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ASYMPTOTIC PROPERTIES OF SOLUTIONS OF PARTIAL OPERATOR-DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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In memory of M. G. Gasymov on his 75th birthday

Abstract. Above we investigate the questions of existence λ asymptotic behavior of solutions of operator-differential equation with partial derivatives in case, when the resolvent $R(\lambda)$ of given equation is regular at all values of the parameter λ. However, in applications we see the cases, when the resolvent has the properties.

In the present paper we prove the theorem on existence asymptotic behavior of operator equations in case when the resolvent has the properties at the point $\lambda = 0$.

1. Introduction

Let $H_0 \supset H_1 \supset H_2 \supset ... \supset H_m$ be a family of Hilbert space, where all the imbeddings are compact.

Consider the equation

$$Lu = \sum_{|\alpha| \leq m} A_\alpha D^\alpha u = f(x), \tag{1}$$

where $x = (x_1, x_2, ..., x_n) \in R^n$, $u(x) \in H_m$, $f(x) \in H_0$, $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, $D^\alpha u \in H_{m-|\alpha|}$, $A_\alpha : H_{m-|\alpha|} \to H_0$ are linear bounded operators.

The solution of equation (1) is the function $u(x) \in H_m$ possessing all strong derivatives $D^\alpha u$ for $|\alpha| \leq m$, and $D^\alpha u \in H_{m-|\alpha|}$ are such that equality (1) is everywhere valid. The asymptotic property of the solutions of equation (1) is established in the case when $f(x)$ belongs to the Schwartz space and has a compact support.

By $R(\lambda)$ denote the operator

$$R(\lambda) = \left[\sum_{|\alpha| \leq m} A_\alpha (i\lambda)^\alpha \right]^{-1}$$

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acting from H_0 to H_m. Here $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, $\lambda_1, \lambda_2, ..., \lambda_n$ are complex numbers, and $\lambda^\alpha = \lambda_1^{\alpha_1} - \lambda_2^{\alpha_2} - ... - \lambda_n^{\alpha_n}$.

Earlier we have proved that the boundedness of the operator-function $R(\lambda)$ for all $\lambda \in R^n$ and fulfilment of the inequality

$$\sum_{j=0}^{m} \left(1 + |\lambda|^j\right) \|R(\lambda)\|_{H_0 \to H_{m-j}} \leq C \tag{2}$$

is a necessary and sufficient condition for the existence and uniqueness of the solution of equation (1). Further we consider the case when the coefficients of equation (1) depend on variable x. Using the Banach theorem on inversibility of the sum of the operator inversible and small with respect to the norm it is established that if there exists η such that $\|A_\alpha(x) - A_\alpha\|_{H_{m-|\alpha|} \to H_0} \leq \eta$ and condition (2) is fulfilled, then there exists a unique solution of equation (1) with variable coefficients. Theorems on normal and Fredholm solvability of equation (1) with variable coefficients (see. [2-8]) are proved.

In the proved existence theorems it was assumed that the operator function $R(\lambda)$ is regular for all $\lambda \in R^n$. However, in applications there are cases when the operator-function has peculiarities. For example, in the Neumann problem for second order elliptic equations in unbounded domains, $R(\lambda)$ has a peculiarity for $\lambda = 0$. Taking into account this circumstance, we consider the case when $R(\lambda)$ has a peculiarity for $\lambda = 0$ that is for a second order elliptic equation.

Second order differential equations with operator coefficients were considered in a lot of papers [1-9, 12-15]. The results of these papers have a number of applications in theory of boundary value problems. The main applications relate to problems of behavior of solutions in an infinite cylinder [10], [11]. The papers [11], [14] deal with ordinary differential equations in functional spaces. The partial equations with operator coefficients were considered in the papers [13], [15]. In these papers there are theorems on coercive and unique solvability of the equation in the respective space. Such equations have applications in theory of boundary value problems as well.

