of inverse problems [10, 13, 29, 30, 31, 37, 42], for relativistic scattering problems was first suggested in [8] and [52]. Various inverse scattering problems for the case $\rho(x) = 1$, $\beta = 1$ on the half line and full line was investigated in [4, 25, 32, 39, 40, 44, 45, 50]. Direct and inverse spectral problems in a finite interval for the case $\rho(x) = 1$, $\beta = 1$ was first investigated in [15, 16, 17]. For further discussing of the inverse spectral theory for equation (1) in a finite interval with $\rho(x) = 1$, $\beta = 1$ we refer to works [18, 21, 38, 43, 46].

Note that, in the case $p(x) = 0$ direct and inverse problems boundary-value problems for equation of type (1) in various formulations have been studied in [3, 5, 6, 20, 27, 48, 49] and other works. Inverse scattering problem for equation (1) with $p(x) = 0$ on the half line $[0, +\infty)$ was investigated and the complete solution of this problem was given in [19] where the new integral representation, similar to transformation operators [35] , was obtained for the Jost solution of the discontinuous Sturm-Liouville equation. Direct and inverse scattering problems on the half-line for the equation of type (1) with various boundary conditions also has been investigated in [33, 34]. The direct and inverse spectral problem for the equation (1) in the case $p(x) = 0$ with some separated boundary conditions on the interval $(0, \pi)$ recently has been investigated in [1, 2, 22, 41], where the new integral representations for solutions have been also constructed.

The inverse spectral problem of recovering pencils of second-order differential operators on the half-axis with turning points was studied in [47], where the properties of spectral characteristics were established, formulation of the inverse problem was given and a uniqueness theorem for solution of the inverse problem is proven. But the spectral problems for equation (1) in a finite interval, especially, inverse spectral problems and full-line inverse scattering problems requiring the recovery of the potential functions by the Marchenko methods have not been studied yet and there isn't any serious work published in this direction.

In this work, as a first stage, we construct useful new integral representations for the fundamental solutions of the equation (1) and study some significant properties of the kernels of these integral representations for the solutions. The constructed integral representations allow us to apply and modify the methods in classical theory for the solution of the inverse spectral problems for the equation (1) . The authors plan to examine these problems in other studies.İntegral representations for solutions of the Sturm-Liouville equation with the discontinuous coefficient.

2. Derivation the integral representations for the solutions

We seek a couple of linearly-independent solutions $y_j(x, \lambda)$ $(j = 1; 2)$ of Eq. (1) satisfying the initial conditions

$$y_j\left(0, \lambda\right) = 1, y_j^{'}\left(0, \lambda\right) = \left(-1\right)^{j+1} i\lambda. \tag{3}$$

İt is not difficult to show that when $q(x) \equiv p(x) \equiv 0$ the initial value problem $(1) , (3)$ has solution

$$e_j\left(x, \lambda\right) = r^{+}(x)e^{\omega_j \lambda \mu^{+}(x)} + r^{-}(x)e^{\omega_j \lambda \mu^{-}(x)}, \tag{4}$$

where

$$\begin{aligned} \mu^{\pm}(x) &= \pm x\sqrt{\rho(x)} + a\left(1 \mp \sqrt{\rho(x)}\right), \\ r^{\pm}(x) &= \frac{1}{2}\left(\beta \pm \frac{1}{\beta\sqrt{\rho(x)}}\right) \end{aligned}$$

and $\omega_j = (-1)^{j+1} i$.

Consider the integral equation

$$y_j(x,\lambda) = e_j(x,\lambda) + \int_0^x \Phi(x,t,\lambda)\left[q(t) + 2\lambda p(t)\right] y_j(t,\lambda)dt \qquad (j = 1; 2) \tag{5}$$

which is equivalent to the problem $(1), (3)$. Here

$$\Phi(x,t,\lambda) = \frac{e_1(x,\lambda)\, e_2(t,\lambda) - e_1(t,\lambda)\, e_2(x,\lambda)}{2i\lambda}. \tag{6}$$

By using (4) it is easily obtained that

$$\Phi(x,t,\lambda) = p^{+}(x,t)\frac{\sin\lambda\sigma^{+}(x,t)}{\lambda} - p^{-}(x,t)\frac{\sin\lambda\sigma^{-}(x,t)}{\lambda}, \; 0 \le t \le x, \tag{7}$$

where

$$p^{\pm}(x,t) = \frac{1}{2}\left(\frac{1}{\sqrt{\rho(x)}} \pm \frac{1}{\sqrt{\rho(t)}}\right), \;\; \sigma^{\pm}(x,t) = \mu^{\pm}(x) - \mu^{+}(t).$$

It is easy to obtain that

$$\begin{gathered} 2i\lambda\Phi(x,t,\lambda)e^{\omega_j\lambda\mu^{\pm}(t)} = \\ (-1)^{j+1} p^{+}(x,t)\left[e^{\omega_j\lambda\left(\mu^{+}(x)-\mu^{+}(t)\right)} - e^{-\omega_j\lambda\left(\mu^{+}(x)-\mu^{+}(t)\right)}\right] e^{\omega_j\lambda\mu^{\pm}(t)} - \\ (-1)^{j+1} p^{-}(x,t)\left[e^{\omega_j\lambda\left(\mu^{-}(x)-\mu^{+}(t)\right)} - e^{-\omega_j\lambda\left(\mu^{-}(x)-\mu^{+}(t)\right)}\right] e^{\omega_j\lambda\mu^{\pm}(t)} = \\ (-1)^{j+1} p^{\pm}(x,t)e^{\omega_j\lambda\mu^{+}(x)} - (-1)^{j+1} p^{\mp}(x,t)e^{\omega_j\lambda\mu^{-}(x)} + \\ (-1)^{j} p^{\pm}(x,t)e^{\omega_j\lambda\left(2\mu^{\pm}(t)-\mu^{+}(x)\right)} + (-1)^{j+1} p^{\mp}(x,t)e^{\omega_j\lambda\left(2\mu^{\pm}(t)-\mu^{-}(x)\right)}, \end{gathered}$$

i.e.

$$\begin{gathered} 2i\lambda\Phi(x,t,\lambda)e^{\omega_j\lambda\mu^{\pm}(t)} = \\ (-1)^{j+1} p^{\pm}(x,t)\left[e^{\omega_j\lambda\mu^{+}(x)} - e^{\omega_j\lambda\left(2\mu^{\pm}(t)-\mu^{+}(x)\right)}\right] + \\ (-1)^{j} p^{\mp}(x,t)\left[e^{\omega_j\lambda\mu^{-}(x)} - e^{\omega_j\lambda\left(2\mu^{\pm}(t)-\mu^{-}(x)\right)}\right]. \end{gathered} \tag{8}$$

The formula (8) is also written as

$$\begin{gathered} \Phi(x,t,\lambda)e^{\omega_j\lambda\mu^{\pm}(t)} = \\ \frac{1}{2}p^{\pm}(x,t)\int_{2\mu^{\pm}(t)-\mu^{+}(x)}^{\mu^{+}(x)} e^{\omega_j\lambda s}ds - \frac{1}{2}p^{\mp}(x,t)\int_{2\mu^{\pm}(t)-\mu^{-}(x)}^{\mu^{-}(x)} e^{\omega_j\lambda s}ds. \end{gathered} \tag{8$'$}$$

Consider the integral equation (5) and substitute

$$y_j(x,\lambda) = R_j^+(x)e^{\omega_j\lambda\mu^+(x)} + R_j^-(x)e^{\omega_j\lambda\mu^-(x)} + z_j(x,\lambda), \tag{9}$$

where $R_j^{\pm}(x)$ will be defined below and $z_j(x,\lambda)$ is a new unknown function. We have

$$R_j^+(x)e^{\omega_j\lambda\mu^+(x)} + R_j^-(x)e^{\omega_j\lambda\mu^-(x)} + z_j(x,\lambda) = r^+(x)e^{\omega_j\lambda\mu^+(x)} + r^-(x)e^{\omega_j\lambda\mu^-(x)} +$$

$$\int_0^x \Phi(x,t,\lambda)\left[q(t)+2\lambda p(t)\right]\left[R_j^+(t)e^{\omega_j\lambda\mu^+(t)} + R_j^-(t)e^{\omega_j\lambda\mu^-(t)}\right]dt +$$

$$\int_0^x \Phi(x,t,\lambda)\left[q(t)+2\lambda p(t)\right] z_j(t,\lambda)dt. \tag{10}$$

Taking into our account (8) and the second integral in the right hand side of (10) we require

$$R_j^+(x)e^{\omega_j\lambda\mu^+(x)} + R_j^-(x)e^{\omega_j\lambda\mu^-(x)} =$$

$$r^+(x)e^{\omega_j\lambda\mu^+(x)} + r^-(x)e^{\omega_j\lambda\mu^-(x)} + i\,(-1)^j\, e^{\omega_j\lambda\mu^+(x)} \times$$

$$\int_0^x p(t)\, R_j^+(t)p^+(x,t)dt - i\,(-1)^j\, e^{\omega_j\lambda\mu^-(x)} \int_0^x p(t)\, R_j^+(t)p^-(x,t)dt +$$

$$i\,(-1)^j\, e^{\omega_j\lambda\mu^+(x)} \int_0^x p(t)\, R_j^-(t)p^-(x,t)dt - i\,(-1)^j\, e^{\omega_j\lambda\mu^-(x)} \int_0^x p(t)\, R_j^-(t)p^+(x,t)dt,$$

to be satisfied. Obviously, the last equality will be satisfied if we choose

$$R_j^{\pm}(x) = r^{\pm}(x) \mp \omega_j \int_0^x p(t)\, R_j^+(t)p^{\pm}(x,t)dt \mp \omega_j \int_0^x p(t)\, R_j^-(t)p^{\mp}(x,t)dt. \tag{11}$$

From (11) we immediately have

$$R_j^{\pm}(x) = r^{\pm}(x)e^{\pm\omega_j \int_0^x sgn(t\pm a)\frac{p(t)}{\sqrt{\rho(t)}}dt}. \tag{12}$$

