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FUNCTION SPACES AND INTEGRAL OPERATORS ASSOCIATED WITH SCHRÖDINGER OPERATORS: AN OVERVIEW

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In memory of M. G. Gasymov on his 75th birthday

Abstract. In this paper we overview known and recently obtained results on function spaces and integral operators associated with Schrödinger operators with respect to the properties of the spaces themselves, that is, we touch the study of operators in these spaces. In particular, we overview equivalent definitions of various versions of the spaces and the boundedness of some Schrödinger type operators on these spaces related to certain nonnegative potentials belonging to the reverse Hölder class.

1. Introduction

We started our studies of various Schrödinger type operators in spaces related to certain nonnegative potentials belonging to the reverse Holder class.

In this paper, we consider the Schrödinger differential operator

$$\mathcal{L} = -\Delta + V(x) \text{ on } \mathbb{R}^n,\ n \geq 3,$$

where $V(x)$ is a nonnegative potential belonging to the reverse Hölder class B_q for $q \geq n/2$.

A nonnegative locally L^q integrable function $V(x)$ on $\mathbb{R}^n$ is said to belong to B_q $(1 < q \leq \infty)$ if there exists $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|B(x,r)|}\int_{B(x,r)} V^q(y)dy\right)^{1/q} \leq \left(\frac{C}{|B(x,r)|}\int_{B(x,r)} V(y)dy\right) \tag{1.1}$$

holds for every $x \in \mathbb{R}^n$ and $0 < r < \infty$, where $B(x,r)$ denotes the ball centered at x with radius r. In particular, if V is a nonnegative polynomial, then $V \in B_\infty$. It is worth pointing out that the B_q class is such that, if $V \in B_q$ for some $q > 1$, then there exists an $\epsilon > 0$, which depends only n and the constant C in (1.1), such that $V \in B_{q+\epsilon}$. Throughout this paper, we always assume that $0 \neq V \in B_{n/2}$.

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For $x \in \mathbb{R}^n$, the function $\rho(x)$ is defined by

$$\rho(x) := \frac{1}{m_V(x)} = \sup_{r>0} \left\{ r : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \le 1 \right\}.$$

Obviously, $0 < m_V(x) < \infty$ if $V \ne 0$. In particular, $m_V(x) = 1$ with $V = 1$ and $m_V(x) \sim 1 + |x|$ with $V(x) = |x|^2$.

Lemma 1.1. [53] *There exists $k_0 > 0$ and $C_0 > 1$ such that*

$$\frac{1}{C_0}\left(1 + \frac{|x-y|}{\rho(x)}\right)^{-k_0} \le \frac{\rho(y)}{\rho(x)} \le C_0 \left(1 + \frac{|x-y|}{\rho(x)}\right)^{k_0/(k_0+1)}.$$

In particular, $\rho(x) \sim \rho(y)$ if $|x-y| < C\rho(x)$.

In this paper, we write $\Psi_\theta(B) = (1 + r/\rho(x_0))^\theta$, where $\theta > 0$, x_0 and r denotes the center and radius of B respectively.

1.1. Operators under consideration.

The maximal operator $M_V \equiv M_V^\theta$ associated with Schrödinger operators ($\mathcal{L}$–*maximal operator*) is defined by

$$M_V f(x) \equiv M_V^\theta f(x) = \sup_{t>0} \left(1 + \frac{|x-y|}{\rho(x)}\right)^{-\theta} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy.$$

When $V = 0$ and $\theta = 0$, we denote $M_0 f(x)$ by $Mf(x)$ (the standard Hardy-Littlewood maximal function). It is easy to see that $|f(x)| \le M_V^\theta f(x) \le Mf(x)$ for *a.e.* $x \in \mathbb{R}^n$ and $\theta \ge 0$.

From [53, 58], we first consider a class Schrödinger type operators, such as

(1) $\nabla(-\Delta + V)^{-1}\nabla$ with $V \in B_n$,
(2) $\nabla(-\Delta + V)^{-1/2}$ with $V \in B_n$,
(3) $(-\Delta + V)^{-1/2}\nabla$ with $V \in B_n$,
(4) $(-\Delta + V)^{i\gamma}$ with $\gamma \in \mathbb{R}$ and $V \in B_{n/2}$, and
(5) $\nabla^2(-\Delta + V)^{-1}$ with V is a nonnegative polynomial,

that are standard Calderón-Zygmund operators; see [56]. In particular, the kernels K of operators above all satisfy the following conditions for some $\delta_0 > 0$ and any $l \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\Big|k(x,y)\Big| \le \frac{C_l}{\Big(1 + |x-y|\big(m_V(x) + m_V(y)\big)\Big)^l} \frac{1}{|x-y|^n}$$

and

$$\Big|k(x+h,y) - k(x,y)\Big| \le \frac{C_l}{\Big(1 + |x-y|\big(m_V(x) + m_V(y)\big)\Big)^l} \frac{|h|^{\delta_0}}{|x-y|^{n+\delta_0}},$$

whenever x, y, $h \in \mathbb{R}^n$ and $|h| < |x-y|/2$.

Let $\mathcal{L} = -\Delta + V$ with $V \in B_q$ for $q \ge n/2$ and its associated semigroup:

$$T_t f(x) = e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^n} k_t(x,y) f(y)dy, \;\; f \in L^2(\mathbb{R}^n),\; t > 0. \qquad (1.2)$$

Lemma 1.2. [15] *Let $k_t(x,y)$ be as in* (1.2). *For every nonnegative integer k, there is a constant C_k such that*

$$0 \le k_t(x,y) \le C_k t^{-n/2} \exp\left(-\frac{|x-y|^2}{5t}\right)\left(1 \ + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^k. \tag{1.3}$$

Lemma 1.3. [51] *Let $k_t(x,y)$ be as in* (1.2). *For any $N \in \mathbb{N}$, there exists a C_N such that*

$$\int_0^\infty t^{-(n-\beta)/2-1} k_t(x,y)dt \le \frac{C_N}{\left(1+\frac{|x_0-y|}{\rho(x_0)}\right)^N} \frac{1}{|x_0-y|^{n-\beta}} \tag{1.4}$$

for all $x \in B(x_0,r)$ and $y \in \mathbb{R}^n \backslash B(x_0,2r)$.

Maximal operator of the diffusion semi-group is defined by

$$T^* f(x) = \sup_{t>0} e^{-t\mathcal{L}} f(x).$$

Next, we discuss the Littlewood–Paley g function related to Schrödinger operators defined by

$$g(f)(x) = \left(\int_0^\infty \Big|\frac{d}{dt} e^{-t\mathcal{L}}(f)(x)\Big|^2 tdt\right)^{1/2},$$

and the commutator g_b of g with $b \in BMO(\rho)$ is defined by

$$g_b(f)(x) = \left(\int_0^\infty \Big|\frac{d}{dt} e^{-t\mathcal{L}}((b(x)-b(\cdot))f)(x)\Big|^2 tdt\right)^{1/2}.$$

The Riesz transform associated with the Schrödinger operator $\mathcal{L}$ is defined by $R = \nabla \mathcal{L}^{-\frac{1}{2}}$ and the commutator operator

$$[b,\ R](f)(x) = R(bf)(x) - b(x)Rf(x),\ x \in \mathbb{R}^n,$$

where f is a suitable integral function. Also, the dual Riesz transform associated with the Schrödinger operator $\mathcal{L}$ is defined by $R^* = \mathcal{L}^{-\frac{1}{2}}\nabla$ and the commutator operator

$$[b,\ R^*](f)(x) = R^*(bf)(x) - b(x)R^* f(x),\ x \in \mathbb{R}^n.$$

The fractional integral operators associated with Schrödinger operators ($\mathcal{L}$–*fractional integral operator*) is defined by

$$\mathcal{I}_\beta f(x) = \mathcal{L}^{-\beta/2} f(x) = \int_0^\infty e^{-t\mathcal{L}} f(x)\ t^{\beta/2-1}\, dt$$
$$= \int_{\mathbb{R}^n} k_\beta(x,y) f(y)dy \text{ for } 0 < \beta < n.$$

Using Proposition 2.4 in [13], we can get the following result for the fractional integral associated with Schrödinger operator.

Lemma 1.4. *If $V \in B_q$, $q \ge n/2$ and $0 < \beta < n$, k_β denotes the kernel of the fractional integral $\mathcal{I}_\beta$ as above, then there exists $\delta_0 = \delta_0(q) > 0$ such that for every $l > 0$ there is a constant $C_l >$ so that*

$$\Big|k_\beta(x,y)\Big| \le \frac{C_l}{\Big(1+|x-y|\big(m_V(x)+m_V(y)\big)\Big)^l} \frac{1}{|x-y|^{n-\beta}}$$

and

$$\Big|k_\beta(x+h,y)-k_\beta(x,y)\Big| \le \frac{C_l}{\Big(1+|x-y|\big(m_V(x)+m_V(y)\big)\Big)^l}\,\frac{|h|^{\delta_0}}{|x-y|^{n-\beta+\delta_0}},$$

whenever $x,\ y,\ h\in\mathbb{R}^n$ *and* $|h|<|x-y|/2$.