2. Main Results

Theorem 1. *Let* $R(\lambda) = \left[\sum_{|\alpha| \leq m} (i\lambda)^\alpha A_\alpha\right]^{-1}$ *be a bounded operator:* $H_0 \to H_m$ *for all* $\lambda \in R^n|_0$. *Suppose that there exists a homogeneous polynomial* $P(\lambda)$ *of degree* k *such that* $P(\lambda) \neq 0$ *for* $\lambda \neq 0$ *and*

$$P(\lambda) R(\lambda) = A(\lambda) + P(\lambda) R_1(\lambda),$$

$R_1(\lambda)$ *is an analytic function in the layer* $|Jm\lambda_j| \leq c$ *such that* $\|R_1(\lambda)\|_{H_0 \to H_m} \leq c(1 + |\lambda|)^s$ *for some* s, $A(\lambda)$ *is a bounded operator-function on all* c^n.

Let $u(x)$ *be the solution of the equation such that*

$$\int_{R^n} \|u(x)\|_{H_m}^2 dx < \infty.$$

Then

$$u(x) = u_1(x) + u_2(x), \tag{3}$$

$$\|u_2(x)\|_{H_m} \leq c_1 e^{-c_2|x|}, \tag{4}$$

$$P\left[i\frac{\partial}{\partial x}\right] u_1 = 0 \ for \ |x| \geq N \ and \ \|u_1(x)\|_{H_m} \leq c. \tag{5}$$

Proof. Having applied in (5) the Fourier transform, we get:

$$\left[\sum_{|\alpha|\leq m} (i\lambda)^\alpha A_\alpha\right] \tilde{u}(\lambda) = \tilde{f}(\lambda).$$

Hence

$$P(\lambda)\left[\sum_{|\alpha|\leq m} (i\lambda)^\alpha A_\alpha\right] \tilde{u}(\lambda) = P(\lambda)\tilde{f}(\lambda).$$

Consequently,

$$P(\lambda)\tilde{u}(\lambda) = R(\lambda)P(\lambda)\tilde{f}(\lambda) = A(\lambda)\tilde{f}(\lambda) + R_1(\lambda)P(\lambda)\tilde{f}(\lambda). \tag{6}$$

Assume

$$\tilde{u}_2(\lambda) = R_1(\lambda)\tilde{f}(\lambda).$$

Then $u_2(x)$ will satisfy (4). From (6) it follows that $P(\lambda)\tilde{u}_1(\lambda) = A(\lambda)\tilde{f}(\lambda)$. Consequently,

$$P\left[i\frac{\partial}{\partial x}\right] u_1(x) = F^{-1}\left[A(\lambda)\tilde{f}(\lambda)\right],$$

where F^{-1} is the inverse Fourier transformation of the entire function. It is known that such a function has a compact support. So, $u_1(x)$ satisfies conditions (5). The theorem is proved.

The operator of the Neumann problem in the infinite layer satisfies the conditions of Theorem 1.

Let $\Pi = \left\{x : \ 0 < x_n < 1, \ (x_1, x_2, ..., x_{n-1}) \in R^{n-1}\right\}$. Consider the equation

$$\sum_{i=1}^{n-1} \frac{\partial^2 u}{\partial x_i^2} + a(x_n)\frac{\partial^2 u}{\partial x_n^2} = f(x), x \in \Pi,$$

with boundary conditions

$$\left.\frac{\partial u}{\partial x_n}\right|_{\partial\Pi} = 0.$$

Assume $H_0 = W_2^2(\Pi)$, $H_1 = W_2'(\Pi)$, $H_2 = L_2[0,1]$.
In this case $P(\lambda) = |\lambda|^2$. The function $R(\lambda)$ will be regular in the layer $|Jm\lambda_k| \leq \pi$.

Theorem 2. *Assume that $R(\lambda)$ is a bounded operator $H_0 \to H_m$ for $\lambda \in R^n|_0$ and*

$$\|D^\alpha R(\lambda)\|_{H_0\to H_m} \leq C_\alpha |\lambda|^{m_\alpha}, \quad \forall\alpha, \tag{7}$$

for $|\lambda| > 1$ and some m_α.