Then (10) implies that

$$z_j(x,\lambda) = J^{(j)}(x,\lambda) + \int_0^x \Phi(x,t,\lambda)q(t)\left[R_j^+(t)e^{\omega_j\lambda\mu^+(t)} + R_j^-(t)e^{\omega_j\lambda\mu^-(t)}\right]dt +$$

$$\int_0^x \Phi(x,t,\lambda)\left[q(t)+2\lambda p(t)\right] z_j(t,\lambda)dt, \tag{13}$$

$$J^{(j)}(x,\lambda) = \int_{-\mu^+(x)}^{\mu^+(x)} A_j(x,t)e^{\omega_j\lambda t}dt, \tag{14}$$

where

$$A_j(x,t)=\frac{\omega_j}{2}p\left(\frac{x+t}{2}\right)R_j^+\left(\frac{x+t}{2}\right),\ 0\le x\le a,$$

and

$$\begin{aligned}A_j(x,t) &= \frac{r^+(x)\omega_j}{2}p\left(\frac{t+\mu^+(x)}{2}\right)R_j^+\left(\frac{t+\mu^+(x)}{2}\right)\\ &+\frac{r^-(x)\omega_j}{2}p\left(\frac{t+\mu^-(x)}{2}\right)R_j^+\left(\frac{t+\mu^-(x)}{2}\right)\\ &+\frac{r^+(x)\omega_j}{2\alpha}p\left(\frac{t-\mu^-(x)}{2\alpha}+a\right)R_j^+\left(\frac{t-\mu^-(x)}{2\alpha}+a\right)\\ &-\frac{r^-(x)\omega_j}{2\alpha}p\left(a-\frac{t+\mu^-(x)}{2\alpha}\right)R_j^+\left(a-\frac{t+\mu^-(x)}{2\alpha}\right),\end{aligned}$$

$\mu^-(x)\le t\le\mu^+(x),x>a,$

$$\begin{aligned}A_j(x,t) &= \frac{r^+(x)\omega_j}{2}p\left(\frac{t+\mu^+(x)}{2}\right)R_j^+\left(\frac{t+\mu^+(x)}{2}\right)\\ &+\frac{r^-(x)\omega_j}{2}p\left(\frac{t+\mu^-(x)}{2}\right)R_j^+\left(\frac{t+\mu^-(x)}{2}\right),\end{aligned}$$

$-\mu^-(x)<t<\mu^-(x),x>a,$

$$A_j(x,t) = \frac{r^+(x)\omega_j}{2}p\left(\frac{t+\mu^+(x)}{2}\right)R_j^+\left(\frac{t+\mu^+(x)}{2}\right),$$

$-\mu^+(x)\le t<-\mu^-(x),\ x>a.$

We require that the integral equation (13) has the solution

$$z_j(x,\lambda)=\int\limits_{-\mu^+(x)}^{\mu^+(x)}K_j(x,t)e^{\omega_j\lambda t}dt, \tag{15}$$

where $K_j(x,t)$ is an unknown function. Substituting the expression (15) of the solution $z_j(x,\lambda)$ in the equation (13) we have

$$\begin{aligned}&\int\limits_{-\mu^+(x)}^{\mu^+(x)}K_j(x,t)e^{\omega_j\lambda t}dt=\int\limits_{-\mu^+(x)}^{\mu^+(x)}A_j(x,t)e^{\omega_j\lambda t}dt\\ &+\int\limits_0^x\Phi(x,t,\lambda)e^{\omega_j\lambda\mu^+(t)}q(t)R_j^+(t)dt+\int\limits_0^x\Phi(x,t,\lambda)e^{\omega_j\lambda\mu^-(t)}q(t)R_j^-(t)dt\\ &+\int\limits_0^x[q(t)+2\lambda p(t)]\int\limits_{-\mu^+(t)}^{\mu^+(t)}K_j(t,s)\Phi(x,t;\lambda)e^{\omega_j\lambda s}dsdt.\end{aligned} \tag{16}$$

Now using the formulas $(8),(8')$ we transform the right hand side of Eq.(15) to the form of the Fourier integral.

First consider the case $0\le x\le a$ for which the equation (16) is written as

$$\int_{-x}^{x} K_j(x,t)e^{\omega_j\lambda t}dt = \int_{-x}^{x} \tfrac{\omega_j}{2}p\left(\frac{x+t}{2}\right)R_j^{+}\left(\frac{x+t}{2}\right)e^{\omega_j\lambda t}dt$$
$$+\tfrac{1}{2}\int_{0}^{x} q(t)\,R_j^{+}(t)dt\int_{2t-x}^{x} e^{\omega_j\lambda\xi}d\xi + \tfrac{1}{2}\int_{0}^{x} q(t)\int_{-t}^{t} K_j(t,s)\int_{s-x+t}^{s+x-t} e^{\omega_j\lambda\xi}d\xi ds dt \tag{17}$$
$$-\omega_j\int_{0}^{x}\left[e^{\omega_j\lambda(x-t)} - e^{-\omega_j\lambda(x-t)}\right]p(t)\int_{-t}^{t} K_j(t,s)e^{\omega_j\lambda s}ds dt.$$

Supposing $K_j(x,t)$ to be zero as $|t| > x$ and changing orders of integrations at the right hand side of Eq.(17) we obtain

$$\int_{-x}^{x} K_j(x,t)e^{\omega_j\lambda t}dt =$$

$$\int_{-x}^{x} e^{\omega_j\lambda t}\left\{\frac{1}{2}\int_{0}^{\frac{x+t}{2}} q(s)\,R_j^{+}(s)ds + \frac{\omega_j}{2}p\left(\frac{x+t}{2}\right)R_j^{+}\left(\frac{x+t}{2}\right)+\right.$$

$$\frac{1}{2}\int_{0}^{x} q(s)\,ds\int_{t-x+s}^{t+x-s} K_j(s,\xi)\,d\xi -$$

$$\left.\omega_j\left[\int_{\frac{x-t}{2}}^{x} p(s)\,K_j(s,t-x+s)\,ds - \int_{\frac{x+t}{2}}^{x} p(s)\,K_j(s,t+x-s)\,ds\right]\right\}dt. \tag{18}$$

According to the uniqueness properties of the Fourier transformation, Eq.(18) implies that

$$K_j(x,t) = \frac{1}{2}\int_{0}^{\frac{x+t}{2}} q(s)\,R_j^{+}(s)ds + \frac{\omega_j}{2}p\left(\frac{x+t}{2}\right)R_j^{+}\left(\frac{x+t}{2}\right)+$$

$$\frac{1}{2}\int_{0}^{x} q(s)\,ds\int_{t-x+s}^{t+x-s} K_j(s,\xi)\,d\xi -$$

$$\omega_j\left[\int_{\frac{x-t}{2}}^{x} p(s)\,K_j(s,t-x+s)\,ds - \int_{\frac{x+t}{2}}^{x} p(s)\,K_j(s,t+x-s)\,ds\right], |t| \le x. \tag{19}$$

Now consider the case $x > a$. In this case, according to formulas (8) and (8′), the equation (16) yields

$$\int_{-\mu^+(x)}^{\mu^+(x)} K_j(x,t)e^{\omega_j\lambda^n t}dt = \int_{-\mu^+(x)}^{\mu^+(x)} A_j(x,t)e^{\omega_j\lambda t}dt$$
$$+\frac{1}{2}\int_0^a q(t)\,R_j^+(t)dt\left[r^+(x)\int_{2t-\mu^+(x)}^{\mu^+(x)} e^{\omega_j\lambda s}ds + r^-(x)\int_{2t-\mu^-(x)}^{\mu^-(x)} e^{\omega_j\lambda s}ds\right]$$
$$+\frac{1}{2\alpha}\int_a^x q(t)\left[R_j^+(t)\int_{2\mu^+(t)-\mu^+(x)}^{\mu^+(x)} e^{\omega_j\lambda s}ds + R_j^-(t)\int_{\mu^-(x)}^{2\mu^-(t)-\mu^-(x)} e^{\omega_j\lambda s}ds\right]dt$$
$$+\frac{1}{2}\int_0^a q(t)\,dt\int_{-t}^{t} K_j(t,s)ds\left[r^+(x)\int_{t-\mu^+(x)+s}^{\mu^+(x)-t+s} e^{\omega_j\lambda\xi}d\xi + r^-(x)\int_{t-\mu^-(x)+s}^{\mu^-(x)-t+s} e^{\omega_j\lambda\xi}d\xi\right]$$
$$-\omega_j r^+(x)\int_0^a p(t)\,dt\int_{-t}^{t} K_j(t,s)\left[e^{\omega_j\lambda\left(s+\mu^+(x)-t\right)} - e^{\omega_j\lambda\left(s-\mu^+(x)+t\right)}\right]ds$$
$$-\omega_j r^-(x)\int_0^a p(t)\,dt\int_{-t}^{t} K_j(t,s)\left[e^{\omega_j\lambda\left(s+\mu^-(x)-t\right)} - e^{\omega_j\lambda\left(s-\mu^-(x)+t\right)}\right]ds$$
$$+\frac{1}{2\alpha}\int_a^x q(t)\,dt\int_{-\mu^+(t)}^{\mu^+(t)} K_j(t,s)ds\int_{s+\mu^+(t)-\mu^+(x)}^{s+\mu^+(x)-\mu^+(t)} e^{\omega_j\lambda\xi}d\xi$$
$$-\omega_j\int_a^x p(t)\,dt\int_{-\mu^+(t)}^{\mu^+(t)} K_j(t,s)\left[e^{\omega_j\lambda\left(s+\mu^+(x)-\mu^+(t)\right)} - e^{\omega_j\lambda\left(s-\mu^+(x)+\mu^+(t)\right)}\right]ds. \tag{20}$$

Now, similar to previous case we obtain from the equation (20) that the function $K_j(x,t)\,(x > a)$, continued as zero for $|t| > \mu^+(x)$, satisfies some integral equations of type (19) in the corresponding regions. Namely we have the following:

(1) if $-\mu^+(x) \le t < -\mu^-(x)$, then

$$K_j(x,t) = \frac{\omega_j r^+(x)}{2}p\left(\frac{t+\mu^+(x)}{2}\right)R_j^+\left(\frac{t+\mu^+(x)}{2}\right)+$$
$$\frac{1}{2}r^+(x)\int_0^{\frac{\mu^+(x)+t}{2}} q(s)\,R_j^+(s)ds + \frac{1}{2}r^+(x)\int_{\frac{\mu^+(x)-t}{2}}^{a} q(s)\,ds\int_{t+s-\mu^+(x)}^{t-s+\mu^+(x)} K_j(s,\xi)\,d\xi-$$
$$\frac{1}{2}r^+(x)\int_0^{\frac{\mu^+(x)+t}{2}} q(s)\,ds\int_{t+s+\mu^+(x)}^{s} K_j(s,\xi)\,d\xi+$$
$$\frac{1}{2}r^-(x)\int_0^{\frac{\mu^-(x)-t}{2\alpha}} q(s)\,ds\int_{-s}^{t-s+\mu^+(x)} K_j(s,\xi)\,d\xi+$$
$$+\frac{1}{2}r^-(x)\int_{\frac{\mu^+(x)-t}{2}}^{a} q(s)\,ds\int_{t+s-\mu^-(x)}^{t-s+\mu^-(x)} K_j(s,\xi)\,d\xi+$$

$$\frac{1}{2\alpha}\int\limits_{a}^{a-\frac{\mu^{-}(x)+t}{2\alpha}} q\left(s\right)ds \int\limits_{-\mu^{-}(s)}^{t-\mu^{+}(x)+\mu^{+}(s)} K_{j}\left(s,\xi\right)d\xi+$$

$$\frac{1}{2\alpha}\int\limits_{a}^{x} q\left(s\right)ds \int\limits_{t-\mu^{+}(x)+\mu^{+}(s)}^{t+\mu^{+}(x)-\mu^{+}(s)} K_{j}\left(s,\xi\right)d\xi+$$

$$\omega_{j}r^{+}(x)\int\limits_{\frac{t+\mu^{+}(x)}{2}}^{a} p\left(s\right)K_{j}(s,t+\mu^{+}\left(x\right)-s)ds-$$

$$\omega_{j}r^{-}(x)\int\limits_{0}^{\frac{\mu^{-}(x)-t}{2}} p\left(s\right)K_{j}(s,t-\mu^{+}\left(x\right)+s)ds-$$

$$\omega_{j}\int\limits_{a}^{\frac{\mu^{-}(x)+t}{2\alpha}-a} p\left(s\right)K_{j}(s,t-\mu^{+}\left(x\right)+\mu^{+}\left(s\right))ds. \tag{21}$$

(2) if $-\mu^{-}\left(x\right)\leq t<\mu^{-}\left(x\right)$, then

$$K_{j}(x,t)=\frac{r^{+}(x)}{2}\int\limits_{0}^{\frac{\mu^{+}(x)+t}{2}} q\left(s\right)R_{j}^{+}(s)ds+\frac{1}{2}r^{-}(x)\int\limits_{0}^{\frac{\mu^{-}(x)+t}{2}} q\left(s\right)R_{j}^{+}(s)ds+$$

$$\frac{1}{2}r^{-}(x)\int\limits_{d}^{\frac{\mu^{-}(x)+t}{2}} q\left(s\right)R_{j}^{+}(s)ds+\frac{\omega_{j}r^{+}(x)}{2}p\left(\frac{t+\mu^{+}\left(x\right)}{2}\right)R_{j}^{+}\left(\frac{t+\mu^{+}\left(x\right)}{2}\right)+$$

$$\frac{\omega_{j}r^{-}(x)}{2}p\left(\frac{t+\mu^{-}\left(x\right)}{2}\right)R_{j}^{+}\left(\frac{t+\mu^{-}\left(x\right)}{2}\right)-$$

$$\omega_{j}r^{+}(x)\int\limits_{0}^{\frac{\mu^{+}(x)-t}{2}} p\left(s\right)K_{j}\left(s,t+s-\mu^{+}(x)\right)ds+$$

$$\omega_{j}r^{+}(x)\int\limits_{\frac{\mu^{+}(x)+t}{2}}^{a} p\left(s\right)K_{j}\left(s,t-s+\mu^{+}(x)\right)ds-$$

$$\omega_{j}r^{-}(x)\int\limits_{0}^{\frac{\mu^{-}(x)-t}{2}} p\left(s\right)K_{j}\left(s,t+s-\mu^{-}(x)\right)ds+$$

$$\omega_{j}r^{-}(x)\int\limits_{\frac{\mu^{-}(x)+t}{2}}^{a} p\left(s\right)K_{j}\left(s,t-s+\mu^{-}(x)\right)ds+$$

$$\frac{1}{2}r^{+}(x)\int\limits_{0}^{\frac{\mu^{-}(x)-t}{2}}q(s)\,ds\int\limits_{-s}^{t+\mu^{+}(x)-s}K_j(s,\xi)\,d\xi+$$

$$\frac{1}{2}r^{+}(x)\int\limits_{0}^{\frac{\mu^{+}(x)+t}{2}}q(s)\,ds\int\limits_{t+\mu^{+}(x)-s}^{t-\mu^{+}(x)+s}K_j(s,\xi)\,d\xi-$$

$$\frac{1}{2}r^{+}(x)\int\limits_{\frac{\mu^{+}(x)-t}{2}}^{a}q(s)\,ds\int\limits_{t-\mu^{+}(x)+s}^{t+\mu^{+}(x)-s}K_j(s,\xi)\,d\xi+$$

$$\frac{1}{2}r^{+}(x)\int\limits_{\frac{\mu^{+}(x)+t}{2}}^{a}q(s)\,ds\int\limits_{t-\mu^{+}(x)+s}^{t+\mu^{+}(x)-s}K_j(s,\xi)\,d\xi+$$

$$\frac{1}{2}r^{+}(x)\int\limits_{0}^{\frac{\mu^{+}(x)+t}{2}}q(s)\,ds\int\limits_{t-\mu^{+}(x)+s}^{a}K_j(s,\xi)\,d\xi+$$

$$\frac{r^{-}(x)}{2}\int\limits_{0}^{\frac{\mu^{-}(x)-t}{2}}q(s)\,ds\int\limits_{-s}^{t+\mu^{-}(x)-s}K_j(s,\xi)\,d\xi+$$

$$\frac{r^{-}(x)}{2}\int\limits_{0}^{\frac{\mu^{-}(x)+t}{2}}q(s)\,ds\int\limits_{t-\mu^{-}(x)+s}^{t+\mu^{-}(x)-s}K_j(s,\xi)\,d\xi+$$

$$\frac{r^{-}(x)}{2}\int\limits_{0}^{\frac{\mu^{-}(x)-t}{2}}q(s)\,ds\int\limits_{t-\mu^{-}(x)+s}^{t+\mu^{-}(x)-s}K_j(s,\xi)\,d\xi+$$

$$\frac{r^{-}(x)}{2}\int\limits_{\frac{\mu^{-}(x)-t}{2}}^{a}q(s)\,ds\int\limits_{t-\mu^{-}(x)+s}^{t+\mu^{-}(x)-s}K_j(s,\xi)\,d\xi-$$

$$\frac{r^{-}(x)}{2}\int\limits_{\frac{\mu^{-}(x)+t}{2}}^{a}q(s)\,ds\int\limits_{t-\mu^{-}(x)+s}^{t+\mu^{-}(x)-s}K_j(s,\xi)\,d\xi+$$

$$\frac{r^{-}(x)}{2}\int\limits_{0}^{\frac{\mu^{-}(x)+t}{2}}q(s)\,ds\int\limits_{t-\mu^{-}(x)+s}^{s}K_j(s,\xi)\,d\xi+$$

$$\int_a^x p(s) K_j\left(s, t+\mu^+(x)-\mu^+(s)\right) ds + \frac{1}{2\alpha}\int_a^x q(s)\, ds \int_{t-\mu^+(x)+\mu^-(x)}^{t+\mu^+(x)-\mu^-(x)} K_j(s,\xi)\, d\xi. \tag{22}$$

(3) if $\mu^-(x) \le t < \mu^+(x)$, then

$$K_j(x,t) = \frac{1}{2} r^+(x) \int_0^{\frac{\mu^+(x)+t}{2}} q(s) R_j^+(s) ds +$$

$$\frac{\omega_j r^+(x)}{2} p\left(\frac{t+\mu^+(x)}{2}\right) R_j^+\left(\frac{t+\mu^+(x)}{2}\right) - \frac{1}{2} r^-(x) \times$$

$$\int_0^{\frac{\mu^+(x)+t}{2}} q(s) R_j^+(s) ds + \frac{\omega_j r^-(x)}{2\alpha} p\left(\frac{t-\mu^-(x)}{2\alpha}+a\right) R_j^+\left(\frac{t-\mu^-(x)}{2\alpha}+a\right) +$$

$$\frac{1}{2} r^+(x) \int_a^{\frac{\mu^+(x)+t}{2}} q(s) R_j^+(s) ds + \frac{\omega_j r^-(x)}{2} p\left(\frac{t+\mu^-(x)}{2}\right) R_j^+\left(\frac{t+\mu^-(x)}{2}\right) -$$

$$\frac{\omega_j r^-(x)}{2\alpha} p\left(a - \frac{t+\mu^-(x)}{2\alpha}\right) R_j^+\left(a - \frac{t+\mu^-(x)}{2\alpha}\right) + \frac{r^+(x)}{2\alpha} \times$$

$$\int_{\frac{t-\mu^-(x)}{2}+a\alpha}^{x} q(s) R_j^+(s) ds + \frac{r^-(x)}{2\alpha} \int_a^{a\alpha - \frac{t+\mu^-(x)}{2}} q(s) R_j^-(s) ds +$$

$$\frac{1}{2} r^+(x) \int_0^{\frac{\mu^+(x)-t}{2}} q(s)\, ds \int_{-s}^{t+\mu^+(x)-s} K_j(s,\xi)\, d\xi +$$

$$\frac{1}{2} r^+(x) \int_0^{\frac{\mu^+(x)-t}{2}} q(s)\, ds \int_{t-\mu^+(x)+s}^{t+\mu^+(x)-s} K_j(s,\xi)\, d\xi -$$

