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ABSOLUTE CONVERGENCE OF SPECTRAL EXPANSION IN EIGENFUNCTIONS OF THIRD ORDER ORDINARY DIFFERENTIAL OPERATOR

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In memory of M. G. Gasymov on his 75th birthday

Abstract. In the paper we consider a third order ordinary differential operator with summable coefficients. We study absolute convergence of spectral expansion of the function from the class $W_1^1(G)$ in eigenfunction of the given operator and establish uniform convergence rate of this expansion.

1. Introduction

Consider on the interval $G=(0,1)$ a formal differential operator

$$Lu = u^{(3)} + P_1(x)\,u^{(2)} + P_2(x)\,u^{(1)} + P_3(x)\,u$$

with summable coefficients $P_1(x)\in L_2(G)$, $P_\ell(x)\in L_1(G)$, $l=2,3$.

Denote by $D(G)$ the class of functions absolutely continuous together with its derivatives to second order inclusively, on the closed interval $\overline{G}=[0,1]$.

Under the eigenfunction of the operator L responding to the eigenvalue λ, we understand any function $u(x)\in D(G)$ identically not equal to zero and satisfying almost everywhere in G the equation (see [1])

$$Lu+\lambda u=0.$$

Let $\{u_n(x)\}_{n=1}^{\infty}$ be a complete system, orthonormalized in $L_2(G)$, consisting of eigenfunctions of the operator L, and $\{\lambda_n\}_{n=1}^{\infty}$ be a corresponding system of eigenvalues ($\operatorname{Re}\lambda_n=0$).

Denote

$$\mu_k = \begin{cases} (-i\lambda_n)^{1/3}, & if \quad \operatorname{Im}\lambda_n \ge 0, \\ (i\lambda_n)^{1/3}, & if \quad \operatorname{Im}\lambda_n < 0, \end{cases}$$

and consider the partial sum of orthogonal expansion of the function $f(x)\in W_1^1(G)$ in the system $\{u_n(x)\}_{n=1}^{\infty}$:

$$\sigma_\nu(x,f) = \sum_{\mu_n\le\nu} f_n u_n(x)\,, \nu>0,$$

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where $f_n = (f, u_n) = \int_0^1 f(x)\,\overline{u}_n(x)\,dx$.

Study the behavior of the difference $R_\nu(x,f) = \sigma_\nu(x,f) - f(x)$.

In the paper, the following theorem is proved.

Theorem. *Let the function $f(x)$ belong to the class $W_1^1(G)$, the system $\{u_n(x)\}_{n=1}^\infty$ by uniformly bounded, and the following conditions be fulfilled:*

$$\left| f(x) u_n^{(2)}(x) \Big|_0^1 \right| \le C(f)\,\mu_n^\alpha, \quad 0 \le \alpha < 2, \quad \mu_n \ge 4\pi; \tag{1}$$

$$\sum_{k=1}^\infty k^{-1}\omega_1\left(f', k^{-1}\right) < \infty, \qquad \sum_{k=1}^\infty k^{-1}\omega_1\left(f\overline{P}_1, k^{-1}\right) < \infty. \tag{2}$$

Then the expansion of the function $f(x)$ in the system $\{u_n(x)\}_{n=1}^\infty$ converges absolutely and uniformly on $\overline{G} = [0,1]$, and the following estimation is valid:

$$\sup_{x\in\overline{G}} |R_\nu(x,f)| \le$$

$$const\left\{ C(f)\,\nu^{\alpha-2} + (1 + \|P_1\|_1)\left[\sum_{k=[\nu]}^\infty k^{-1}\omega_1\left(f\overline{P}_1, k^{-1}\right) + \right.\right.$$

$$\left. + \sum_{k=[\nu]}^\infty \omega_1\left(f', k^{-1}\right)k^{-1} + \nu^{-1}\left(\left\|f'\right\|_1 + \left\|f\overline{P}_1\right\|_1\right)\right] +$$

$$\left. + \left(\left\|f'\right\|_1 + \left\|f\overline{P}_1\right\|_1 + \|f\|_\infty\right)\sum_{r=2}^3 \nu^{1-r}\,\|P_r\|_1 \right\}, \quad \nu \ge 2, \tag{3}$$

where $\omega_1(g,\delta)$ is the integral modulus of continuity of the function $g(x) \in L_1(G)$, $\|P_l\|_1 = \int_0^1 |P_l(x)|\,dx$, const is independent of $f(x)$.

For a second order operator such theorems were proved in the papers [5], [6], [8].

2. Preliminary results and proof of the theorem

For proving the theorem it is necessary to estimate the Fourier coefficients $f(x)$ in the system $\{u_n(x)\}_{n=1}^\infty$. To this end, we establish a representation for the eigenfunction $u_n(x)$.