Let $S^{n-1}=\{x\in\mathbb{R}^n:|x|=1\}$ is the unit sphere of $\mathbb{R}^n$ $(n\ge 2)$ equipped with the normalized Lebesgue measure $d\sigma=d\sigma(x')$. Let $\Omega\in L_s(S^{n-1})$, $s\ge 1$ be a homogeneous function of degree zero on $\mathbb{R}^n$ and satisfy the cancellation condition

$$\int_{S^{n-1}}\Omega(x)d\sigma(x)=0. \tag{1.5}$$

The Marcinkiewicz integral operator of higher dimension μ_Ω is defined by

$$\mu_\Omega(f)(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}f(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}.$$

It is well known that the Littlewood-Paley g-function is a very important tool in harmonic analysis and the Marcinkiewicz integral is essentially a Littlewood-Paley g-function. In this paper, we will also consider the commutator $\mu_{\Omega,b}$ which is given by the following expression

$$\mu_{\Omega,b}f(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1}}[b(x)-b(y)]f(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}.$$

Similar to the classical Marcinkiewicz function, we define the Marcinkiewicz functions $\mu_{j,\Omega}$ associated with the Schrödinger operator $\mathcal{L}$ by

$$\mu^{\mathcal{L}}_{j,\Omega}f(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}|\Omega(x-y)|K^{\mathcal{L}}_j(x,y)f(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2},$$

where $K^L_j(x,y)=\widetilde{K^L_j}(x,y)|x-y|$ and $\widetilde{K^L_j}(x,y)$ is the kernel of $R_j=\frac{\partial}{\partial x_j}L^{-\frac12}$, $j=1,\dots,n$. In particular, when $V=0$, $K^\Delta_j(x,y)=\widetilde{K^\Delta_j}(x,y)|x-y|=\frac{(x-y)_j/|x-y|}{|x-y|^{n-1}}$ and $\widetilde{K^\Delta_j}(x,y)$ is the kernel of $R_j=\frac{\partial}{\partial x_j}\Delta^{-\frac12}$, $j=1,\dots,n$. In this paper, we write $K_j(x,y)=K^\Delta_j(x,y)$ and

$$\mu_{j,\Omega}f(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}|\Omega(x-y)|K_j(x,y)f(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}.$$

The commutator of the classical Marcinkiewicz function with rough kernel is defined by

$$\mu_{j,\Omega,b}f(x)=\left(\int_0^\infty\left|\int_{|x-y|\le t}|\Omega(x-y)|K_j(x,y)[b(x)-b(y)]f(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}.$$

The commutator $\mu^L_{j,\Omega,b}$ formed by $b \in L_1^{\rm loc}(\mathbb{R}^n)$ and the Marcinkiewicz function with rough kernel $\mu^L_{j,\Omega}$ is defined by

$$\mu^L_{j,\Omega,b}f(x) = \left(\int_0^\infty \left|\int_{|x-y|\le t} |\Omega(x-y)| K_j^{\mathcal{L}}(x,y)[b(x)-b(y)]f(y)dy\right|^2 \frac{dt}{t^3}\right)^{1/2}.$$

Lemma 1.5. [53] *Let $V \in B_q$ with $q \ge n/2$. For any $l > 0$, there exists $C_l > 0$ such that*

$$\left|K_j^L(x,y)\right| \le \frac{C_l}{\left(1+\frac{|x-y|}{\rho(x)}\right)^l} \frac{1}{|x-y|^{n-1}},$$

and

$$\left|K_j^L(x,y) - K_j(x-y)\right| \le \ C\frac{\rho(x)^{-1}}{|x-y|^{n-2}}.$$

2. Weighted norm inequalities for Schrödinger type operators

We first recall some notation. Given $B = B(x,r)$ and $\lambda > 0$, we will write λB for the λ-dilate ball, which is the ball with the same center x and with radius λr. Given a Lebesgue measurable set E and a weight ω, $|E|$ will denote the Lebesgue measure of E and $\omega(E) = \int_E \omega(x)dx$. For $0 < p < \infty$, $\|f\|_{L^p_\omega}$ will denote $\left(\int_{\mathbb{R}^n} |f(y)|^p \omega(y)dy\right)^{1/p}$.

A weight will always mean a nonnegative function which is locally integrable. As in [10], we say that a weight ω belongs to the class $A_p^{\rho,\theta}$ for $1 < p < \infty$, if there is a constant C such that for all ball $B = B(x,r)$

$$\left(\frac{1}{\Psi_\theta(B)|B|}\int_B \omega(y)dy\right)\left(\frac{1}{\Psi_\theta(B)|B|}\int_B \omega^{-\frac{1}{p-1}}(y)dy\right)^{p-1} \le C.$$

We also say that a nonnegative function ω satisfies the $A_1^{\rho,\theta}$ condition if there exists a constant C for all balls B

$$M_V^\theta(\omega)(x) \le C\omega(x), \text{ a.e. } x \in \mathbb{R}^n,$$

where

$$M_V^\theta f(x) = \sup_{x\in B} \frac{1}{\Psi_\theta(B)|B|}\int_B |f(y)|dy.$$

Since $\Psi_\theta(B) \ge 1$, obviously, $A_p \subset A_p^{\rho,\theta}$ for $1 \le p < \infty$, where A_p denote the classical Muckenhoupt weights; see [18], [38], [45]. We will see that $A_p \subset\subset A_p^{\rho,\theta}$ for $1 \le p < \infty$ in some cases. In fact, let $\theta > 0$ and $0 \le \gamma \le \theta$, it is easy to check that

$$\omega(x) = (1+|x|)^{-(n+\gamma)} \notin A_\infty = \bigcup_{p\ge 1} A_p$$

and $\omega(x)dx$ is not a doubling measure, but

$$\omega(x) = (1+|x|)^{-(n+\gamma)} \in A_1^{\rho,\theta}$$

provided that $V = 1$ and $\Psi_\theta(B(x_0,r)) = (1+r)^\theta$.

For convenience, we always assume that $\Psi(B)$ denotes $\Psi_\theta(B)$, $A_p^{\rho,\infty} = \bigcup_{\theta>0} A_p^{\rho,\theta}$ and $A^{\rho,\infty} = \bigcup_{p\geq 1} A_p^{\rho,\infty}$.

Lemma 2.1. [50] *Let $0 < \theta < \infty$, then*

(i) if $1 \leq p_1 < p_2 < \infty$, then $A_{p'}^{\rho_1\theta} \subset A_{p'}^{\rho_2\theta}$;

(ii) $\omega \in A_p^{\rho,\theta}$ if and only if $\omega^{-\frac{1}{p-1}} \in A_p^{\rho,\theta}$, where $1/p + 1/p' = 1$;

(iii) if $\omega \in A_p^{\rho,\theta}$ for $1 \leq p < \infty$, then there exists a constant such that for any $\lambda > 1$

$$\omega\left(\lambda B(x_0,r)\right) \leq C\left(1+\frac{\lambda r}{\rho(x_0)}\right)^{(k_0+1)\theta} \omega\left(B(x_0,r)\right).$$

Lemma 2.2. [50] *Let $0 < \theta < \infty$, $1 \leq p < \infty$. If $\omega \in A_p^{\rho,\theta}$, then there exists positive constants 6, η and C such that*

$$\left(\frac{1}{|B|}\int_B \omega(y)^{1+\delta}dy\right)^{1/(1+\delta)} \leq C\frac{1}{|B|}\int_B \omega(y)dy\left(1+\frac{r}{\rho(x_0)}\right)^{\eta}$$

for all ball $B(x_0,r)$.

As a consequence of Lemma 2.3, we have the following result.

Corollary 2.1. [50] *Let $0 < \theta < \infty$, $1 \leq p < \infty$. If $\omega \in A_p^{\rho,\theta}$, then there exist positive constants $q > 1$, η and C such that*

$$\frac{\omega(E)}{\omega(B)} \leq C\left(\frac{|E|}{|B|}\right)^{1/q}\left(1+\frac{r}{\rho(x_0)}\right)^{\eta}$$

for all ball any measurable subset E of a ball $B(x_0,r)$.

As in [50], we say that a weight ω belongs to the class $A_{(p,q)}^{\rho} \equiv A_{(p,q)}^{\rho,\theta}$ for $1 \leq p < \infty$ and $1 \leq p < \infty$, if there is a constant C such that for all ball $B = B(x,r)$

$$\left(\frac{1}{\Psi_\theta(B)|B|}\int_B [\omega(y)]^q dy\right)^{1/q}\left(\frac{1}{\Psi_\theta(B)|B|}\int_B \omega^{-p'}(y)dy\right)^{1/p'} \leq C,$$

where $p' = p/(p-1)$. Obviously,

$$\omega^{1/p} \in A_{(p,p)}^{\rho} \Longleftrightarrow \omega \in A_p^{\rho} \text{ for } 1 \leq p < \infty.$$

Proposition 2.1. [50] *Let $1 < p < \infty$ and suppose that A_p^{ρ}. If $p < p_1 < \infty$, then the estimate*

$$\int_{\mathbb{R}^n}\Big|M_V f(x)\Big|^{p_1}\omega(x)dx \leq C\int_{\mathbb{R}^n}\Big|f(x)\Big|^{p_1}\omega(x)dx$$

holds. Further, let $1 \leq p < \infty$. Then $\omega \in A_p^{\rho}$ if and only if

$$\omega\big(\{x\in\mathbb{R}^n : M_V f(x) > \lambda\}\big) \leq \frac{C}{\lambda^p}\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx.$$

For the fractional maximal operator the following property is valid.

Proposition 2.2. [50] *Let* $0 < \beta < n$, $1 \le p < n/\beta$ *and* $1/q = 1/p - \beta/n$. *Then*

$$\Big(\int_{\big\{x\in\mathbb{R}^n: M_{\beta,V}f(x)>\lambda\big\}}[\omega(y)]^q dy\Big)^{1/q} \le \frac{C}{\lambda}\Big(\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx\Big)^{1/p}$$

for each $\omega \in A^{\rho}_{(p,q)}$.