$$\frac{1}{2} r^-(x) \int_{\frac{\mu^-(x)+t}{2}}^{a} q(s)\, ds \int_{t-\mu^-(x)+s}^{t+\mu^-(x)-s} K_j(s,\xi)\, d\xi$$

$$\frac{r^{-}(x)}{2}\int\limits_{0}^{\frac{\mu^{+}(x)+t}{2}} q(s)\,ds \int\limits_{t-\mu^{-}(x)+s}^{s} K_j(s,\xi)\,d\xi+$$
$$\frac{1}{2\alpha}\int\limits_{a}^{x} q(s)\,ds \int\limits_{t-\mu^{+}(x)-\mu^{+}(s)}^{t+\mu^{+}(x)-\mu^{+}(s)} K_j(s,\xi)\,d\xi+$$
$$\frac{1}{2\alpha}\int\limits_{a}^{\frac{t-\mu^{-}(x)}{2}+a\alpha} q(s)\,ds \int\limits_{t+\mu^{+}(x)-\mu^{+}(s)}^{\mu^{+}(s)} K_j(s,\xi)\,d\xi- \tag{23}$$
$$\omega_j r^{+}(x)\int\limits_{0}^{\frac{\mu^{+}(x)-t}{2}} p(s)\,K_j(s,t-\mu^{+}(x)+s)ds+$$
$$\omega_j r^{-}(x)\int\limits_{\frac{\mu^{+}(x)+t}{2}}^{a} p(s)\,K_j(s,t+\mu^{-}(x)-s)ds-$$
$$\omega_j \int\limits_{\frac{t-\mu^{-}(x)}{2}+a}^{x} p(s)\,K_j(s,t+\mu^{+}(x)-\mu^{-}(s))ds.$$

Now we use the method of the successive approximation to show that for every fixed $x\in[0,\pi]$ the integral equation $(19),(21)-(23)$ has a unique solution $K_j(x,t)$ belonging to $L_1\left(-\mu^{+}(x),\mu^{+}(x)\right)$. For this reason let us define

$$K_j^{(0)}(x,t)=\frac{1}{2}\int\limits_{0}^{\frac{x+t}{2}} q(s)\,R_j^{+}(s)ds+\frac{\omega_j}{2}p\left(\frac{x+t}{2}\right)R_j^{+}\left(\frac{x+t}{2}\right), \tag{24}$$

$$K_j^{(n)}(x,t)=$$
$$\frac{1}{2}\int\limits_{0}^{x} q(s)\,ds \int\limits_{-s}^{\min(s,t+x-s)} K_j^{(n-1)}(s,\xi)\,d\xi-\frac{1}{2}\int\limits_{\frac{x-t}{2}}^{x} q(s)\,ds\int\limits_{-s}^{t-x+s} K_j^{(n-1)}(s,\xi)\,d\xi-$$
$$\omega_j\left[\int\limits_{\frac{x-t}{2}}^{x} p(s)\,K_j^{(n-1)}(s,t-x+s)\,ds-\int\limits_{\frac{x+t}{2}}^{x} p(s)\,K_j^{(n-1)}(s,t+x-s)\,ds\right],$$

$|t|\le x\le a, n=1,2,...,$

$$K_j^{(0)}(x,t)=$$
$$\frac{1}{2}r^{+}(x)\int\limits_{0}^{\frac{\mu^{+}(x)+t}{2}} q(s)\,R_j^{+}(s)ds+\frac{\omega_j r^{+}(x)}{2}p\left(\frac{t+\mu^{+}(x)}{2}\right)R_j^{+}\left(\frac{t+\mu^{+}(x)}{2}\right),$$

$$K_j^{(n)}(x,t)=\frac{1}{2}r^{+}(x)\int\limits_{0}^{a} q(s)\,ds\int\limits_{-s}^{\min\left(s,t+\mu^{+}(x)-s\right)} K_j^{(n-1)}(s,\xi)\,d\xi$$

$$-\frac{1}{2}r^{-}(x)\int\limits_{\frac{\mu^{-}(x)-t}{2}}^{a} q(s)\,ds\int\limits_{-s}^{t-\mu^{-}(x)+s} K_j^{(n-1)}(s,\xi)\,d\xi+$$

$$\frac{1}{2\alpha}\int\limits_{a}^{x} q(s)\,ds\int\limits_{-\mu^{+}(s)}^{t+\mu^{+}(x)-\mu^{+}(s)} K_j^{(n-1)}(s,\xi)\,d\xi-$$

$$\frac{1}{2\alpha}\int\limits_{a-\frac{\mu^{-}(x)+t}{2\alpha}}^{x} q(s)\,ds\int\limits_{-\mu^{+}(s)}^{t-\mu^{+}(x)+\mu^{+}(s)} K_j^{(n-1)}(s,\xi)\,d\xi+$$

$$\omega_j r^{+}(x)\int\limits_{\frac{t+\mu^{+}(x)}{2}}^{a} p(s)\,K_j^{(n-1)}(s,t+\mu^{+}(x)-s)ds-$$

$$\omega_j r^{-}(x)\int\limits_{\frac{\mu^{-}(x)-t}{2}}^{a} p(s)\,K_j^{(n-1)}(s,t-\mu^{-}(x)+s)ds-$$

$$\frac{\omega_j}{\alpha}\int\limits_{a-\frac{\mu^{-}(x)+t}{2\alpha}}^{x} p(s)\,K_j^{(n-1)}(s,t-\mu^{+}(x)+\mu^{+}(s))ds+$$

$$\frac{\omega_j}{\alpha}\int\limits_{a}^{x} p(s)\,K_j^{(n-1)}(s,t+\mu^{+}(x)-\mu^{+}(s))ds,\ x>a,$$

$$-\mu^{+}(x)\le \mathbf{t}<-\mu^{-}(x),\, n=1,2,... \tag{25}$$

$$K_j^{(0)}(x,t)=\frac{1}{2}r^{+}(x)\int\limits_{0}^{\frac{\mu^{+}(x)+t}{2}} q(s)\,R_j^{+}(s)ds+\frac{1}{2}r^{-}(x)\int\limits_{0}^{\frac{\mu^{-}(x)+t}{2}} q(s)\,R_j^{+}(s)ds+$$
$$\frac{\omega_j r^{+}(x)}{2}p\left(\frac{t+\mu^{+}(x)}{2}\right)R_j^{+}\left(\frac{t+\mu^{+}(x)}{2}\right)+$$
$$\frac{\omega_j r^{-}(x)}{2}p\left(\frac{t+\mu^{-}(x)}{2}\right)R_j^{+}\left(\frac{t+\mu^{-}(x)}{2}\right), \tag{26}$$

$$K_j^{(n)}(x,t) = \frac{1}{2}r^{+}(x)\int\limits_{0}^{a} q(s)\,ds\int\limits_{-s}^{\min\left(s,t+\mu^{+}(x)-s\right)} K_j^{(n-1)}(s,\xi)\,d\xi-$$
$$\frac{1}{2}r^{+}(x)\int\limits_{\frac{\mu^{+}(x)-t}{2}}^{a} q(s)\,ds\int\limits_{-s}^{t-\mu^{+}(x)+s} K_j^{(n-1)}(s,\xi)\,d\xi-$$

$$\frac{1}{2}r^{-}(x)\int\limits_{\frac{\mu^{-}(x)-t}{2}}^{a}q\left(s\right)ds\int\limits_{-s}^{t-\mu^{-}(x)+s}K_{j}^{(n-1)}\left(s,\xi\right)d\xi+$$

$$\frac{1}{2}r^{-}(x)\int\limits_{0}^{a}q\left(s\right)ds\int\limits_{-s}^{\min(s,t+\mu^{-}(x)-s)}K_{j}^{(n-1)}\left(s,\xi\right)d\xi+$$

$$\frac{1}{2\alpha}\int\limits_{a}^{x}q\left(s\right)ds\int\limits_{t-\mu^{+}(x)+\mu^{+}(s)}^{t+\mu^{+}(x)-\mu^{+}(s)}K_{j}^{(n-1)}\left(s,\xi\right)d\xi+$$

$$\omega_{j}r^{+}(x)\int\limits_{\frac{\mu^{+}(x)+t}{2}}^{a}p\left(s\right)K_{j}^{(n-1)}(s,t+\mu^{+}\left(x\right)-s)ds$$

$$-\omega_{j}r^{+}(x)\int\limits_{\frac{\mu^{+}(x)-t}{2}}^{a}p\left(s\right)K_{j}^{(n-1)}(s,t-\mu^{+}\left(x\right)+s)ds+$$

$$\omega_{j}r^{-}(x)\int\limits_{\frac{\mu^{-}(x)+t}{2}}^{a}p\left(s\right)K_{j}^{(n-1)}(s,t+\mu^{-}\left(x\right)-s)ds-$$

$$\omega_{j}r^{-}(x)\int\limits_{\frac{\mu^{-}(x)-t}{2}}^{a}p\left(s\right)K_{j}^{(n-1)}(s,t-\mu^{-}\left(x\right)+s)ds-$$

$$\frac{\omega_{j}}{\alpha}\int\limits_{a}^{x}p\left(s\right)K_{j}^{(n-1)}(s,t-\mu^{+}\left(x\right)+\mu^{+}\left(s\right))ds$$

$$+\frac{\omega_{j}}{\alpha}\int\limits_{a}^{x}p\left(s\right)K_{j}^{(n-1)}(s,t+\mu^{+}\left(x\right)-\mu^{+}\left(s\right))ds,$$

$$x>a,-\mu^{-}\left(x\right)\leq\mathbf{t}<\mu^{-}\left(x\right),n=1,2,...$$

$$K_{j}^{(0)}(x,t)=\frac{1}{2}r^{+}(x)\int\limits_{0}^{a}q\left(s\right)R_{j}^{+}(s)ds-$$

$$\frac{1}{2}r^{-}(x)\int\limits_{\frac{\mu^{-}(x)+t}{2}}^{a}q\left(s\right)R_{j}^{+}(s)ds+\frac{1}{2\alpha}\int\limits_{a}^{a+\frac{t-\mu^{-}(x)}{2\alpha}}q\left(s\right)R_{j}^{+}(s)ds+\frac{1}{2\alpha}\times$$