Furthermore, there exists a $P(\lambda)$ homogeneous polynomial of degree k such that $P(\lambda) \neq 0$ for $\lambda \in R^n|_0$ and $R(\lambda)P(\lambda)$ is a infinitely differentiable operator function λ for $|\lambda| \leq 1$ as well.

Then if $f(x)$ has a compact support, $D^\alpha f(x) \in H_0$ for all α, then there exists the solution of equation (1) such that

$$u(x) = \sum_{|\alpha| \leq N} e_\alpha D^\alpha \Gamma(x) + O\left(|x|^{k-n-N}\right), e_\alpha \in H_m, \tag{8}$$

as $|x| \to \infty$, where $\Gamma(x) = \Phi(x)|x|^{k-n}$, $\Phi(x)$ is a homogeneous function of zero degree if $k < n$ or n is odd, $\Gamma(x) = \Phi_1(x)|x|^{k-n}\ln|x|$, $\Phi_1(x)$ is a homogeneous function of zero degree if $k \geq n$, n is even.

Proof. At first consider the case $k < n$.

Determine the function $\upsilon(\lambda)$, having put $\upsilon(\lambda) = R(\lambda)\tilde{f}(\lambda)$, where $\tilde{f}(\lambda)$ is the Fourier transformation of the function $f(x)$. The function $\upsilon(\lambda)$ is determined for $\lambda \in R^n|_0$, $\upsilon(\lambda) \in H_m$ for $\lambda \neq 0$. Since $f(x)$ has a compact support and is infinitely differentiable, then $\tilde{f}(\lambda) \in H_0$ is such that

$$(1 + |\lambda|^p)\left\|D^\alpha \tilde{f}(\lambda)\right\|_{H_0} \leq c_{p,\alpha} \tag{9}$$

for any p, α.

For each $x \in R^n$ the function $R^n \to H_m$ is determined

$$u(x) = (2\pi)^{-n/2}\int_{R^n} e^{i\lambda x}R(\lambda)\tilde{f}(\lambda)\,d\lambda. \tag{10}$$

The theorem condition yields the boundedness of $\|R(\lambda)P(\lambda)\|_{H_0 \to H_m}$ for $|\lambda| < 1$, whence

$$\|R(\lambda)\|_{H_0 \to H_m} \leq \frac{1}{|\lambda|^k}$$

for $|\lambda| < 1$. Since $n > k$, the integral in (10) converges for $\lambda = 0$. At infinity it converges by virtue of (7) and (9). Furthermore, all the derivatives $D^\alpha u$ exist and are bounded:

$$\|D^\alpha u(x)\|_{H_m} \leq C_\alpha, \ C_\alpha \text{ is independent of } x.$$

The function $u(x)$ is the solution of equation (1).

Consider $g(\lambda) = R(\lambda)P(\lambda)\tilde{f}(\lambda)$.

Expanding $g(\lambda)$ in Taylor series for $|\lambda| < \frac{1}{2}$, we get

$$g(\lambda) = \sum_{|\alpha| \leq N} g_\alpha \lambda^\alpha + g_N(\lambda), \tag{11}$$

where $g_\alpha \in H_m$ and

$$\left\|\partial^\beta g_N(\lambda)\right\|_{H_m} \leq c_N |\lambda|^{N+1-|\beta|} \ \ for \ \ |\lambda| < \frac{1}{2}. \tag{12}$$

Using formula (11), rewrite formula (10) in the form:

$$u(x) = (2\pi)^{-n/2}\int_{R^n} e^{i\lambda x}\frac{\theta(\lambda)P(\lambda)R(\lambda)\tilde{f}(\lambda)}{P(\lambda)}d\lambda +$$

$$(2\pi)^{-n/2}\int_{R^n} e^{i\lambda x}\left(1-\theta\left(\lambda\right)\right)R\left(\lambda\right)\tilde{f}\left(\lambda\right)d\lambda = u_1\left(x\right)+u_2\left(x\right),$$

where $\theta(\lambda) \in \overset{\circ}{C}{}^{\infty}(R^n)$, $\theta(\lambda) : R^n \to R^1$, $\theta(\lambda) \equiv 1$ for $|\lambda| < \frac{1}{4}$ $\theta(\lambda) \equiv 0$ for $|\lambda| > \frac{1}{2}$.