Theorem 2.1. [50] *Let* T *denote one of the Schrödinger type operators, such as* $\nabla(-\Delta+V)^{-1}\nabla$, $\nabla(-\Delta+V)^{-1/2}$, $(-\Delta+V)^{-1/2}\nabla$ *with* $V \in B_n$. *Let* $1 < p < \infty$ *and suppose that* $\omega \in A^{\rho}_p$. *Then*

$$\int_{\mathbb{R}^n}|Tf(x)|^p\omega(x)dx \le C\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx,$$

where C *is independent of* f.

Further, suppose that $\omega \in A^{\rho}_1$. *Then there exists a constant* C *such that for all* $\lambda > 0$

$$\omega\big(\big\{x\in\mathbb{R}^n : |Tf(x)| > \lambda\big\}\big) \le \frac{C}{\lambda^p}\int_{\mathbb{R}^n}|f(x)|^p\omega(x)dx.$$

We remark that the weighted boundedness of $\nabla(-\Delta+V)^{-1/2}$, $(-\Delta+V)^{-1/2}\nabla$ with $V \in B_n$ is proved in [10].

Next we consider another class $V \in B_q$ for $n/2 \le q$ for Riesz transforms associated to Schrödinger operators. Let $T_1 = (-\Delta+V)^{-1}V$, $T_2 = (-\Delta+V)^{-1/2}V^{1/2}$, $T_3 = (-\Delta+V)^{-1/2}\nabla$.

Theorem 2.2. [50] *Suppose* $V \in B_q$ *for* $q \ge n/2$. *Then*

(i) if $q' \le p < \infty$ *and* $\omega \in A^{\rho}_{p/q'}$, *then*

$$\|T_1 f\|_{L^p_\omega} \le C\,\|f\|_{L^p_\omega},$$

where C *is independent of* f;

(ii) if $(2q)' \le p < \infty$ *and* $\omega \in A^{\rho}_{p/(2q)'}$, *then*

$$\|T_2 f\|_{L^p_\omega} \le C\,\|f\|_{L^p_\omega},$$

where C *is independent of* f;

(iii) if $p_0' \le p < \infty$ *and* $\omega \in A^{\rho}_{p/p_0'}$, *where* $1/p_0' = 1/q - 1/n$ *and* $n/2 \le q < n$, *then*

$$\|T_3 f\|_{L^p_\omega} \le C\,\|f\|_{L^p_\omega},$$

where C *is independent of* f.

Let

$$T_1^* = V(-\Delta+V)^{-1},\; T_2^* = V^{1/2}(-\Delta+V)^{-1/2},\; T_3^* = \nabla(-\Delta+V)^{-1/2}.$$

By duality, easily get the following results.

Theorem 2.3. [50] *Suppose* $V \in B_q$ *for* $q \ge n/2$. *Then*

(i) if $1 < p \le q$ *and* $\omega^{-\frac{1}{p-1}} \in A^{\rho}_{p'/q'}$, *then*

$$\|T_1^* f\|_{L^p_\omega} \le C\,\|f\|_{L^p_\omega},$$

where C *is independent of* f;

(ii) if $1 < p \le 2q$ *and* $\omega^{-\frac{1}{p-1}} \in A^{\rho}_{p'/(2q)'}$*, then*

$$\|T_2^* f\|_{L^p_\omega} \le C\ \|f\|_{L^p_\omega},$$

where C *is independent of* f*;*

(iii) if $1 < p \le p_0$ *and* $\omega^{-\frac{1}{p-1}} \in A^{\rho}_{p'/p'_0}$*, where* $1/p'_0 = 1/q - 1/n$ *and* $n/2 \le q < n$*, then*

$$\|T_3^* f\|_{L^p_\omega} \le C\ \|f\|_{L^p_\omega},$$

where C *is independent of* f*.*

We remark that the weighted L^p_ω boundedness of T_3, T_3^* is proved in [10].

The following result for the fractional integral associated with Schrödinger operator is valid.

Proposition 2.3. [50] *Let* $0 < \beta < n$, $1 \le p < n/\beta$ *and* $1/q = 1/p - \beta/n$*. Then*

$$\Big(\int_{\mathbb{R}^n} \big|I_{\beta,V} f(x)\big|^q [\omega(y)]^q dy\Big)^{1/q} \le \Big(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx\Big)^{1/p}$$

for each $\omega \in A^{\rho}_{(p,q)}$*. Further, suppose that* $\mu = \omega^q \in A^{\rho}_1$ *with* $q = n/(n-\beta)$*. Then there exists a constant* C *such that for all* $\lambda > 0$

$$\mu\big(\{x \in \mathbb{R}^n : I_\beta f(x) > \lambda\}\big)^{1/q} \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| \omega(x) dx.$$

The following theorem was proved in [2] (in the case $w = 1$ in [4] and in the case $s = \infty$ in [29]).

Theorem 2.4. *Suppose that* $\Omega \in L_s(S^{n-1})$, $1 < s \le \infty$ *be a homogeneous function of degree zero on* $\mathbb{R}^n$ *and satisfy the conditions* (1.5) *and* $V \in B_n$*. Then for every* $s' < p < \infty$ *and* $w \in A_{p/s'}$ *or* $1 < p < s$ *and* $w^{1-p'} \in A_{p'/s'}$ *there is a constant* C *independent of* f *such that*

$$\|\mu^{\mathcal{L}}_{j,\Omega}(f)\|_{L_{p,w}} \le C \|f\|_{L_{p,w}}.$$

3. Commutators for Schrödinger type operators

Let $b \in BMO$ (see its definition in [39]), we define the commutator of T by

$$[b,\ T]f = bTf - T(bf).$$

Bongioanni, Harboure and Salinas [8] introduce a new space $BMO_\theta(\rho)$ defined by

$$\|f\|_{BMO_\theta(\rho)} = \sup_{B \subset \mathbb{R}^n} \frac{1}{\Psi_\theta(B)|B|} \int_B |f(x) - f_B|\, dx < \infty,$$

where

$$f_B = \frac{1}{|B|} \int_B f(y) dy \text{ and } \Psi_\theta(B) = \left(1 + \frac{r}{\rho(x_0)}\right)^\theta$$

with $B = B(x_0, r)$ and $\theta > 0$.

In particularly, Bongioanni, etc. [8] proved the following result for $BMO_\theta(\rho)$.

Proposition 3.1. *Let $\theta > 0$ and $1 \le s < \infty$. If $b \in BMO_\theta(\rho)$, then*

$$\left(\frac{1}{|B|}\int_B |b - b_B|^s\right)^{1/s} \le c\, C_0^\theta\, s\, \|b\|_{BMO_\theta(\rho)} \left(1 + \frac{r}{\rho(x)}\right)^{\theta'},$$

for all $B = B(x,r)$ with $x \in \mathbb{R}^n$ and $r > 0$, where $\theta' = (k_0+1)\theta$ and C_0 is defined in Lemma 2.1 and c is a constant depending only on n.

Obviously, the classical BMO (see [39, 55]) is properly contained in $BMO_\theta(\rho)$; for more examples see [8]. For convenience, we let $BMO(\rho) = BMO_\theta(\rho)$.

Tang [51] gave the following result, which is equivalent to Proposition 2.1.

Proposition 3.2. *Suppose that $f \in BMO_\theta(\rho)$. There exist positive constants γ and C such that for any ball $B = B(x_0, r)$*

$$\frac{1}{|B|}\int_B \exp\left\{\frac{\gamma}{\|f\|_{BMO_\theta(\rho)}\Psi_\theta(B)}|f(x) - f_B|\right\} dx \le C.$$

Applying Corollary 2.1 and Proposition 2.1, we can obtain the following result.

Proposition 3.3. [50] *If $f \in BMO_\theta(\rho)$ and $\omega \in A_p^{\rho,\theta}$ $(p > 1)$, then there exist positive constants c_1, c_2 and η such that for every ball $B = B(x,r)$ and every $\lambda > 0$, we have*

$$\omega\left(\{x \in B : |f(x) - f_B| > \lambda\}\right)$$
$$\le c_1 \omega(B) \exp\left\{-\frac{c_2\lambda}{\|f\|_{BMO_\theta(\rho)}\Psi_{\theta'}(B)}\right\}\left(1 + \frac{r}{\rho(x)}\right)^{\eta},$$

where

$$\Psi_{\theta'}(B) = \left(1 + \frac{r}{\rho(x_0)}\right)^{\theta'},\ \theta' = (k_0+1)\theta.$$

From Proposition 3.3, it is easy to see that

Corollary 3.1. *If $f \in BMO_\rho$ and $\omega \in A_{(\infty)}^{\rho,\infty}$, then there exist positive constants C and η such that for every ball $B = B(x,r)$, we have*

$$\frac{1}{\omega(B)}\int_B |f(x) - f_B|\omega(x)dx \le \left(1 + \frac{r}{\rho(x)}\right)^{\eta} \|f\|_{BMO_\rho}^p.$$

Let $b \in BMO_\rho$, we define the commutator of $\mathcal{I}_\beta$ by

$$[b, I_{\beta,V}]f = bI_{\beta,V}f - I_{\beta,V}(bf).$$

We first consider commutators of fractional integrals associated with Schrödinger operators.

Proposition 3.4. [50] *Let $b \in BMO_\rho$, $0 < \beta < n$, $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. If $\omega \in A_{(p,q)}^\rho$, then there exists a constant C such that*

$$\left(\int_{\mathbb{R}^n} \left|[I_{\beta,V}, b]f(x)\right|^q [\omega(y)]^q dy\right)^{1/q} \le C\,\|b\|_{BMO_\theta(\rho)}\left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)dx\right)^{1/p}.$$

The weighted weak-type endpoint estimate for the commutator is the following.