$$\int_{a}^{\frac{\mu^+(x)-t}{2\alpha}+a} q(s)\,R_j^-(s)ds+\frac{\omega_j}{2\alpha^2}p\left(\frac{t-\mu^-(x)}{2\alpha}+a\right)R_j^+\left(\frac{t-\mu^-(x)}{2\alpha}+a\right)-$$

$$\frac{\omega_j}{2\alpha^2}p\left(\frac{\mu^+(x)-t}{2\alpha}+a\right)R_j^-\left(\frac{\mu^+(x)-t}{2\alpha}+a\right)+$$

$$\frac{\omega_j r^-(x)}{2}p\left(\frac{t+\mu^-(x)}{2}\right)R_j^+\left(\frac{t+\mu^-(x)}{2}\right),$$

$$\begin{aligned}
K_j^{(n)}(x,t) \;=\;& \frac{1}{2}r^+(x)\int_0^a q(s)\,ds\int_{-s}^{s} K_j^{(n-1)}(s,\xi)\,d\xi- \\
& \frac{1}{2}r^+(x)\int_{\frac{\mu^+(x)-t}{2}}^{a} q(s)\,ds\int_{-s}^{t-\mu^+(x)+s} K_j^{(n-1)}(s,\xi)\,d\xi-
\end{aligned}$$

$$\frac{1}{2}r^-(x)\int_0^a q(s)\,ds\int_{-s}^{s} K_j^{(n-1)}(s,\xi)\,d\xi+$$

$$\frac{1}{2}r^-(x)\int_0^a q(s)\,ds\int_{-s}^{\min(s,t+\mu^-(x)-s)} K_j^{(n-1)}(s,\xi)\,d\xi+$$

$$\frac{1}{2\alpha}\int_a^x q(s)\,ds\int_{-\mu^+(s)}^{\min(\mu^+(s),t+\mu^+(x)-\mu^+(s))} K_j^{(n-1)}(s,\xi)\,d\xi-$$

$$\frac{1}{2\alpha}\int_a^x q(s)\,ds\int_{-\mu^+(s)}^{t-\mu^+(x)+\mu^+(s)} K_j^{(n-1)}(s,\xi)\,d\xi-$$

$$\omega_j r^+(x)\int_{\frac{\mu^+(x)-t}{2}}^{a} p(s)\,K_j^{(n-1)}(s,t-\mu^+(x)+s)ds+$$

$$\omega_j r^-(x)\int_{\frac{\mu^-(x)+t}{2}}^{a} p(s)\,K_j^{(n-1)}(s,t+\mu^-(x)-s)ds-$$

$$\frac{\omega_j}{\alpha}\int_a^x p(s)\,K_j^{(n-1)}(s,t-\mu^+(x)+\mu^+(s))ds+$$

$$\frac{\omega_j}{\alpha}\int\limits_{a+\frac{t-\mu^-(x)}{2\alpha}}^{x} p\,(s)\,K_j^{(n-1)}(s,t+\mu^+\,(x)-\mu^+\,(s))ds,$$

$$x>a,\ \mu^-\,(x)\le t<\mu^+\,(x)\,,\ n=1,2,\dots. \tag{27}$$

We have

$$\int\limits_{-x}^{x}\left|K_j^{(0)}(x,t)\right|dt\le\int\limits_{0}^{x}\left[(x-s)\left|q(s)\right|+\left|p(s)\right|\right],0\le x\le a,$$

$$\int\limits_{-\mu^+(x)}^{\mu^+(x)}\left|K_j^{(0)}(x,t)\right|dt\le 2\left(r^+(x)+\left|r^-(x)\right|\right)\times$$

$$\int\limits_{0}^{x}\left[\left(\mu^+\,(x)-\mu^+\,(s)\right)\left|q(s)\right|+\left|p(s)\right|\right],\ a<x\le\pi,$$

that is

$$\int\limits_{-\mu^+(x)}^{\mu^+(x)}\left|K_j^{(0)}(x,t)\right|dt\le\sigma\,(x)\,, \tag{28}$$

$$\sigma\,(x)=2C_0\int\limits_{0}^{x}\left[\left(\mu^+\,(x)-\mu^+\,(s)\right)\left|q(s)\right|+\left|p(s)\right|\right], \tag{29}$$

where $C_0=\max(1,(r^+(x)+|r^-(x)|)$. Further we obtain that

$$\int\limits_{-x}^{x}\left|K_j^{(n)}(x,t)\right|dt\le 2\int\limits_{0}^{x}\left[(x-s)\left|q(s)\right|+\left|p(s)\right|\right]ds\int\limits_{-s}^{s}\left|K_j^{(n-1)}(s,\xi)\right|d\xi,\ 0\le x\le a,$$

$$\int\limits_{-\mu^+(x)}^{\mu^+(x)}\left|K_j^{(n)}(x,t)\right|dt\le$$

$$2C_0\int\limits_{0}^{x}\left[\left(\mu^+\,(x)-\mu^+\,(s)\right)\left|q(s)\right|+\left|p(s)\right|\right]ds\int\limits_{-\mu^+(s)}^{\mu^+(s)}\left|K_j^{(n-1)}(s,\xi)\right|d\xi,\ a<x\le\pi,$$

that is

$$\int\limits_{-\mu^+(x)}^{\mu^+(x)}\left|K_j^{(n)}(x,t)\right|dt\le$$

$$2C_0\int\limits_{0}^{x}\left[\left(\mu^+\,(x)-\mu^+\,(s)\right)\left|q(s)\right|+\left|p(s)\right|\right]ds\int\limits_{-\mu^+(s)}^{\mu^+(s)}\left|K_j^{(n-1)}(s,\xi)\right|d\xi, \tag{30}$$

for all $x \in [0, \pi]$. Therefore, we have

$$\int_{-\mu^+(x)}^{\mu^+(x)} \left| K_j^{(n)}(x,t) \right| dt \leq \frac{[\sigma(x)]^{n+1}}{(n+1)!} \tag{31}$$

for all $x \in [0,\pi]$ and $n = 0, 1, 2, ...$ Hence the series

$$\sum_{n=0}^{\infty} K_j^{(n)}(x,.) \tag{32}$$

absolutely and uniformly converges in the space $L_1(-\mu^+(x), \mu^+(x))$ for each $x \in [0,\pi]$, the sum $K_j(x,.)$ of this series is a unique solution of the integral equation $(19), (21) - (23)$ and the solution $K_j(x,.)$ satisfies the inequality

$$\int_{-\mu^+(x)}^{\mu^+(x)} |K_j(x,t)|\, dt \leq e^{\sigma(x)} - 1. \tag{33}$$

Therefore we have proved the following theorem:

Theorem 2.1. *For every λ the solution $y_j(x,\lambda)$ of Equation* (1) *satisfying the jump conditions* (2) *and initial conditions* (3) *can be represented as*

$$y_j(x,\lambda) = R_j^+(x) e^{\omega_j \lambda \mu^+(x)} + R_j^-(x) e^{\omega_j \lambda \mu^-(x)} + \int_{-\mu^+(x)}^{\mu^+(x)} K_j(x,t) e^{\omega_j \lambda t} dt, \tag{34}$$

where

$$R_j^{\pm}(x) = r^{\pm}(x) e^{\pm\omega_j \int_0^x sgn(t\pm a) \frac{p(t)}{\sqrt{\rho(t)}} dt}$$

and the kernel $K_j(x,t)$ satisfies (33).

3. Properties of the kernels

From the integral equation $(19), (21) - (23)$ we easily compute the following boundary relations for $K_j(x,.)$:

i) if $0 \leq x \leq a$, then from Eq.(19) we have

$$K_j(x,-x) = \frac{\omega_j}{2} p(0) + \omega_j \int_0^x p(s) K_j(s,-s) ds$$

which implies

$$K_j(x,-x) = \frac{\omega_j}{2} p(0) e^{\omega_j \int_0^x p(s)ds}. \tag{35}$$

Similarly, we find from Eq.(19) that

$$K_j(x,x) = \frac{\omega_j}{2} p(x) R_j^+(x) + \frac{1}{2} \int_0^x p(s) R_j^+(s) ds - \omega_j \int_0^x p(s) K_j(s,s) ds,$$

that is

$$K_j(x,x)=\frac{1}{2}R_j^{+}(x)\left(\omega_j p(x)+\int\limits_0^x\left[q\left(s\right)+p^2(s)\right]ds\right). \tag{36}$$

ii) Let $x>a$. Then from integral equations $(21)-(23)$ we obtain the equation

$$K_j(x,-\mu^{+}(x))=\frac{\omega_j r^{+}(x)}{2}p(0)+\omega_j\int\limits_0^x p(s)K_j(s,-\mu^{+}(s))ds.$$

Now using (i) we easily find that

$$K_j(x,-\mu^{+}(x))=\frac{\omega_j r^{+}(x)}{2}p(0)e^{\omega_j\int\limits_0^x\frac{p(t)}{\sqrt{\rho(t)}}dt}. \tag{37}$$

Hence, combining the formulas (35) and (37) we obtain

$$K_j(x,-\mu^{+}(x))=\frac{\omega_j p(0)}{2}r^{+}(x)e^{\omega_j\int\limits_0^x\frac{p(t)}{\sqrt{\rho(t)}}dt}. \tag{38}$$

From the integral equations $(21)-(23)$ it is clear that the function $K_j(x,.)$ $(x>a)$ has a jump discontinuity at points $\pm\mu^{-}(x)$. Computing the jumps $K_j(x,\pm\mu^{-}(x)-0)-K_j(x,\pm\mu^{-}(x)+0)$ we have

$$K_j(x,-\mu^{-}(x)-0)-K_j(x,-\mu^{-}(x)+0)=\frac{\omega_j r^{-}(x)p(0)}{2}e^{\omega_j\int\limits_0^a p(t)dt} \tag{39}$$

and

$$K_j(x,\mu^{-}(x)-0)-K_j(x,\mu^{-}(x)+0)=$$

$$=R_j^{-}(x)\left\{\frac{1}{2}\int\limits_0^a\left[q\left(s\right)+p^2(s)\right]ds-\frac{1}{2\alpha}\int\limits_a^x\left[q\left(s\right)+\frac{p^2(s)}{\alpha^2}\right]ds+\right.$$

$$\left.+\frac{\omega_j}{2\alpha^2}p(x)+2\omega_j\left(r^{+}(x)\right)^2p(a)\right\}. \tag{40}$$