Introduce some denotations:

$$R_1(\xi - x - t) = \begin{cases} \sum\limits_{j=1}^3 \omega_j e^{i\omega_j\mu_n(\xi - x + t)} & as \quad \operatorname{Im}\lambda_n \ge 0, \\ \sum\limits_{j=1}^3 \omega_j e^{-i\omega_j\mu_n(\xi - x + t)} & as \quad \operatorname{Im}\lambda_n < 0, \end{cases}$$

where $\omega_1 = -1$, $\omega_2 = e^{-i\pi/3}$, $\omega_3 = e^{i\pi/3}$;

$$M(u(\xi)) = \frac{1}{3\mu_n^2}\sum_{r=1}^3 P_r(\xi)\,u^{(3-r)}(\xi);$$

$$X_j^{\mp}(x)=\frac{1}{3}\sum_{m=0}^{2}(\mp i\mu_n)^{m-2}\,\omega_j^{m+1}u^{(2-m)}(x)\,.$$

Lemma 1. *For the eigenfunction $u_n(x)$ the following representations ($\lambda_n\neq 0$) are valid:*

$$u_n(x+t)=\sum_{j=1}^{3}X_j^{-}(x)\,e^{i\omega j\mu_n t}-\int_x^{x+t}M(u_n(\xi))\,R_1(\xi-x-t)\,d\xi, \tag{4}$$

if $\operatorname{Im}\lambda_n>0$*;*

$$u_n(x+t)=\sum_{j=1}^{3}X_j^{+}(x)\,e^{i\omega j\mu_n t}-\int_x^{x+t}M(u_n(\xi))\,R_1(\xi-x-t)\,d\xi, \tag{5}$$

if $\operatorname{Im}\lambda_n<0$.

Proof. For definiteness consider the case $\operatorname{Im}\lambda_n<0$. Multiply each side of the equation

$$Lu_n(\xi)+\lambda_n u_n(\xi)=0$$

by the function $R_1(\xi-x-t)$ and integrate with respect to ξ from x to $x+t$ $(x,\,x+t\in G)$

$$\sum_{j=1}^{3}\omega_j\int_x^{x+t}u_n^{(3)}(\xi)\,e^{-i\omega_j\mu_n(\xi-x-t)}d\xi+$$

$$\int_x^{x+t}\Big\{P_1(\xi)\,u_n^{(2)}(\xi)+P_2(\xi)\,u_n^{(1)}(\xi)+P_3(\xi)\,u_n(\xi)\Big\}\,R_1(\xi-x-t)\,d\xi+$$

$$\lambda_n\int_x^{x+t}u_n(\xi)\,R_1(\xi-x-t)\,d\xi=0. \tag{6}$$

We integrate by parts and use $\sum\limits_{j=1}^{3}\omega_j^{s}=-3\delta_{3s}$ (δ_{3s} is the Kronecker symbol) as follows

$$\sum_{j=1}^{3}\omega_j\int_x^{x+t}u_n^{(3)}(\xi)\,e^{-i\omega_j\mu_n(\xi-x-t)}d\xi=$$

$$\sum_{j=1}^{3}\omega_j\sum_{m=0}^{2}(-1)^m(-i\omega_j\mu_n)^m\left[u_n^{(3-m-1)}(x+t)-u_n^{(3-m-1)}(x)\,e^{i\omega_j\mu_n t}\right]+$$

$$\sum_{j=1}^{2}(-1)^3\,\omega_j\,(-i\omega_j\mu_n)^3\int_x^{x+t}u_n(\xi)\,e^{-i\omega_j\mu_n(\xi-x-t)}d\xi=3\,(i\mu_n)^2\,u_n(x+t)-$$

$$\sum_{j=1}^{3}e^{i\omega_j\mu_n}\left(\sum_{m=0}^{2}(i\mu_n)^m\,\omega^{m+1}u^{(3-m-1)}(x)\right)+(-i\mu_n)^3\int_x^{x+t}u_n(\xi)\,R_1(\xi-x-t)\,d\xi.$$

Here we take into account $(-i\mu_n)^3 = -\lambda_n$ and get

$$\sum_{j=1}^{3}\omega_j \int\limits_x^{x+t} u_n^{(3)}(\xi)\, e^{-i\omega_j\mu_n(\xi-x-t)}d\xi = 3\,(i\mu_n)^2\, u_n(x+t) -$$

$$\sum_{j=1}^{3} e^{i\omega_j\mu_n t}\left(\sum_{m=0}^{2}(i\mu_n)^m\,\omega_j^{m+1}u_n^{(3-m-1)}(x)\right) - \lambda_n \int\limits_x^{x+t} u_n(\xi)\, R_1(\xi-x-t)\, d\xi.$$