Proposition 3.5. *Let $b \in BMO_\rho$, $0 < \beta < n$, $1 < p < n/\beta$ and $1/q = 1/p - \beta/n$. Let*

$$B(t) = t\log(e+t),$$

$$\Lambda(t) = \Big[t\log\big(e + t^{\beta/n}\big)\Lambda\Big]^{n/(n-\beta)},$$

$$\Theta(t) = t^{\beta/n}\Big[t\log\big(e + t^{\beta/n}\big)\Lambda\Big]^{n/(n-\beta)}.$$

If $\omega \in A_1^\rho$, then there exists a constant C such that for any $\lambda > 0$

$$\omega\left(\{x \in \mathbb{R}^n : \big|[I_{\beta,V}, b]f(x)\big| > \lambda\}\right) \le C\Lambda\Big(\int_{\mathbb{R}^n} B\Big(\frac{\|b\|_{BMO_\theta(\rho)|f(x)|}}{\lambda}\Big)\Theta(\omega(x))dx\Big).$$

Let $b \in BMO_\rho$, we define the commutator of T by

$$[b,\ T]f = bTf - T(bf).$$

Theorem 3.1. [50] *Let T denote one of the Schrödinger type operators, such as $\nabla(-\Delta+V)^{-1}\nabla$, $\nabla(-\Delta+V)^{-1/2}$, $(-\Delta+V)^{-1/2}\nabla$ with $V \in B_n$, $(-\Delta+V)^{i\gamma}$ with $\gamma \in \mathbb{R}$ and $V \in B_{n/2}$, and $\nabla^2(-\Delta+V)^{-1}$ with V is a nonnegative polynomial. Let $b \in BMO_\rho$, $1 < p < \infty$ and suppose that $\omega \in A_p^\rho$. Then*

$$\|[b, Tf]\|_{L^p_\omega} \le C\,\|b\|_{BMO_\theta(\rho)}\,\|f\|_{L^p_\omega},$$

where C is independent of f.

Further, suppose that $\omega \in A_1^\rho$. Then there exists a constant C such that for all $\lambda > 0$

$$\omega\big(\{x \in \mathbb{R}^n : |[b, Tf]| > \lambda\}\big) \le C\int_{\mathbb{R}^n}\frac{|f(x)|}{\lambda}\Big(1 + \log^+\Big(\frac{|f(x)|}{\lambda}\Big)\Big)\omega(x)dx.$$

Finally, we consider another class $V \in B_q$ for $n/2 \le q$ for Riesz transforms associated to Schrödinger operators. Let $T_1 = (-\Delta+V)^{-1}V$, $T_2 = (-\Delta + V)^{-1/2}V^{1/2}$, $T_3 = (-\Delta+V)^{-1/2}\nabla$.

Theorem 3.2. [50] *Suppose $V \in B_q$ for $q \ge n/2$. Let $b \in BMO_\rho$. Then*

(i) if $q' \le p < \infty$ and $\omega \in A^\rho_{p/q'}$, then

$$\|[b, T_1]f\|_{L^p_\omega} \le C\,\|b\|_{BMO_\theta(\rho)}\,\|f\|_{L^p_\omega},$$

where C is independent of f;

(ii) if $(2q)' \le p < \infty$ and $\omega \in A^\rho_{p/(2q)'}$, then

$$\|[b, T_2]f\|_{L^p_\omega} \le C\,\|b\|_{BMO_\theta(\rho)}\,\|f\|_{L^p_\omega},$$

where C is independent of f;

(iii) if $p_0' \le p < \infty$ and $\omega \in A^\rho_{p/p_0'}$, where $1/p_0' = 1/q - 1/n$ and $n/2 \le q < n$, then

$$\|[b, T_3]f\|_{L^p_\omega} \le C\,\|b\|_{BMO_\theta(\rho)}\,\|f\|_{L^p_\omega},$$

where C is independent of f.

Let

$$T_1^* = V(-\Delta+V)^{-1},\ T_2^* = V^{1/2}\,(-\Delta+V)^{-1/2},\ T_3^* = \nabla(-\Delta+V)^{-1/2}.$$

By duality, easily get the following results.

Theorem 3.3. [50] *Suppose* $V \in B_q$ *for* $q \geq n/2$. *Let* $b \in BMO_\rho$. *Then*

(*i*) *if* $1 < p \leq q$ *and* $\omega^{-\frac{1}{p-1}} \in A^\rho_{p'/q'}$, *then*

$$\|T_1^* f\|_{L^p_\omega} \leq C\, \|b\|_{BMO_\theta(\rho)}\, \|f\|_{L^p_\omega},$$

where C *is independent of* f;

(*ii*) *if* $1 < p \leq 2q$ *and* $\omega^{-\frac{1}{p-1}} \in A^\rho_{p'/(2q)'}$, *then*

$$\|T_2^* f\|_{L^p_\omega} \leq C\, \|b\|_{BMO_\theta(\rho)}\, \|f\|_{L^p_\omega},$$

where C *is independent of* f;

(*iii*) *if* $1 < p \leq p_0$ *and* $\omega^{-\frac{1}{p-1}} \in A^\rho_{p'/p'_0}$, *where* $1/p'_0 = 1/q - 1/n$ *and* $n/2 \leq q < n$, *then*

$$\|T_3^* f\|_{L^p_\omega} \leq C\, \|b\|_{BMO_\theta(\rho)}\, \|f\|_{L^p_\omega},$$

where C *is independent of* f.

4. Morrey spaces related to nonnegative potentials

Let $p \in [1, \infty)$, $\alpha \in (-\infty, \infty)$ and $\lambda \in [0, n)$. For $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ and $V \in B_q$ $(q > 1)$, we say $f \in L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$ (*Morrey spaces related to the nonnegative potential* V) provided that

$$\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} = \sup_{B(x,r)\subset\mathbb{R}^n} \left(1 + \frac{r}{\rho(x)}\right)^\alpha r^{-\lambda} \int_{B(x,r)} |f(y)|^p dy < \infty,$$

where $B = B(x, r)$ denotes a ball with centered at x and radius r. In particular, when $\alpha = 0$ or $V = 0$ and $0 < \lambda < n$, the space $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$ is the class Morrey space $L^{p,\lambda}(\mathbb{R}^n)$($L^{p,\lambda}(\mathbb{R}^n)$ was first introduced in [44], some new properties of $L^{p,\lambda}(\mathbb{R}^n)$ have been studied in [1, 12, 57]). It is easy to see that $L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n) \subset L^{p,\lambda}(\mathbb{R}^n)$ for $\alpha > 0$, and $L^{p,\lambda}(\mathbb{R}^n) \subset L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$ for $\alpha < 0$.

It is well known that the boundedness of the standard Calderón-Zygmund operators and their commutators have been established on the class Morrey spaces (see [17]). Hence, it will be an interesting question whether in [51] were establish the boundedness of Schrödinger type operators on the Morrey spaces related to certain nonnegative potentials. More precisely, was obtained the following results.

Theorem 4.1. [51] *Suppose* $\alpha \in (-\infty, \infty)$ *and* $\lambda \in (0, n)$.

(*i*) *If* $1 < p < \infty$, *when*

$$\|Tf\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} \leq C\, \|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)},$$

where C *is independent of* f.

(*ii*) *If* $p = 1$, *then for any* $t > 0$,

$$t\Big(1 + \frac{r}{\rho(x)}\Big)^\alpha \Big|\{y \in B(x, r) : |Tf(y)| > t\}\ \Big| \leq C\, r^\lambda \|f\|_{L^{1,\lambda}_{\alpha,V}(\mathbb{R}^n)}$$

holds for all balls B, *where* C *is independent of* x, r, t *and* f.

Theorem 4.2. [51] *Suppose $b \in BMO_\rho$, $\alpha \in (-\infty, \infty)$ and $\lambda \in (0, n)$.*
(i) If $1 < p < \infty$, then

$$\|[b,\ T]f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} \le C\ \|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)},$$

where C is independent of f.
(ii) If $p = 1$, then for any $t > 0$,

$$r^{-\lambda}\Big(1 + \frac{r}{\rho(x)}\Big)^{\alpha} \Big|\{y \in B(x,r) : |[b,\ T]f(y)| > t\}\Big|$$
$$\le C \sup_{B(x,r)\subset\mathbb{R}^n} \Big(1 + \frac{r}{\rho(x)}\Big)^{\alpha} r^{-\lambda} \int_{B(x,r)} \frac{|f(y)|}{t} \ln\Big(2 + \frac{|f(y)|}{t}\Big)\, dy$$

holds for all balls B, where C is independent of x, r, t and f.

Theorem 4.3. [51] *Suppose $V \in B_{n/2}$, $\alpha \in (-\infty, \infty)$ and $0 < \beta < n$.*
(i) If $1 < p < n/\beta$, $1/q = 1/p - \beta/n$, $\theta = q/p$ and $0 < \lambda < n/\theta$, then

$$\|\mathcal{I}_\beta f\|_{L^{q,\theta\lambda}_{\alpha,V}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)},$$

where C is independent of f.
(ii) If $p = 1$ and $q = n/(n - \beta)$, then for any $t > 0$,

$$t\Big(1 + \frac{r}{\rho(x)}\Big)^{\alpha} \Big|\{y \in B(x,r) : \big|\mathcal{I}_\beta f(y)\big| > t\}\Big|^{1/q} \le C\, r^{\lambda}\|f\|_{L^{1,\lambda}_{\alpha,V}(\mathbb{R}^n)}$$

holds for all balls B, where C is independent of x, r, t and f.