Finally, from (23) we find that

$$K_j(x,\mu^{+}(x))=$$

$$=R_j^{+}(x)\left\{\frac{1}{2}\int\limits_0^a\left[q\left(s\right)+p^2(s)\right]ds+\frac{1}{2\alpha}\int\limits_a^x\left(q\left(s\right)+\frac{p^2(s)}{\alpha^2}\right)ds+\right.$$

$$\left.+\frac{\omega_j}{2\alpha^2}p(x)+2\omega_j\left(r^{-}(x)\right)^2p(a)\right\}. \tag{41}$$

Hence, combining the formulas (36) and (41) we obtain

$$K_j(x,\mu^{+}(x))=\frac{1}{2}R_j^{+}(x)\left\{\int\limits_0^x\left(\frac{q\left(s\right)}{\sqrt{\rho\left(s\right)}}+\frac{p^2(s)}{\rho\left(s\right)\sqrt{\rho\left(s\right)}}\right)ds+\right.$$

$$\left.+\frac{\omega_j}{\rho\left(x\right)}p(x)+2\omega_j\left(r^{-}(x)\right)^2p(a)\right\}. \tag{42}$$

Now we investigate the additional properties of the function $K_j(x,.)$. Consider the successive approximation $(24)-(26)$. By the differentiation with respect to the variable t we find

$$D_tK_j^{(0)}(x,t)=\frac{1}{4}q\left(\frac{x+t}{2}\right)R_j^{+}\left(\frac{x+t}{2}\right)+$$
$$\frac{\omega_j}{4}\left(p'\left(\frac{x+t}{2}\right)-\omega_j p^2\left(\frac{x+t}{2}\right)\right)R_j^{+}(\left(\frac{x+t}{2}\right),\ -x<t<x,$$
$$D_tK_j^{(n)}(x,t)=\frac{1}{2}\int\limits_{\frac{x+t}{2}}^{x}q\left(s\right)K_j^{(n-1)}\left(s,t+x-s\right)ds-$$
$$\frac{1}{2}\int\limits_{\frac{x-t}{2}}^{x}q\left(s\right)K_j^{(n-1)}\left(s,t-x+s\right)ds-\frac{\omega_j}{2}p\left(\frac{x-t}{2}\right)K_j^{(n-1)}\left(\frac{x-t}{2},-\frac{x-t}{2}\right)-$$
$$\frac{\omega_j}{2}p\left(\frac{x+t}{2}\right)K_j^{(n-1)}\left(\frac{x+t}{2},\frac{x+t}{2}\right)-\omega_j\int\limits_{\frac{x-t}{2}}^{x}q\left(s\right)\,D_\xi K_j^{(n-1)}\left(s,\xi\right)\Big|_{t-x+s}ds+$$
$$\omega_j\int\limits_{\frac{x+t}{2}}^{x}q\left(s\right)\,D_\xi K_j^{(n-1)}\left(s,\xi\right)\Big|_{t+x-s}ds,\ -x<t<x,$$
$$D_tK_j^{(0)}(x,t)=\frac{1}{4}r^{+}(x)q\left(\frac{\mu^{+}\left(x\right)+t}{2}\right)R_j^{+}\left(\frac{\mu^{+}\left(x\right)+t}{2}\right)+$$
$$\frac{\omega_j r^{+}(x)}{4}p'\left(\frac{\mu^{+}\left(x\right)+t}{2}\right)R_j^{+}\left(\frac{t+\mu^{+}\left(x\right)}{2}\right)+$$
$$\frac{1}{4}r^{+}(x)p^2\left(\frac{\mu^{+}\left(x\right)+t}{2}\right)R_j^{+}\left(\frac{t+\mu^{+}\left(x\right)}{2}\right),$$
$$D_tK_j^{(n)}(x,t)=\frac{1}{2}r^{+}(x)\int\limits_{\frac{\mu^{+}(x)+t}{2}}^{a}q\left(s\right)K_j^{(n-1)}\left(s,t+\mu^{+}\left(x\right)-s\right)ds-$$
$$\frac{1}{2}r^{-}(x)\int\limits_{\frac{\mu^{-}(x)-t}{2}}^{a}q\left(s\right)K_j^{(n-1)}\left(s,t-\mu^{-}\left(x\right)+s\right)ds+$$
$$\frac{1}{2\alpha}\int\limits_{a}^{x}q\left(s\right)K_j^{(n-1)}\left(s,t+\mu^{+}\left(x\right)-\mu^{+}\left(s\right)\right)ds-$$
$$\frac{1}{2\alpha}\int\limits_{a-\frac{\mu^{-}(x)+t}{2\alpha}}^{x}q\left(s\right)K_j^{(n-1)}\left(s,t-\mu^{+}\left(x\right)+\mu^{+}\left(s\right)\right)ds-$$
$$\frac{\omega_j r^{+}(x)}{2}p\left(\frac{t+\mu^{+}\left(x\right)}{2}\right)K_j^{(n-1)}\left(\frac{t+\mu^{+}\left(x\right)}{2},\frac{t+\mu^{+}\left(x\right)}{2}\right)+$$
$$\omega_j r^{+}(x)\int\limits_{\frac{t+\mu^{+}(x)}{2}}^{a}p\left(s\right)\,D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t+\mu^{+}(x)-s}ds-$$

$$\omega_j r^-(x) \int\limits_{\frac{\mu^-(x)-t}{2}}^{a} p(s)\ D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t-\mu^-(x)+s} ds-$$

$$\frac{\omega_j r^-(x)}{2} p\left(\frac{\mu^-(x)-t}{2}\right) K_j^{(n-1)}\left(\frac{\mu^-(x)-t}{2}, -\frac{\mu^-(x)-t}{2}\right)-$$

$$\frac{\omega_j}{\alpha} \int\limits_{a-\frac{\mu^-(x)+t}{2\alpha}}^{x} p(s)\ D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t-\mu^+(x)+\mu^+(s)} ds+$$

$$\frac{\omega_j}{\alpha} \int\limits_{a}^{x} p(s)\ D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t+\mu^+(x)-\mu^+(s)} ds,$$

$x > a, -\mu^+(x) \le t < -\mu^-(x), n = 1, 2, ...,$

$$D_t K_j^{(0)}(x,t) = \frac{1}{4} r^+(x) q\left(\frac{\mu^+(x)+t}{2}\right) R_j^+\left(\frac{\mu^+(x)+t}{2}\right)+$$

$$\frac{1}{4} r^-(x) q\left(\frac{\mu^-(x)+t}{2}\right) R_j^+\left(\frac{\mu^-(x)+t}{2}\right)+$$

$$\frac{\omega_j r^+(x)}{2}\left[p'\left(\frac{t+\mu^+(x)}{2}\right)-\omega_j p^2\left(\frac{t+\mu^+(x)}{2}\right)\right] R_j^+\left(\frac{t+\mu^+(x)}{2}\right)+$$

$$\frac{\omega_j r^-(x)}{2}\left[p'\left(\frac{t+\mu^-(x)}{2}\right)-\omega_j p^2\left(\frac{t+\mu^-(x)}{2}\right)\right] R_j^+\left(\frac{t+\mu^-(x)}{2}\right),$$

$$D_t K_j^{(n)}(x,t) = \frac{1}{2} r^+(x) \int\limits_{\frac{\mu^+(x)+t}{2}}^{a} q(s)\, K_j^{(n-1)}\left(s, t+\mu^+(x)-s\right) ds-$$

$$\frac{1}{2} r^+(x) \int\limits_{\frac{\mu^+(x)-t}{2}}^{a} q(s)\, K_j^{(n-1)}\left(s, t-\mu^+(x)+s\right) ds-$$

$$\frac{1}{2} r^-(x) \int\limits_{\frac{\mu^-(x)-t}{2}}^{a} q(s)\, K_j^{(n-1)}\left(s, t-\mu^-(x)+s\right) ds+$$

$$\frac{1}{2} r^-(x) \int\limits_{\frac{\mu^-(x)+t}{2}}^{a} q(s)\, K_j^{(n-1)}\left(s, t+\mu^-(x)-s\right) ds+\frac{1}{2\alpha}\times$$

$$\int\limits_{a}^{x} q(s)\left[K_j^{(n-1)}\left(s, t+\mu^+(x)-\mu^+(s)\right) - K_j^{(n-1)}\left(s, t-\mu^+(x)+\mu^+(s)\right)\right] ds-$$

$$\frac{\omega_j r^+(x)}{2} p\left(\frac{t+\mu^+(x)}{2}\right) K_j^{(n-1)}\left(\frac{t+\mu^+(x)}{2}, \frac{t+\mu^+(x)}{2}\right)+$$

$$\omega_j r^+(x) \int_{\frac{t+\mu^+(x)}{2}}^{a} p(s)\ D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t+\mu^+(x)-s} ds -$$

$$\omega_j r^+(x) \int_{\frac{\mu^+(x)-t}{2}}^{a} p(s)\ D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t-\mu^+(x)+s} ds -$$

$$\omega_j r^-(x) \int_{\frac{\mu^-(x)-t}{2}}^{a} p(s)\ D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t-\mu^-(x)+s} +$$

$$\omega_j r^-(x) \int_{\frac{\mu^-(x)+t}{2}}^{a} p(s)\ D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t+\mu^-(x)-s} -$$

$$\frac{\omega_j r^+(x)}{2} p\left(\frac{\mu^+(x)-t}{2}\right) K_j^{(n-1)}\left(\frac{\mu^+(x)-t}{2}, -\frac{\mu^+(x)-t}{2}\right) -$$

$$\frac{\omega_j r^-(x)}{2} p\left(\frac{\mu^-(x)-t}{2}\right) K_j^{(n-1)}\left(\frac{\mu^-(x)-t}{2}, -\frac{\mu^-(x)-t}{2}\right) -$$