Having, substituted it in equality (6) and finding $u_n(x+t)$ from the obtained equality we get

$$u_n(x+t) = \sum_{j=1}^{3} e^{i\omega_j\mu_n t}\left(\frac{1}{3}\sum_{m=0}^{2}(i\mu_n)^{m-2}\,\omega_j^{m+1}u_n^{(2-m)}(x)\right) +$$

$$\frac{1}{3\mu_n^2}\int\limits_x^{x+t}\left\{P_1(\xi)\, u_n^{(2)}(\xi) + P_2(\xi)\, u_n^{(1)}(\xi) + P_3(\xi)\, u_n(\xi)\right\} R_1(\xi-x-t)\, d\xi.$$

Thus, formula (5) is established. The lemma 1 is proved.

For $x=0$ differentiating with respect to variable t ℓ times $\left(\ell=\overline{0,2}\right)$ formulas (4), (5), we find

$$\mu_n^{-\ell}u_n^{(\ell)}(t) = \sum_{j=1}^{3}(-i\omega_j)^\ell\, X_j^-(0)\, e^{-i\omega_j\mu_n t} -$$

$$\sum_{j=1}^{3}\omega_j(-i\omega_j)^\ell \int\limits_0^t M(u_n(\xi))\, e^{i\omega_j\mu_n(\xi-t)}d\xi \tag{7}$$

for $\operatorname{Im}\lambda_n > 0$;

$$\mu_n^{-\ell}u_n^{(\ell)}(t) = \sum_{j=1}^{3}(i\omega_j)^\ell\, X_j^+(0)\, e^{i\omega_j\mu_n t} +$$

$$\sum_{j=1}^{3}\omega_j(i\omega_j)^\varepsilon \int\limits_0^t M(u_n(\xi))\, e^{-i\omega_j\mu_n(\xi-t)}d\xi \tag{8}$$

for $\operatorname{Im}\lambda_n < 0$.

Write formulas (7) and (8) in a more convenient form:

$$\mu_n^{-\ell}u_n^{(\ell)} = \sum_{j=1}^{2} X_j^-(0)\,(-i\omega_j)^\ell\, e^{-i\omega_j\mu_n t} +$$

$$(-i\omega_3)^\ell\, B_3^-\, e^{i\omega_3\mu_n(1-t)} - \sum_{j=1}^{2}(-i)^\ell\,\omega_j^{\ell+1}\int\limits_0^t M(u_n(\xi))\, e^{i\omega_j\mu_n(\xi-t)}d\xi +$$

$$(-i)^\ell\,\omega_3^{\ell+1}\int\limits_t^1 M(u_n(\xi))\, e^{i\omega_3\mu_n(\xi-t)}d\xi, \quad \ell=\overline{0,2}, \tag{9}$$

where $B_3^- = X_3^-(0)\, e^{-i\omega_3\mu_n} - \omega_3 \int\limits_0^t M(u_n(\xi))\, e^{-i\omega_3\mu_n(\xi-t)} d\xi$;

$$\mu_n^{-\ell} u_n^{(\ell)}(t) = \sum_{j=1,\ j\neq 2}^{3} (i\omega_j)^\ell X_j^+(0)\, e^{i\omega_j\mu_n t} +$$

$$(i\omega_2)^\ell B_2^+ e^{-i\omega_2\mu_n(1-t)} + \sum_{j=1, j\neq 2}^{3} (i)^\ell \omega_j^{\ell+1} \int\limits_0^t M(u_n(\xi))\, e^{-i\omega_j\mu_n(\xi-t)} d\xi -$$

$$(i)^\ell \omega_2^{\ell+1} \int\limits_t^1 M(u_n(\xi))\, e^{-i\omega_2\mu_n(\xi-t)} d\xi, \quad \ell = \overline{0,2}, \tag{10}$$

where $B_2^+ = X_2^+(0)\, e^{i\omega_2\mu_n} + \omega_2 \int\limits_0^t M(u_n(\xi))\, e^{-i\omega_2\mu_n(\xi-t)} d\xi$.

Now, using formulas (9), (10), we estimate the Fourier coefficients of the function $f(x) \in W_1^1(G)$ in the system $\{u_n(x)\}_{n=1}^\infty$.