Theorem 4.4. [51] *Let $b \in BMO_\rho$, $V \in B_{n/2}$, $\alpha \in (-\infty, \infty)$ and $0 < \beta < n$.*
(i) If $1 < p < n/\beta$, $1/q = 1/p - \beta/n$, $\theta = q/p$ and $0 < \lambda < n/\theta$, then

$$\|[b,\ \mathcal{I}_\beta]f\|_{L^{q,\theta\lambda}_{\alpha,V}(\mathbb{R}^n)} \le C\ \|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)},$$

where C is independent of f.
(ii) If $p = 1$ and $q = n/(n - \beta)$, then for any $t > 0$,

$$r^{-\lambda}\ \Big(1 + \frac{r}{\rho(x)}\Big)^{\alpha} \Big|\{y \in B(x,r) : |[b,\ \mathcal{I}_\beta]f(y)| > t\}\Big|$$
$$\le C \sup_{B(x,r)\subset\mathbb{R}^n} \Big(1 + \frac{r}{\rho(x)}\Big)^{\alpha} r^{-\lambda}\Phi\left(\int_{B(x,r)} \frac{|f(y)|}{t} \ln(2 + \frac{|f(y)|}{t})dy\right)$$

holds for all balls B, where $\Phi(t) = (t\ \log(2 + t^{\beta/n}))^{n/(n-\beta)}$, and C is independent of x, r, t and f.

We remark that even in the classical Morrey space, the above results about the case $p = 1$ in Theorems 4.2 and 4.4 are also new; see [17].

5. Generalized Morrey spaces related to nonnegative potentials

In 1938, Morrey considered regularity of the solution of elliptic differential equations in terms of the solutions themselves and their derivatives. This is a very famous work by Morrey [44].

We recall the definitions. Let $1 \le p < \infty$ and $0 \le \lambda \le n$. Then define

$$\|f\|_{L^{p,\lambda}} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\frac{\lambda}{p}}\ \|f\|_{L^p(B(x,r))} \equiv \sup_B r^{-\frac{\lambda}{p}}\ \|f\|_{L^p(B)},$$

where $B = B(x, r)$ runs over all balls.

Later many researchers studied Morrey spaces from various points of view. After studying Morrey spaces in detail, researchers passed to generalized Morrey spaces, weighted Morrey spaces and generalized weighted Morrey spaces.

Definition 5.1. Let $\varphi(x, r)$ be a positive measurable function on $\mathbb{R}^n \times (0, \infty)$ and $1 \le p < \infty$. We denote by $M^{p,\varphi}(\mathbb{R}^n)$ the generalized Morrey space, the space of all functions $f \in L^{p,\mathrm{loc}}(\mathbb{R}^n)$ with finite quasinorm

$$\|f\|_{M^{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x, r)^{-1} \, \|f\|_{L^p(B(x,r))}. \tag{5.1}$$

The generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ with norm (5.1) introduced by Mizuhara in [43], which was later extended and studied by many authors (see [48]- [52]). Note that, the generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ with normalized norm

$$\|f\|_{M^{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x, r)^{-1} \, |B(x, r)|^{-\frac{1}{p}} \, \|f\|_{L_p(B(x,r))} \tag{5.2}$$

first defined by Guliyev in [21].

Also, in [21], there was defined the weak generalized Morrey space $WM^{p,\varphi} \equiv WM^{p,\varphi}(\mathbb{R}^n)$ of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM^{p,\varphi}} = \sup_{x \in \mathbb{R}^n, r>0} \varphi(x, r)^{-1} \, |B(x, r)|^{-\frac{1}{p}} \, \|f\|_{WL_p(B(x,r))} < \infty.$$

According to this definition, we recover the Morrey spaces $L^{p,\lambda}$ and $WL^{p,\lambda}$ under the choice $\varphi(x, r) = r^{\frac{\lambda-n}{p}}$:

$$L^{p,\lambda} = M^{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}, \quad WL^{p,\lambda} = WM^{p,\varphi}\Big|_{\varphi(x,r)=r^{\frac{\lambda-n}{p}}}.$$

There are many papers where there discussed the conditions on $\varphi(x, r)$ to obtain the boundedness of operators on the generalized Morrey spaces. For example, in [21] (see also [19, 20]) by Guliyev the following condition was imposed on the pair (φ_1, φ_2) :

$$\int_r^\infty t^\alpha \, \varphi_1(x, t) \frac{dt}{t} \le C \varphi_2(x, r), \tag{5.3}$$

where $C > 0$ does not depend on x and r. Under the above condition, in [21] obtained the boundedness of Riesz potential operators from $M^{p,\varphi_1}(\mathbb{R}^n)$ to $M^{q,\varphi_2}(\mathbb{R}^n)$, where $1 < p < q < \infty$ and $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$. Also, in [3] (see also [26]), introduced a weaker condition:

$$\int_r^\infty \frac{\operatorname*{ess\,inf}\limits_{t<s<\infty} \varphi_1(x, s) s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \le C \, \varphi_2(x, r), \tag{5.4}$$

$1 \le p < \infty$, for all $x \in \mathbb{R}^n$ and $r > 0$. If the pair (φ_1, φ_2) satisfies the condition (5.3), then (φ_1, φ_2) satisfied condition (5.4). But the opposite is not true, see remark 4.7 in [26] for details (see also [28, 35, 36, 34, 30, 31]).

In the present section we give the definitions of the generalized Morrey spaces related to nonnegative potentials V.

Definition 5.2. Let $p \in [1, \infty)$, $\alpha \in (-\infty, \infty)$ and $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. For $f \in L^p_{\rm loc}(\mathbb{R}^n)$ and $V \in B_q$ $(q > 1)$, we say $f \in M^{p,\varphi}_{\alpha,V}(\mathbb{R}^n)$ (*generalized Morrey spaces related to the nonnegative potential* V) provided that

$$\|f\|^p_{M^{p,\varphi}_{\alpha,V}(\mathbb{R}^n)} = \sup_{B(x,r)\subset\mathbb{R}^n} \Big(1 + \frac{r}{\rho(x)}\Big)^\alpha \varphi(x,r)^{-p}|B(x,r)|^{-1} \int_{B(x,r)} |f(y)|^p dy \ < \infty,$$

where $B = B(x,r)$ denotes a ball with centered at x and radius r. In particular, when $\alpha = 0$ or $V = 0$ and $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function, the space $M^{p,\varphi}_{\alpha,V}(\mathbb{R}^n)$ is the class generalized Morrey space $M^{p,\varphi}(\mathbb{R}^n)$.

It is well known that the boundedness of the standard Calderón-Zygmund operators and their commutators have been established on the generalized Morrey spaces (see [3, 26]). Hence, it will be an interesting question whether in [23] were establish the boundedness of Schrödinger type operators on the generalized Morrey spaces related to certain nonnegative potentials. More precisely, was obtained the following results.

Theorem 5.1. [23] *Suppose* $1 \le p < \infty$, $V \in B_n$, $\alpha \in (-\infty, \infty)$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\, \varphi_2(x,r), \tag{5.5}$$

where C *does not depend on* x *and* r.

(i) If $1 < p < \infty$, *then*

$$\|Tf\|_{L^{p,\varphi_2}_{\alpha,V}(\mathbb{R}^n)} \le C\ \|f\|_{L^{p,\varphi_1}_{\alpha,V}(\mathbb{R}^n)},$$

where C *is independent of* f.

(ii) If $p = 1$, *then for any* $t > 0$,

$$t\Big(1 + \frac{r}{\rho(x)}\Big)^\alpha \Big|\{y \in B(x,r) : |Tf(y)| > t\}\ \Big| \le C\, \varphi(x,r)|B(x,r)|\|f\|_{L^{1,\varphi_1}_{\alpha,V}(\mathbb{R}^n)}$$

holds for all balls B, *where* C *is independent of* x, r, t *and* f.

Theorem 5.2. [23] *Suppose* $1 \le p < \infty$, $V \in B_n$, $b \in BMO_\rho$, $\alpha \in (-\infty, \infty)$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \ln\Big(e + \frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{p}+1}} dt \le C\, \varphi_2(x,r), \tag{5.6}$$

where C *does not depend on* x *and* r.

(i) If $1 < p < \infty$, *then*

$$\|[b,\ T]f\|_{L^{p,\varphi_2}_{\alpha,V}(\mathbb{R}^n)} \le C\ \|f\|_{L^{p,\varphi_1}_{\alpha,V}(\mathbb{R}^n)},$$

where C *is independent of* f.

(ii) If $p = 1$, *then for any* $t > 0$,

$$\varphi_2(x,r)^{-1}\Big(1 + \frac{r}{\rho(x)}\Big)^\alpha\ \Big|\{y \in B(x,r) : |[b,\ T]f(y)| > t\}\Big|$$
$$\le C \sup_{B(x,r)\subset\mathbb{R}^n} \Big(1 + \frac{r}{\rho(x)}\Big)^\alpha \varphi_1(x,r)^{-1} \int_{B(x,r)} \frac{|f(y)|}{t} \ln\Big(2 + \frac{|f(y)|}{t}\Big)\, dy$$

holds for all balls B, *where* C *is independent of* x, r, t *and* f.

Theorem 5.3. [23] *Suppose* $1 \le p < \infty$, $V \in B_n$, $\alpha \in (-\infty,\infty)$, $0<\beta<n$ *and* (φ_1,φ_2) *satisfies the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \le C\,\varphi_2(x,r), \tag{5.7}$$

where C *does not depend on* x *and* r.