$$\frac{\omega_j}{\alpha} \int_a^x p(s)\ D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t-\mu^+(x)+\mu^+(s)} ds +$$

$$\frac{\omega_j}{\alpha} \int_a^x p(s)\ D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t+\mu^+(x)-\mu^+(s)} ds,$$

$$x > a, -\mu^-(x) \le \mathbf{t} < \mu^-(x), n = 1, 2, ...,$$

$$D_t K_j^{(0)}(x,t) = \frac{1}{4} r^-(x) q\left(\frac{\mu^-(x)+t}{2}\right) R_j^+\left(\frac{\mu^-(x)+t}{2}\right) +$$

$$\frac{1}{4\alpha^2} q\left(a + \frac{t-\mu^-(x)}{2\alpha}\right) R_j^+\left(a + \frac{t-\mu^-(x)}{2\alpha}\right) -$$

$$\frac{1}{4\alpha^2} q\left(\frac{\mu^+(x)-t}{2\alpha} + a\right) R_j^-\left(\frac{\mu^+(x)-t}{2\alpha} + a\right) +$$

$$\frac{\omega_j}{4\alpha^3}\left[p'\left(\frac{t-\mu^-(x)}{2\alpha} + a\right) - \frac{\omega_j}{\alpha} p^2\left(\frac{t-\mu^-(x)}{2\alpha} + a\right)\right] R_j^+\left(\frac{t-\mu^-(x)}{2\alpha} + a\right) +$$

$$\frac{\omega_j}{4\alpha^3}\left[p'\left(\frac{\mu^+(x)-t}{2\alpha} + a\right) + \frac{\omega_j}{\alpha} p^2\left(\frac{\mu^+(x)-t}{2\alpha} + a\right)\right] R_j^-\left(\frac{\mu^+(x)-t}{2\alpha} + a\right) +$$

$$\frac{\omega_j r^-(x)}{4}\left[p'\left(\frac{t+\mu^-(x)}{2}\right) - \omega_j p^2\left(\frac{t+\mu^-(x)}{2}\right)\right] R_j^+\left(\frac{t+\mu^-(x)}{2}\right),$$

$$D_t K_j^{(n)}(x,t) = -\frac{1}{2} r^+(x) \int_{\frac{\mu^+(x)-t}{2}}^{a} q(s) K_j^{(n-1)}\left(s, t-\mu^+(x)+s\right) ds +$$

$$\frac{1}{2}r^{-}(x)\int\limits_{\frac{\mu^{-}(x)+t}{2}}^{a} q(s)\,K_j^{(n-1)}\left(s,t+\mu^{-}(x)-s\right)ds+$$

$$\frac{1}{2\alpha}\int\limits_{a+\frac{t+\mu^{-}(x)}{2\alpha}}^{x} q(s)\,K_j^{(n-1)}\left(s,t+\mu^{+}(x)-\mu^{+}(s)\right)ds-$$

$$\frac{1}{2\alpha}\int\limits_{a}^{x} q(s)\,K_j^{(n-1)}\left(s,t-\mu^{+}(x)+\mu^{+}(s)\right)ds-$$

$$-\omega_j r^{+}(x)\int\limits_{\frac{\mu^{+}(x)-t}{2}}^{a} p(s)\;D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t-\mu^{+}(x)+s}ds+$$

$$\omega_j r^{-}(x)\int\limits_{\frac{\mu^{-}(x)+t}{2}}^{a} p(s)\;D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t+\mu^{-}(x)-s}-$$

$$-\frac{\omega_j r^{+}(x)}{2}p\left(\frac{\mu^{+}(x)-t}{2}\right)K_j^{(n-1)}\left(\frac{\mu^{+}(x)-t}{2},-\frac{\mu^{+}(x)-t}{2}\right)-$$
$$-\frac{\omega_j r^{-}(x)}{2}p\left(\frac{\mu^{-}(x)+t}{2}\right)K_j^{(n-1)}\left(\frac{\mu^{-}(x)+t}{2},\frac{\mu^{-}(x)+t}{2}\right)-$$

$$\frac{\omega_j}{2\alpha^2}p\left(a+\frac{t+\mu^{-}(x)}{2\alpha}\right)K_j^{(n-1)}\left(a+\frac{t+\mu^{-}(x)}{2\alpha},\frac{\mu^{+}(x)+t}{2}\right)-$$

$$\frac{\omega_j}{\alpha}\int\limits_{a}^{x} p(s)\;D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t-\mu^{+}(x)+\mu^{+}(s)}ds+$$

$$\frac{\omega_j}{\alpha}\int\limits_{a+\frac{t+\mu^{-}(x)}{2\alpha}}^{x} p(s)\;D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t+\mu^{+}(x)-\mu^{+}(s)}ds,$$

$$x\;>\;a,\;\mu^{-}(x)<t<\mu^{+}(x),n=1,2,...$$

Therefore, we have

$$\int\limits_{-x}^{x}\left|D_tK_j^{(0)}(x,t)\right|dt\;\le\;\frac{1}{2}\int\limits_{0}^{x}\left[|q(s)|+\left|p'(s)\right|+p^2(s)\right]ds,$$
$$0\;\le\;x\le a,$$

$$\int\limits_{-\mu^{+}(x)}^{\mu^{+}(x)}\left|D_tK_j^{(0)}(x,t)\right|dt\le\left(r^{+}(x)+\left|r^{-}(x)\right|\right)\int\limits_{0}^{x}\left[|q(s)|+\frac{|p'(s)|}{\sqrt{\rho(s)}}+\frac{p^2(s)}{\rho(s)}\right]ds, x>a.$$

Hence

$$\int_{-\mu^+(x)}^{\mu^+(x)} \left| D_t K_j^{(0)}(x,t) \right| dt \le C_0 \int_0^x \left[|q(s)| + \frac{|p'(s)|}{\sqrt{\rho(s)}} + \frac{p^2(s)}{\rho(s)} \right] ds. \tag{43}$$

Further, because of

$$\int_{-x}^{x} \left| D_t K_j^{(n)}(x,t) \right| dt \le \int_0^x |p(s)| \left| K_j^{(n-1)}(s,-s) \right| ds + \int_0^x |p(s)| \left| K_j^{(n-1)}(s,s) \right| ds +$$

$$\int_0^x |q(s)| \, ds \int_{-s}^{s} \left| K_j^{(n-1)}(s,t) \right| dt + 2 \int_0^x |p(s)| \, ds \int_{-s}^{s} D_t \left| K_j^{(n-1)}(s,t) \right| dt, 0 \le x \le a,$$

$$\int_{-\mu^+(x)}^{\mu^+(x)} \left| D_t K_j^{(n)}(x,t) \right| dt \le \left(r^+(x) + \left| r^-(x) \right| \right) \int_0^x |q(s)| \, ds \int_{-\mu^+(s)}^{\mu^+(s)} \left| K_j^{(n-1)}(s,t) \right| dt +$$

$$\left(r^+(x) + \left| r^-(x) \right| \right) \int_0^x |p(s)| \left| K_j^{(n-1)}(s, -\mu^+(s)) \right| ds +$$

$$\left(r^+(x) + \left| r^-(x) \right| \right) \int_0^x |p(s)| \left| K_j^{(n)}(s, \mu^+(s)) \right| ds +$$

$$2 \left(r^+(x) + \left| r^-(x) \right| \right) \int_0^x |p(s)| \, ds \int_{-\mu^+(s)}^{\mu^+(s)} \left| D_t K_j^{(n)}(s,t) \right| dt, \ \ x > a,$$

we can write for all $x \in [0, \pi]$

$$\int_{-\mu^+(x)}^{\mu^+(x)} \left| D_t K_j^{(n)}(x,t) \right| dt \le C_0 \int_0^x |q(s)| \, ds \int_{-\mu^+(s)}^{\mu^+(s)} \left| K_j^{(n-1)}(s,t) \right| dt +$$

$$C_0 \int_0^x |p(s)| \left| K_j^{(n-1)}(s, -\mu^+(s)) \right| ds +$$

$$C_0 \int_0^x |p(s)| \left| K_j^{(n-1)}(s, \mu^+(s)) \right| ds + 2C_0 \int_0^x |p(s)| \, ds \int_{-\mu^+(s)}^{\mu^+(s)} \left| D_t K_j^{(n-1)}(s,t) \right| dt. \tag{44}$$

Note that

$$\int_{-\mu^+(x)}^{\mu^+(x)} \left| K_j^{(n-1)}(x,t) \right| dt \le \frac{\sigma^n(x)}{n!},$$

$$\int_0^x |p(s)| \left|K_j^{(n-1)}(s, \pm\mu^+(s))\right| ds \le AC^n \left(\frac{1}{n!}\int_0^x |p(s)|\, ds\right)^n,$$

where $C = \max\left(r^+(x), \dfrac{1}{\beta\alpha}\right)$ and $A\left(0 < A \le \frac{1}{2}\int_0^\pi [|q(s)| + |p'(s)|]\, ds + \frac{|p(0)|}{2}\right)$ is a constant. We see that $C \le C_0$and from (44) we immediately have

$$\int_{-\mu^+(x)}^{\mu^+(x)} \left|D_t K_j^{(n)}(x,t)\right| dt \le 4C_0 A\frac{\sigma^n(x)}{n!} + 2C_0\int_0^x |p(s)|\, ds \int_{-\mu^+(s)}^{\mu^+(s)} \left|D_t K_j^{(n-1)}(s,t)\right| dt$$

for all $x \in [0,\pi]$ and $n = 1, 2, \dots.$ Consequently,

$$\int_{-\mu^+(x)}^{\mu^+(x)} \left|D_t K_j^{(n)}(x,t)\right| dt \le 4C_0 A\frac{\sigma^{n+1}(x)}{n!} + \frac{\sigma_1^{n+1}(x)}{(n+1)!}, \tag{45}$$

$$x \in [0,\pi],\ n = 0, 1, \dots,$$

where

$$\sigma_1(x) = 2C_0\int_0^x \left[|q(s)| + |p(s)| + \frac{|p'(s)|}{\sqrt{\rho(s)}} + \frac{p^2(s)}{\rho(s)}\right] ds. \tag{46}$$

This means that the series

$$\sum_{n=0}^{\infty} K_j^{(n)}(x,.)$$

can be differentiated term by term in the space $L_1(-\mu^+(x), \mu^+(x))$ and the sum $K_j(x,.)$ is also differentiable in this space with

$$\int_{-\mu^+(x)}^{\mu^+(x)} |D_t K_j(x,t)|\, dt \le 4C_0 A\sigma(x)\, e^{\sigma(x)} + e^{\sigma_1(x)} - 1. \tag{47}$$

Similarly, from the successive approximation $(24)-(26)$ by differentiation with respect to the variable x we have the series

$$\sum_{n=0}^{\infty} D_x K_j^{(n)}(x,.)$$

converges in the space $L_1(-\mu^+(x), \mu^+(x))$ and $D_x K_j(x,.) \in L_1(-\mu^+(x), \mu^+(x))$.