From (11)

$$u_1\left(x\right) = (2\pi)^{-n/2}\int_{R^n} e^{i\lambda x}\sum_{|\alpha|\leq N}\frac{g_\alpha\lambda^\alpha\theta\left(\lambda\right)}{P\left(\lambda\right)}d\lambda +$$

$$(2\pi)^{-n/2}\int_{R^n} e^{i\lambda x}\frac{\theta\left(\lambda\right)g_N\left(\lambda\right)}{P\left(\lambda\right)}d\lambda = u_{11}\left(x\right)+u_{12}\left(x\right). \tag{13}$$

There exists $G(x)$ such that its Fourier transformation is $\theta(\lambda)$. Such a $G(x)$ is determined by the inversion formula and $G(x) \in S(R^n)$ to the Schwartz space. Therefore

$$u_{11}\left(x\right) = \sum_{|\alpha|\leq N} g_\alpha D^\alpha w,$$

where $w(x) : R^n \to G^1$ (a complex valued scalar function) $w(x) = \Gamma(x)G(x)$. $\Gamma(x)$ is a fundamental solution of the elliptic equation $P(D)\Gamma(x) = \delta(x)$.

It is known that [2], in the considered case

$$\Gamma\left(x\right) = \Phi_0\left(x\right)|x|^{k-n}, \tag{14}$$

where $\Phi_0(x)$ is a homogeneous function of zero degree. Thus,

$$(-1)^{|\alpha|}D^\alpha w\left(x\right) = \Gamma\left(x\right)D^\alpha G\left(x\right) = W_\alpha\left(x\right).$$

Represent $w_\alpha(x)$ in the form

$$w_\alpha\left(x\right) = \int_{|y|>2|x|}\Gamma\left(x-y\right)D^\alpha G\left(y\right)dy + \int_{\frac{1}{2}|x|<|y|<2|x|}\Gamma\left(x-y\right)D^\alpha G\left(y\right)dy$$

$$+\int_{|y|<\frac{|x|}{2}}\Gamma\left(x-y\right)D^\alpha G\left(y\right)dy = K_1\left(x\right)+K_2\left(x\right)+K_3\left(x\right).$$

Estimate $K_1(x)$. From (14) we get:

$$K_1\left(x\right) \leq c\int_{|y|>2|x|}\frac{dy}{|y|^M|x-y|^{n-k}} \leq$$

$$c_1|x|^{-\frac{M}{2}}\int_{|y|>2|x|}\frac{dy}{|y|^{M/2+n-k}} \leq O\left(|x|^{-\frac{M}{2}}\right) \tag{15}$$

for any M.

For estimating $K_2(x)$ make a change of variables $y = |x|y'$. Then we get:

$$K_2\left(x\right) = |x|^k\int_{\frac{1}{2}<|y'|<2}\Gamma\left(\frac{x}{|x|}-y'\right)DG\left(y'\right)dy' \leq$$

$$|x|^k\frac{c_2}{|x|^M}\int_{\frac{1}{2}<|y'|<2}\Gamma\left(\frac{x}{|x|}-y'\right)dy' = O\left(|x|^{-M/2}\right) \tag{16}$$

for any M.