(i) If $1<p<n/\beta$, $1/q = 1/p - \beta/n$, *then*

$$\|\mathcal{I}_\beta f\|_{L^{q,\varphi_2}_{\alpha,V}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\varphi_1}_{\alpha,V}(\mathbb{R}^n)},$$

where C *is independent of* f.

(ii) If $p=1$, $q=n/(n-\beta)$, *then for any* $t>0$,

$$t\Big(1+\frac{r}{\rho(x)}\Big)^\alpha \Big|\{y \in B(x,r) : \big|\mathcal{I}_\beta f(y)\big| > t\}\Big|^{1/q} \le C\,\varphi(x,r)|B(x,r)|\|f\|_{L^{1,\varphi_1}_{\alpha,V}(\mathbb{R}^n)}$$

holds for all balls B, *where* C *is independent of* x, r, t *and* f.

Theorem 5.4. [23] *Let* $1 \le p < \infty$, $b \in BMO_\rho$, $V \in B_n$, $\alpha \in (-\infty,\infty)$ *and* (φ_1,φ_2) *satisfies the condition*

$$\int_r^\infty \ln\Big(e+\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}+1}} dt \le C\,\varphi_2(x,r), \tag{5.8}$$

where C *does not depend on* x *and* r.

(i) If $1<p<n/\beta$, $1/q = 1/p - \beta/n$, *then*

$$\|[b,\ \mathcal{I}_\beta]f\|_{L^{q,\varphi_2}_{\alpha,V}(\mathbb{R}^n)} \le C\ \|f\|_{L^{p,\varphi_1}_{\alpha,V}(\mathbb{R}^n)},$$

where C *is independent of* f.

(ii) If $p=1$ *and* $q=n/(n-\beta)$, *then for any* $t>0$,

$$\varphi_2(x,r)^{-1}\Big(1+\frac{r}{\rho(x)}\Big)^\alpha \Big|\{y\in B(x,r) : |[b,\ \mathcal{I}_\beta]f(y)|>t\}\Big|$$

$$\le C \sup_{B(x,r)\subset\mathbb{R}^n} \Big(1+\frac{r}{\rho(x)}\Big)^\alpha \varphi_1(x,r)^{-1}\Phi\left(\int_{B(x,r)} \frac{|f(y)|}{t}\ln(2+\frac{|f(y)|}{t})dy\right)$$

holds for all balls B, *where* $\Phi(t) = (t\ \log(2+t^{\beta/n}))^{n/(n-\beta)}$, *and* C *is independent of* x, r, t *and* f.

6. Weighted Morrey spaces related to certain nonnegative potentials

Let $p\in[1,\ \infty)$, $\alpha\in(-\infty,\ \infty)$ and $\lambda\in[0,1)$. For $f\in L^p_{loc}(\mathbb{R}^n)$ and $V\in B_q$ $(q>1)$, we say $f\in L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ (weighted Morrey spaces related to the potential V) provided that

$$\|f\|^p_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0}\Big(1+\frac{r}{\rho(x)}\Big)^\alpha \omega(B(x,2r))^{-\lambda}\int_{B(x,r)}|f(y)|^p\omega(y)dy<\infty,$$

where $B=B(x,r)$ denotes a ball with centered at x and radius r, and the weight functions $\omega\in A^{\rho,\infty}_p$ which could be viewed as an extension of weighted Lebesgue spaces (that is, when $\alpha=\lambda=0$, $\|f\|_{L^{p,0}_{0,V,\omega}(\mathbb{R}^n)} = \|f\|_{L^p_\omega(\mathbb{R}^n)}$). In particular, when

$\alpha = 0$ or $V = 0$, $\omega = 1$ and $0 < \lambda < 1$, the space $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ is the class Morrey space $L^{p,\lambda}(\mathbb{R}^n)$ (see [44]). When $\alpha = 0$ or $V = 0$ and $0 < \lambda < 1$, $L^{p,\lambda}_{\omega}(\mathbb{R}^n)$ was first introduced in [41], where $\omega \in A_p(\mathbb{R}^n)$ (Muckenhoupt weights class). It is easy to see that $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n) \subset L^{p,\lambda}_{\omega}(\mathbb{R}^n)$ for $\alpha > 0$, and $L^{p,\lambda}_{\omega}(\mathbb{R}^n) \subset L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ for $\alpha < 0$. In addition, when $\omega = 1$, the $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ has been studied in [51].

From [53, 58], we know a class Schrödinger type operators such as $\nabla(-\Delta + V)^{-1}\nabla$ with $V \in B_n$, $\nabla(-\Delta+V)^{-1/2}$ with $V \in B_n$, $(-\Delta+V)^{-1/2}\nabla$ with $V \in B_n$, $(-\Delta+V)^{i\gamma}$ with $\gamma \in \mathbb{R}$ and $V \in B_{n/2}$, and $\nabla^2(-\Delta+V)^{-1}$ with V is a nonnegative polynomial, are standard Calderón-Zygmund operators; see [56]. In particular, the kernels K of operators above all satisfy

$$|K(x,y)| \le \frac{C_k}{\left(1 + \frac{|x-y|}{\rho(x)}\right)^k} \frac{1}{|x-y|^n}$$

for any $k \in \mathbb{N}$. Hence, in the rest of this paper, we always assume that T denotes the above operators.

Recently, Bongioanni, etc, [9] proved $L^p(\mathbb{R}^n)(1 < p < \infty)$ boundedness for commutators of Riesz transforms associated with Schrödinger operator with BMO_ρ functions which include the class BMO function, and they [10] established the weighted boundedness for Riesz transforms, fractional integrals and Littlewood-Paley functions associated with Schrödinger operator with weight $A_p^{\rho,\infty}$ class which includes the Muckenhoupt weight class. Very recently, the author [10, 11] established the weighted norm inequalities for some Schrödinger type operators, which include commutators of Riesz transforms, fractional integrals and Littlewood-Paley functions with BMO_ρ functions; see also [10, 11].

In the following we show the boundedness properties of some Schrödinger type operators on the weighted Morrey spaces $L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)$ [49]. This results are formulated as follows.

Theorem 6.1. [49] *Suppose $\alpha \in (-\infty,\ \infty)$ and $\lambda \in (0,1)$.*

(i) If $1 < p < \infty$ and $\omega \in A_p^{\rho,\infty}$, then

$$\|Tf\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

where C is independent of f.

(ii) If $p = 1$ and $\omega \in A_1^{\rho,\infty}$, then for any $t > 0$,

$$\omega(B(x,2r))^{-\lambda}\left(1 + \frac{r}{\rho(x)}\right)^{\alpha} t\omega\left(\{y \in B(x,r)\ :\ |Tf(y)| > t\}\right) \le C\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}$$

holds for all balls B, where C is independent of x, r, t and f.

Theorem 6.2. [49] *Suppose $b \in BMO_\rho$, $\alpha \in (-\infty,\ \infty)$ and $\lambda \in (0,1)$.*

(i) If $1 < p < \infty$ and $\omega \in A_p^{\rho,\infty}$, then

$$\|[b,\ T]f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

where C is independent of f.

(ii) If $p = 1$ and $\omega \in A_1^{\rho,\infty}$, then for any $t > 0$,

$$\omega\left(B(x,2r)\right)^{-\lambda}\left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \omega\left(\{y \in B(x,r) : |[b,\ T]f(y)| > t\}|\right) \le$$

$$C \sup_{B(x,r)\subset\mathbb{R}^n} \omega\left(B(x,2r)\right)^{-\lambda} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \int_{B(x,r)} \frac{|f(y)|}{t} \ln\left(2+\frac{|f(y)|}{t}\right)\omega(y)dy$$

holds for all balls $B(x,r)$, where C is independent of x, r, t and f.

Theorem 6.3. [49] *Suppose $V \in B_{n/2}$, $\alpha \in (-\infty,\infty)$ and $\lambda \in (0,1)$.*
(i) If $1<p<\infty$ and $\omega \in A_p^{\rho,\infty}$, then

$$\|g(f)\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

where C is independent of f.
(ii) If $p=1$ and $\omega \in A_1^{\rho,\infty}$, then for any $t>0$,

$$\omega\left(B(x,2r)\right)^{-\lambda} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} t\omega\left(\{y \in B(x,r) \ : \ |g(f)(y)|>t\}\right) \le C\|f\|_{L^{1,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)}$$

holds for all balls B, where C is independent of x, r, t and f.

Theorem 6.4. [49] *Suppose $V \in B_{n/2}$, $b \in BMO_\rho$, $\alpha \in (-\infty,\infty)$ and $\lambda \in (0,1)$.*
(i) If $1<p<\infty$ and $\omega \in A_p^{\rho,\infty}$, then

$$\|g_b(f)\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

where C is independent of f.
(ii) If $p=1$ and $\omega \in A_1^{\rho,\infty}$, then for any $t>0$,

$$\omega\left(B(x,2r)\right)^{-\lambda} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \omega\left(\{y \in B(x,r) : |g_b(f)(y)|>t\}|\right) \le$$
$$C \sup_{B(x,r)\subset\mathbb{R}^n} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} \omega\left(B(x,2r)\right)^{-\lambda} \int_{B(x,r)} \frac{|f(y)|}{t} \ln\left(2+\frac{|f(y)|}{t}\right)\omega(y)\,dy$$

holds for all balls B, where C is independent of x, r, t and f.