Further,by differentiation integral equations (19), $(21)-(23)$ we have that

$$D_t K_j(x,t) - D_x K_j(x,t) =$$

$$= -\frac{\omega_j}{2} p\left(\frac{x-t}{2}\right) K_j\left(\frac{x-t}{2}, -\frac{x-t}{2}\right) - \int_{\frac{x-t}{2}}^x q(s)\, K_j(s, t-x+s) ds$$

$$-\omega_j \int_{\frac{x-t}{2}}^x q(s)\, D_\xi K_j(s,\xi)\big|_{t-x+s} ds,\ -x < t < x,\ 0 \le x \le a, \tag{48}$$

$$\alpha D_t K_j(x,t) - D_x K_j(x,t) =$$

$$-\int_{a-\frac{\mu^-(x)+t}{2\alpha}}^{x} q(s)\,K_j\left(s,t-\mu^+(x)+\mu^+(s)\right)ds-$$

$$-2\omega_j\int_{a-\frac{\mu^-(x)+t}{2\alpha}}^{x} p(s)\;D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t-\mu^+(x)+\mu^+(s)}ds-$$

$$-\frac{\omega_j}{\alpha}p\left(a-\frac{\mu^-(x)+t}{2}\right)K_j\left(a-\frac{\mu^-(x)+t}{2},-\frac{\mu^+(x)-t}{2}\right), \tag{49}$$

$$x>a,-\mu^+(x)\le \mathbf{t}<-\mu^-(x),$$

$$\alpha D_tK_j(x,t)-D_xK_j(x,t)=$$

$$\frac{\alpha r^-(x)}{2}q\left(\frac{\mu^-(x)+t}{2}\right)R_j^+\left(\frac{\mu^-(x)+t}{2}\right)$$

$$+\frac{\omega_j\alpha r^-(x)}{2}\left[p'\left(\frac{t+\mu^-(x)}{2}\right)-\omega_jp^2\left(\frac{t+\mu^-(x)}{2}\right)\right]R_j^+\left(\frac{t+\mu^-(x)}{2}\right)$$

$$-\alpha r^+(x)\int_{\frac{\mu^+(x)-t}{2}}^{a} q(s)\,K_j\left(s,t-\mu^+(x)+s\right)ds$$

$$+\alpha r^-(x)\int_{\frac{\mu^-(x)+t}{2}}^{a} q(s)\,K_j\left(s,t+\mu^-(x)-s\right)ds$$

$$-\int_a^x q(s)\,K_j\left(s,t-\mu^+(x)+\mu^+(s)\right)ds$$

$$-2\omega_j\alpha r^+(x)\int_{\frac{\mu^+(x)-t}{2}}^{a} p(s)\;D_\xi K_j(s,\xi)\big|_{t-\mu^+(x)+s}ds$$

$$+2\omega_j\alpha r^-(x)\int_{\frac{\mu^-(x)+t}{2}}^{a} p(s)\;D_\xi K_j(s,\xi)\big|_{t+\mu^-(x)-s}ds$$

$$-2\omega_j\int_a^x p(s)\;D_\xi K_j^{(n-1)}(s,\xi)\Big|_{t-\mu^+(x)+\mu^+(s)}ds$$

$$-\frac{\omega_j\alpha r^+(x)}{2}p\left(\frac{\mu^+(x)-t}{2}\right)K_j\left(\frac{\mu^+(x)-t}{2},-\frac{\mu^+(x)-t}{2}\right)$$

$$-\frac{\omega_j\alpha r^-(x)}{2}p\left(\frac{\mu^-(x)+t}{2}\right)K_j\left(\frac{\mu^-(x)+t}{2},\frac{\mu^-(x)+t}{2}\right), \tag{50}$$

$$x>a,-\mu^-(x)\le \mathbf{t}<\mu^-(x),$$

$$\alpha D_t K_j(x,t) - D_x K_j(x,t) = \frac{\alpha r^-(x)}{4} q\left(\frac{\mu^-(x)+t}{2}\right) R_j^+\left(\frac{\mu^-(x)+t}{2}\right)$$

$$-\frac{1}{2\alpha} q\left(\frac{\mu^+(x)-t}{2\alpha}+a\right) R_j^-\left(\frac{\mu^+(x)-t}{2\alpha}+a\right)$$

$$+\frac{\omega_j}{2\alpha^2}\left[p'\left(\frac{\mu^+(x)-t}{2\alpha}+a\right)+\frac{\omega_j}{\alpha}p^2\left(\frac{\mu^+(x)-t}{2\alpha}+a\right)\right] R_j^-\left(\frac{\mu^+(x)-t}{2\alpha}+a\right)$$

$$+\frac{\omega_j\alpha r^-(x)}{2}\left[p'\left(\frac{t+\mu^-(x)}{2}\right)-\omega_j p^2\left(\frac{t+\mu^-(x)}{2}\right)\right] R_j^+\left(\frac{t+\mu^-(x)}{2}\right)$$

$$-\alpha r^+(x)\int\limits_{\frac{\mu^+(x)-t}{2}}^{a} q(s)\, K_j\left(s, t-\mu^+(x)+s\right) ds$$

$$+\alpha r^-(x)\int\limits_{\frac{\mu^-(x)+t}{2}}^{a} q(s)\, K_j\left(s, t+\mu^-(x)-s\right) ds$$

$$-\int\limits_{a}^{x} q(s) K_j\left(s, t-\mu^+(x)+\mu^+(s)\right) ds$$

$$-2\omega_j\alpha r^+(x)\int\limits_{\frac{\mu^+(x)-t}{2}}^{a} p(s)\, D_\xi K_j(s,\xi)\big|_{t-\mu^+(x)+s}\, ds$$

$$+2\omega_j\alpha r^-(x)\int\limits_{\frac{\mu^-(x)+t}{2}}^{a} p(s)\, D_\xi K_j(s,\xi)\big|_{t+\mu^-(x)-s}$$

$$-2\omega_j\int\limits_{a}^{x} p(s)\, D_\xi K_j(s,\xi)\big|_{t-\mu^+(xds)+\mu^+(s)}$$

$$-\omega_j\alpha r^+(x) p\left(\frac{\mu^+(x)-t}{2}\right) K_j\left(\frac{\mu^+(x)-t}{2}, -\frac{\mu^+(x)-t}{2}\right)$$

$$-\omega_j\alpha r^-(x) p\left(\frac{\mu^-(x)+t}{2}\right) K_j\left(\frac{\mu^-(x)+t}{2}, \frac{\mu^-(x)+t}{2}\right),$$

$$x > a,\ \mu^-(x) < \mathbf{t} < \mu^+(x). \tag{51}$$

These equations with (47) imply that

$$\int\limits_{-\mu^+(x)}^{\mu^+(x)} |D_x K_j(x,t)|\, dt \le 4\sqrt{\rho(x)} C_0 A\sigma(x)\, e^{\sigma(x)} + \sqrt{\rho(x)}\left(e^{\sigma_1(x)}-1\right) + C_1\sigma_1(x), \tag{52}$$

where $C_1 > 0$ is a constant. Differentiating equations (48) – (51) once more we have the following partial differential equation for the kernel $K_j(x,t)$:

$$D_{xx}K_j(x,t) - \rho(x)D_{tt}K_j(x,t) = q(x)K_j(x,t) + 2\omega_j p(x)D_tK_j(x,t). \tag{53}$$

Hence we can formulate the following theorem:

Theorem 3.1. *For all fixed $x \in [0,\pi]$ the kernel of the integral representation* (34) *has the partial derivatives $D_xK_j(x,.)$, $D_tK_j(x,.) \in L_1(-\mu^+(x), \mu^+(x))$ and satisfy the discontinuous partial differential equation* (53) *with the conditions*

$$K_j(x,-\mu^+(x)) = \frac{\omega_j p(0)}{2} r^+(x) e^{\omega_j \int_0^x \frac{p(t)}{\sqrt{\rho(t)}}dt}, \tag{54}$$

$$K_j(x,\mu^+(x)) = \frac{1}{2}R_j^+(x)\left[\int_0^x \left(\frac{q(s)}{\sqrt{\rho(s)}} + \frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)ds + \right.$$

$$\left. + \frac{\omega_j}{\rho(x)}p(x) + 4\omega_j\left(r^-(x)\right)^2 p(a)\right] \tag{55}$$

and the discontinuity conditions

$$K_j(x,-\mu^-(x)-0) - K_j(x,-\mu^-(x)+0) = \frac{\omega_j p(0)}{2} r^-(x) e^{\omega_j \int_0^a p(t)dt} \tag{56}$$

and

$$K_j(x,\mu^-(x)-0) - K_j(x,\mu^-(x)+0) =$$

$$= R_j^-(x)\left\{\frac{1}{2}\int_0^a \left[q(s) + p^2(s)\right]ds - \frac{1}{2\sqrt{\rho(x)}}\int_a^x \left[q(s) + \frac{p^2(s)}{\rho(s)}\right]ds + \right. \tag{57}$$

$$\left. + \frac{\omega_j}{2\rho(x)}p(x) + 2\omega_j\left(r^+(x)\right)^2 p(a)\right\}.$$

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