Finally,

$$K_3(x) = \int_{|y|<\frac{|x|}{2}} \Gamma(x-y) D^\alpha G(y)\, dy =$$

$$\int_{|y|<\frac{|x|}{2}} \left[\sum_{|\alpha|\le N} \frac{(-1)^n D^\alpha \Gamma(x)}{\alpha!} + \Gamma_1(x,y) \right] D^\alpha G(y)\, dy =$$

$$\sum_{|\alpha|\le N} \frac{(-1)^{|\alpha|} D^\alpha \Gamma(x)}{\alpha!} \int_{|y|<\frac{|x|}{2}} D^\alpha G(y)\, dy + \int_{|y|<\frac{|x|}{2}} \frac{dy}{|x-y|^{n-k+N+1}} =$$

$$\sum_{|\alpha|\le N} a_\alpha D^\alpha \Gamma(x) + \sum_{|\alpha|\le N} b_\alpha D^\alpha \Gamma(x) \int_{|y|>\frac{|x|}{2}} D^\alpha G(y)\, dy + O\left(|x|^{-M/2}\right) =$$

$$\sum_{|\alpha|\le N} c_\alpha D^\alpha \Gamma(x) + O\left(|x|^{-N-1}\right). \tag{17}$$

From (15)-(17) it follows that

$$u_{11}(x) = \sum_{|\alpha|\le N} e_\alpha D^\alpha \Gamma(x)\, O\left(|x|^{k-n-N-1}\right). \tag{18}$$

Consider the summand $u_{12}(x)$ in formula (13) for $x \neq 0$. There will be found $j\,(1 \le j \le n)$ such that $|x_j| > \frac{|x|}{\sqrt{n}}$. Without loss of generality, we can assume $j = 1$. Transform $u_{12}(x)$ by means of integration by parts with respect to λ_1 q times ($q \le N$):

$$u_{12}(x) = (-1)^q (2\pi)^{-n/2} \int_{R^n} \frac{e^{i\lambda x}}{(-ix)^q} \frac{\partial^q}{\partial \lambda^q} \left[\frac{\theta(\lambda)\, g_N(\lambda)}{P(\lambda)} \right] d\lambda = O\left(|x|^{-q}\right). \tag{19}$$

Such an estimation holds for $u_2(x)$ as well:

$$u_2(x) = O\left(|x|^{-q}\right), \tag{20}$$

that is obtained by integration by parts in the formula for $u_2(x)$. Estimation (8) follows from (18), (19), (20).

Now let's consider the case $n < k$.

Now it is impossible to define the function $u(x)$ by formula (20) because of divergence of the integral in its right side. We'll construct the solution of equation (1) by means of the formula

$$u(x) = \Gamma(x) (2\pi)^{-n/2} \int_{R^n} e^{i\lambda x} P(\lambda) R(\lambda) \tilde{f}(\lambda)\, d\lambda = \Gamma(x) F(x). \tag{21}$$

The function $F(x)$ accepts the values in H_m and

$$\|D^\alpha F(x)\|_{H_m} \le c_{\alpha,p} (1+|x|)^{-p}.$$

Therefore the convolution (21) exists and

$$u(x) = \int_{R^n} \Gamma(x-y) F(y)\, dy. \tag{22}$$

Having represented $u(x) = J_1 + J_2 + J_3$, where

$$J_1 = \int_{|y|>2|x|} \Gamma(x-y) F(y)\, dy,$$

$$J_2 = \int_{\frac{|x|}{2}<y<2|x|} \Gamma(x-y) F(y)\, dy,$$

$$J_3 = \int_{|y|>\frac{|x|}{2}} \Gamma(x-y) F(y)\, dy$$

and estimated J_1, J_2, J_3 by formula (15)-(17), we get a representation for $u(x)$.

Show that $u(x)$ is the solution of equation (1). We have:

$$Lu = \sum_{|\alpha|\le m} A_\alpha D^\alpha u = \Gamma(x) LF(x) = \Gamma(x) (2\pi)^{-\frac{n}{2}} \int_{R^n} e^{i\lambda x} LF(\lambda)\, d\lambda =$$

$$\Gamma(x) (2\pi)^{-n/2} \int_{R^n} e^{i\lambda x} P(\lambda) \tilde{f}(\lambda)\, d\lambda = \Gamma(x) [P(D) f] = f(x).$$

Thus, equation (1) has a solution in the form (8). Note that $\Gamma(x)$ in the case $n > k$, n is odd, has the form $|x|^{k-n} \Phi_0(x) \ln|x|$ if n is even and $n \le k$. The theorem is proved.

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