Theorem 6.5. [49] *Suppose $V \in B_{n/2}$, $\alpha \in (-\infty,\ \infty)$ and $0<\beta<n$.*
(i) If $1<p<n/\beta$, $1/q=1/p-\beta/n$, $l=q/p$, $0<\lambda<1/l$ and $\omega^q \in A^{\rho,\infty}_{1+q/p'}$, where $p'=p/(p-1)$, then

$$\|\mathcal{I}_\beta f\|_{L^{q,\nu\lambda}_{\alpha,V,\omega^q}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega^p}(\mathbb{R}^n)},$$

where C is independent of f.
(ii) If $p=1$, $q=n/(n-\beta)$, $0<\lambda<1$ and $\omega \in A_1^{\rho,\infty}$, then for any $t>0$,

$$\omega\left(B(x,2r)\right)^{-\lambda} \left(1+\frac{r}{\rho(x)}\right)^{\alpha} t\omega\left(\{y \in B(x,r) : |\mathcal{I}_\beta f(y)|>t\}\right)^{1/q} \le$$
$$C\|f\|_{L^{1,\lambda}_{\alpha,V,\omega^{1/q}}(\mathbb{R}^n)}$$

holds for all balls $B(x,r)$, where C is independent of x, r, t and f.

Let $b \in BMO_\rho$, we define the commutator of $\mathcal{I}_\beta$ by

$$[b,\mathcal{I}_\beta]f = b\mathcal{I}_\beta f - \mathcal{I}_\beta(bf).$$

Theorem 6.6. *Let* $b \in BMO,\ V \in B_{n/2},\ \alpha \in (-\infty,\ \infty)$ *and* $0 < \beta < n$.

(i) If $1 < p < n/\beta,\ 1/q = 1/p - \beta/n, \nu = q/p,\ 0 < \lambda < 1/nu$, *and* $\omega^q \in A^{\rho,\infty}_{1+q/p}$, *then*

$$\|[b, \mathcal{I}_\beta]f\|_{L^{q,\nu\lambda}_{\alpha,V,\omega^q}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega^p}(\mathbb{R}^n)},$$

where C *is independent of* f.

(ii) If $p = 1,\ q = n/(n-\beta),\ 0 < \lambda < 1$ *and* $\omega \in A^{\rho,\infty}_1$, *then for any* $t > 0$,

$$\omega\left(B(x,2r)\right)^{-\lambda}\left(1 + \frac{r}{\rho(x)}\right)^{\alpha}\omega\left(\{y \in B(x,r)\ :\ |[b,\mathcal{I}_\beta]f(y)| > t\}\right) \le$$

$$C \sup_{B(x,r)\subset\mathbb{R}^n} \omega\left(B(x,2r)\right)^{-\lambda}\left(1 + \frac{r}{\rho(x)}\right)^{\alpha}\Phi\left(\int_B \frac{|f(y)|}{t}\ln(2 + \frac{|f(y)|}{t})\Theta(\omega(y))dy\right)$$

holds for all balls $B = B(x,r)$, *where* $\Phi(t) = [t\log(2 + t^{\beta/n})]^{n/(n-\beta)}$ *and* $\Theta(t) = t^{1-\beta/n}\log(e + t^{-\beta/n})$, *and* C *is independent of* $x,\ r,\ t$ *and* f.

For the dual Riesz transform $T^* = \mathcal{L}^{-\frac{1}{2}}\nabla$ and its commutator operator $[b,\ T^*]$ associated with the Schrödinger operator $\mathcal{L}$ the following results was proved in [42].

Theorem 6.7. *([42]) Suppose* $V \in B_q$ *for* $n/2 \le q < n,\ \alpha \in (-\infty,\ \infty),\ \lambda \in (0,\ 1)$, *and* $1/p_0 = 1/q - 1/n$. *Then, for* $p_0' \le p < \infty$ *and* $\omega \in A^{\rho,\infty}_{p/p_0'}$,

$$\|T^*f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

where C *is independent off.*

Theorem 6.8. *([42]) Suppose* $V \in B_q$ *for* $n/2 \le q < n,\ b \in BMO_\rho,\ \alpha \in (-\infty,\ \infty),\ \lambda \in (0,\ 1)$, *and* p_0 *so that* $1/p_0 = 1/q - 1/n$. *Then, for* $p_0' \le p < \infty$ *and* $\omega \in A^{\rho,\infty}_{p/p_0'}$,

$$\|[b,\ T^*]f\|_{L^{p\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{L^{p\lambda}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

where C *is independent off.*

7. Generalized weighted Morrey spaces related to certain nonnegative potentials

Recently, Komori and Shirai [41] defined the weighted Morrey spaces $L^{p,\kappa}(w)$ and studied the boundedness of some classical operators such as the Hardy-Littlewood maximal operator, the Calderón-Zygmund operator on these spaces. Also, Guliyev in [22] first introduced the generalized weighted Morrey spaces $M^{p,\varphi}_w$ and studied the boundedness of the sublinear operators and their higher order commutators generated by Calderón-Zygmund operators and Riesz potentials in these spaces (see, also [40, 32, 47]). Note that, Guliyev [22] gave the concept of generalized weighted Morrey space which could be viewed as an extension of both $M^{p,\varphi}_w$ and $L^{p,\kappa}(w)$.

The generalized weighed Morrey spaces introduced in [22] are defined as follows.

Definition 7.1. Let $1 \le p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{R}^n$. We denote by $M_w^{p,\varphi}$ the generalized weighted Morrey space, the space of all functions $f \in L_w^{p,\mathrm{loc}}(\mathbb{R}^n)$ with finite norm

$$\|f\|_{M_w^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{L_{p,w}(B(x,r))},$$

where

$$\|f\|_{L_w^p(B(x,r))} = \left(\int_{B(x,r)} |f(y)|^p w(y)dy\right)^{\frac{1}{p}}.$$

Furthermore, $WM_w^{p,\varphi}$ is the weak generalized weighted Morrey space of all functions $f \in WL_w^{p,\mathrm{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{WM_w^{p,\varphi}} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, w(B(x,r))^{-\frac{1}{p}}\, \|f\|_{WL_w^p(B(x,r))} < \infty,$$

where $WL_w^p(B(x,r))$ denotes the weak L_w^p-space of measurable functions f for which

$$\|f\|_{WL_w^p(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{WL_w^p(\mathbb{R}^n)} = \sup_{t>0} t \left(\int_{\{y\in B(x,r):\,|f(y)|>t\}} w(y)dy\right)^{\frac{1}{p}}.$$

Remark 7.1. (1) If $w \equiv 1$, then $M^{p,\varphi}(1) = M^{p,\varphi}$ is the generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_w^{p,\varphi} = L^{p,\kappa}(w)$ is the weighted Morrey space.

(3) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M^{p,\varphi}(1) = L^{p,\lambda}(\mathbb{R}^n)$ is the classical Morrey space and $WM^{p,\varphi}(1) = WL^{p,\lambda}(\mathbb{R}^n)$ is the weak Morrey space.

(4) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(w) = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

Recently, in [22] (see, also [32]), introduced a weighted condition: If $1 \le p < q < \infty$, there exits a constant $C > 0$, such that, for any $x \in \mathbb{R}^n$ and $t > 0$,

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{q}}} \frac{dt}{t} \le C\, \varphi_2(x,r). \tag{7.1}$$

In [3], [26] obtained the boundedness results for the fractional operators $T_{\Omega,\alpha}$ with rough kernel and its commutators $[b, T_{\Omega,\alpha}]$ on generalized Morrey spaces when the pair (φ_1,φ_2) satisfies condition (5.4) and

$$\int_r^\infty \ln\left(e + \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) s^{\frac{n}{p}}}{t^{\frac{n}{q}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{7.2}$$

respectively.

In the paper [22] was proved the boundedness of $T_{\Omega,\alpha}$, $(\Omega \equiv 1)$ and their k-th order commutators $[b, T_{\Omega,\alpha}]^k$, $(\Omega \equiv 1)$ on generalized weighted Morrey spaces when $w \in A_p$ and the pair (φ_1, φ_2) satisfies the condition (7.1) or the following inequalities,

$$\int_r^\infty \ln^k \left(e + \frac{t}{r}\right) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{q}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{7.3}$$

where C does not depend on x and r. See also, [27] for the case of Calderón-Zygmund operators and [33] for the case of intrinsic square functions.

Let $p \in [1, \infty)$, $\alpha \in (-\infty, \infty)$ and $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function. For $f \in L^p_{loc}(\mathbb{R}^n)$ and $V \in B_q$ $(q > 1)$, we say $f \in M^{p,\varphi}_{\alpha,V,\omega}(\mathbb{R}^n)$ (generalized weighted Morrey spaces related to the potential V) provided that

$$\|f\|^p_{M^{p,\varphi}_{\alpha,V,\omega}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \left(1 + \frac{r}{\rho(x)}\right)^\alpha \varphi(x,r)^{-1}\, \omega(B(x,2r))^{-\frac{1}{p}} \int_{B(x,r)} |f(y)|^p \omega(y) dy < \infty,$$

where $B = B(x,r)$ denotes a ball with centered at x and radius r, and the weight functions $\omega \in A^{\rho,\infty}_p$ which could be viewed as an extension of weighted Lebesgue spaces. In particular, when $\alpha = 0$ or $V = 0$, the space $M^{p,\varphi}_{\alpha,V,\omega}(\mathbb{R}^n)$ is the generalized weighted Morrey spaces was first introduced in [22], where $\omega \in A_p(\mathbb{R}^n)$ (Muckenhoupt weights class).

Remark 7.2. (1) If $w \equiv 1$, then $M^{p,\varphi}_{\alpha,V,1} = M^{p,\varphi}_{\alpha,V}$ is the generalized Morrey space related to certain nonnegative potentials.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M^{p,\varphi}_{\alpha,V,w} = L^{p,\lambda}_{\alpha,V,\omega}$ is the weighted Morrey space related to certain nonnegative potentials.

(3) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ with $0 < \lambda < n$, then $M^{p,\varphi}_{\alpha,V,1} = L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)$ is the Morrey space related to certain nonnegative potentials.

(4) If $\varphi(x,r) \equiv \left(1 + \frac{r}{\rho(x)}\right)^{-\alpha} w(B(x,2r))^{-\frac{1}{p}}$, then $M^{p,\varphi}_{\alpha,V,\omega} = L_{p,w}(\mathbb{R}^n)$ is the weighted Lebesgue space.

It is well known that the boundedness of the standard Calderón-Zygmund operators and their commutators have been established on the generalized weighted Morrey spaces (see [22]). Hence, it will be an interesting question whether in [23] were establish the boundedness of Schrödinger type operators on the generalized weighted Morrey spaces related to certain nonnegative potentials. More precisely, was obtained the following results.

Theorem 7.1. [24] *Suppose* $1 \le p < \infty$, $V \in B_n$, $\alpha \in (-\infty,\infty)$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{q}}} \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{7.4}$$

where C *does not depend on* x *and* r.

(i) If $1 < p < \infty$ *and* $\omega \in A^{\rho,\infty}_p$*,then*

$$\|Tf\|_{M^{p,\varphi_2}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C \|f\|_{M^{p,\varphi_1}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

where C *is independent of* f.

(ii) If $p = 1$ *and* $\omega \in A_1^{\rho,\infty}$, *then for any* $t > 0$,

$$\varphi_2(x,r)^{-1}\,\omega(B(x,2r))^{-\frac{1}{p}}\left(1+\frac{r}{\rho(x)}\right)^{\alpha} t\omega\left(\{y \in B(x,r)\ :\ |Tf(y)| > t\}\right)$$
$$\le C\|f\|_{M^{1,\varphi_1}_{\alpha,V,\omega}(\mathbb{R}^n)}$$

holds for all balls $B = B(x,r)$, *where* C *is independent of* x, r, t *and* f.

Theorem 7.2. [24] *Suppose* $1 \le p < \infty$, $b \in BMO_\rho$, $V \in B_n$, $\alpha \in (-\infty,\infty)$ *and* (φ_1,φ_2) *satisfy the condition*

$$\int_r^\infty \ln\left(e+\frac{t}{r}\right)\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{q}}}\frac{dt}{t} \le C\,\varphi_2(x,r), \tag{7.5}$$

where C *does not depend on* x *and* r.

(i) If $1 < p < \infty$ *and* $\omega \in A_p^{\rho,\infty}$, *then*

$$\|[b,\ T]f\|_{M^{p,\varphi_2}_{\alpha,V,\omega}(\mathbb{R}^n)} \le C\|f\|_{M^{p,\varphi_1}_{\alpha,V,\omega}(\mathbb{R}^n)},$$

where C *is independent of* f.

(ii) If $p = 1$ *and* $\omega \in A_1^{\rho,\infty}$, *then for any* $t > 0$,

$$\varphi_2(x,r)^{-1}\,\omega(B(x,2r))^{-1}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\omega\left(\{y \in B(x,r) : |[b,\ T]f(y)| > t\}|\right) \le C\times$$

$$\sup_{B(x,r)\subset\mathbb{R}^n}\varphi_1(x,r)^{-1}\,\omega(B(x,2r))^{-1}\left(1+\frac{r}{\rho(x)}\right)^{\alpha}\int_{B(x,r)}\frac{|f(y)|}{t}\ln\left(2+\frac{|f(y)|}{t}\right)\omega(y)dy$$

holds for all balls $B(x,r)$, *where* C *is independent of* x, r, t *and* f.

8. A priori estimates for solution of the Schrödinger equations : the Calderón-Zygmund inequality

For the open set $\Omega \subset \mathbb{R}^n$, $\omega \in A_p^{\rho,\infty}(\mathbb{R}^n)$ $(1 \le p < \infty)$ and $V \in B_n$, we say $f \in L^{p,\lambda}_{\alpha,V,\omega}(\Omega)$, if

$$\|f\|^p_{L^{p,\lambda}_{\alpha,V}(\Omega)} = \sup_{B(x_0,r)\subset\mathbb{R}^n}\left(1+\frac{r}{\rho(x_0)}\right)^{\alpha} r^{-\lambda}\int_{B(x_0,r)\cap\Omega}|f(x)|^p dx < \infty.$$

In this section, we consider the behavior of the solution of the following Schöndinger equation

$$(-\Delta+V)u = f(x),\ \text{a.e. } x \in \Omega,$$

where $f \in L^{p,\lambda}_{\alpha,V,\omega}(\Omega)$, $1 < p < \infty$, $0 < \lambda < 1$, $\alpha \in (-\infty,\ \infty)$ and $\omega \in A_\infty^{\rho,\infty}$.

Theorem 8.1. *([49]) Let* Ω *be an open set in* $\mathbb{R}^n$ *and* $a \in (-\infty,\ \infty)$. *If* $f \in L^{p,\lambda}_{\alpha,V,\omega}(\Omega)$, *then there exists a function* $u \in L^{q,\nu\lambda}_{\alpha,V,\omega}(\Omega)$, *where* $1 < p < n/2, 1/p - 1/q = 2/n, \nu = q/p$, $0 < \lambda < 1/\nu$, *and* $\omega^q \in A^{\rho,\infty}_{1+q/p'}$, *such that*

$$(-\Delta+V)u = f(x),\ \text{a.e. } x \in \Omega.$$

Furthermore,

$$\|D^2u\|_{L^{p,\lambda}_{\alpha,V,\omega}(\Omega)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega}(\Omega)},$$

where $1 < p < \infty,\ 0 < \lambda < n$ *and* $\omega \in A_p^{\rho,\infty}$*;*

$$\|Du\|_{L^{q,\nu_1\lambda}_{\alpha,V,\omega^q}(\Omega)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega^p}(\Omega)},$$

where $1 < p < n,\ 1/p - 1/q = 1/n, \nu_1 = q/p,\ 0 < \lambda < n/\nu_1$ *and* $\omega^q \in A^{\rho,\infty}_{1+q/p'}$*;*

$$\|u\|_{L^{q,\nu\lambda}_{\alpha,V,\omega^q}(\Omega)} \le C\|f\|_{L^{p,\lambda}_{\alpha,V,\omega^p}(\Omega)},$$

where $1 < p < n/2, 1/p - 1/q = 2/n, \nu = q/p,\ 0 < \lambda < 1/\nu$ *and* $\omega^q \in A^{\rho,\infty}_{1+q/p'}$*.*

For the open set $\Omega \subset \mathbb{R}^n$, $\omega \in A_p^{\rho,\infty}(\mathbb{R}^n)$ $(1 \le p < \infty)$, $\varphi : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function and $V \in B_n$, we say $f \in M^{p,\varphi}_{\alpha,V,\omega}(\Omega)$, if

$$\|f\|^p_{M^{p,\lambda}_{\alpha,V}(\Omega)} =$$

$$\sup_{B(x_0,r)\subset\mathbb{R}^n} \left(1 + \frac{r}{\rho(x)}\right)^{\alpha} \varphi(x,r)^{-1}\, \omega(B(x,2r))^{-\frac{1}{p}} \int\limits_{B(x_0,r)\cap\Omega} |f(x)|^p dx < \infty.$$

In this section, we consider the behavior of the solution of the following Schöndinger equation

$$(-\Delta + V)u = f(x),\ \text{a.e. } x \in \Omega,$$

where $f \in M^{p,\varphi}_{\alpha,V,\omega}(\Omega)$, $1 < p < \infty$, $\alpha \in (-\infty,\ \infty)$, $\omega \in A_\infty^{\rho,\infty}$ and (φ, φ) satisfy the condition (7.4).

Theorem 8.2. *(*[24]*) Let* Ω *be an open set in* $\mathbb{R}^n$*,* $a \in (-\infty,\ \infty)$ *and* (φ, φ) *satisfy the condition* (7.4)*. If* $f \in M^{p,\varphi}_{\alpha,V,\omega}(\Omega)$*, then there exists a function* $u \in M^{q,\varphi^\nu}_{\alpha,V,\omega}(\Omega)$*, where* $1 < p < n/2,\ 1/p - 1/q = 2/n,\ \nu = q/p,$ *and* $\omega^q \in A^{\rho,\infty}_{1+q/p'}$*, such that*

$$(-\Delta + V)u = f(x)\ \ a.e.\ x \in \Omega.$$

Furthermore,

$$\|D^2u\|_{M^{p,\varphi}_{\alpha,V,\omega}(\Omega)} \le C\|f\|_{L^{p,\varphi}_{\alpha,V,\omega}(\Omega)},$$

where $1 < p < \infty$ *and* $\omega \in A_p^{\rho,\infty}$*;*

$$\|Du\|_{L^{q,\varphi^\nu}_{\alpha,V,\omega^q}(\Omega)} \le C\|f\|_{L^{p,\varphi}_{\alpha,V,\omega^p}(\Omega)},$$

where $1 < p < n,\ 1/p - 1/q = 1/n$ *and* $\omega^q \in A^{\rho,\infty}_{1+q/p'}$*;*

$$\|u\|_{L^{q,\varphi^\nu}_{\alpha,V,\omega^q}(\Omega)} \le C\|f\|_{L^{p,\varphi}_{\alpha,V,\omega^p}(\Omega)},$$

where $1 < p < n/2, 1/p - 1/q = 2/n$ *and* $\omega^q \in A^{\rho,\infty}_{1+q/p'}$*.*

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