Lemma 2. *Let the system $\{u_n(x)\}_{n=1}^\infty$ be uniformly bounded, the function $f(x) \in W_1^1(G)$ and the system $\{u_n(x)\}_{n=1}^\infty$ satisfy conditions (1). Then for the Fourier coefficients f_n of the function $f(x)$ the estimation ($\mu_n \geq 4\pi$) is valid*

$$|f_n| \leq \left\{ C(f)\, \mu_n^{\alpha-3} + \mu_n^{-1}(1 + \|P_1\|_1) \left[\omega_1\left(f', \mu_n^{-1}\right) + \omega_1\left(f\overline{P}_1, \mu_n^{-1}\right) + \right.\right.$$

$$\left.\left. \mu_n^{-1}\left\|f'\right\|_1 + \mu_n^{-1}\left\|f'\overline{P}_1\right\|_1 \right] + \mu_n^{-2}\left(\left\|f'\right\|_1 + \left\|f'\overline{P}_1\right\|_1 + \|f\|_\infty\right) \sum_{r=2}^{3} \|P_r\|_1 \mu_n^{2-r} \right\}. \tag{11}$$

Proof. By definition of the eigenfunction $u_n(x)$ the Fourier coefficients f_n for $\mu_n \geq 1$ are calculated by the formula

$$f_n = (f, u_n) = \left(f, -\frac{Lu_n}{\lambda_n}\right) = -\frac{1}{\overline{\lambda}_n}(f, Lu_n) =$$

$$-\frac{1}{\overline{\lambda}_n}\left(f, u_n^{(3)} + P_1(x)\, u_n^{(2)} + P_2(x)\, u_n^{(1)} + P_3(x)\, u_n\right) =$$

$$-\frac{1}{\overline{\lambda}_n}\left(f, u_n^{(3)}\right) - \frac{1}{\overline{\lambda}_n}\left(f, P_1(x)\, u_n^{(2)}\right) - \frac{1}{\overline{\lambda}_n}\left(f, P_2(x)\, u_n^{(1)}\right) - \frac{1}{\overline{\lambda}_n}(f, P_3(x)\, u_n). \tag{12}$$

Here, using the estimations (see [4])

$$\left\|u_n^{(s)}\right\|_\infty \leq const\,(1+\mu_n)^{s+\frac{1}{p}} \|u_n\|_p, \quad p \geq 1, \quad s = \overline{0,2}, \tag{13}$$

and taking into account the uniform boundedness of the system $\{u_n(x)\}_{n=1}^\infty$, we get

$$\frac{1}{|\lambda_n|}\left|\left(f, P_2(x)\, u_n^{(1)}\right)\right| \leq \frac{const}{\mu_n^3} \|f\|_\infty \|u_n\|_\infty \|P_2\|_1 \mu_n \leq const\mu_n^{-2} \|f\|_\infty \|P_2\|_1; \tag{14}$$

$$\frac{1}{|\lambda_n|}|(f, P_3(x)\, u_n)| \leq \frac{const}{\mu_n^3} \|f\|_\infty \|P_3\|_1 \|u_n\|_\infty \leq const\mu_n^{-3} \|f\|_\infty \|P_3\|_1. \tag{15}$$

Perform integration by parts in the first term in the right side of equality (12):

$$-\frac{1}{\overline{\lambda}_n}\left(f,u_n^{(3)}\right)=-\frac{1}{\overline{\lambda}_n}\left[f\left(t\right)\overline{u_n^{(2)}\left(t\right)}\Big|_0^1-\int_0^1 f'\left(t\right)\overline{u_n^{(2)}\left(t\right)}dt\right].$$

Hence from condition (1) we have

$$\frac{1}{|\lambda_n|}\left|\left(f,u_n^{(3)}\right)\right|\leq C\left(f\right)\mu_n^{\alpha-3}+\frac{1}{|\lambda_n|}\left|\left(f',u_n^{(2)}\right)\right|. \tag{16}$$

Now estimate the expression $|\lambda_n|^{-1}\left|\left(f',u_n^{(2)}\right)\right|$. For that we use formulas (9) or (10) subject to the sign of $\operatorname{Im}\lambda_n$. For definiteness consider the case $\operatorname{Im}\lambda_n<0$. From formula (10)

$$\frac{1}{\lambda_n}\left(f',u_n^{(2)}\right)=\frac{-1}{i\mu_n^3}\left(f',u_n^{(2)}\right)=\frac{i}{\mu_n}\left(f',\frac{u_n^{(2)}}{\mu_n^2}\right)=$$

$$\frac{i}{\mu_n}\sum_{j=1,j\neq 2}^{3}\left(f',X_j^+\left(0\right)e^{i\omega_j\mu_nt}\left(i\omega_j\right)^2\right)+\frac{i}{\mu_n}\overline{\left(i\omega_2\right)^2B_2^+}\times$$

$$\left(f',e^{-i\omega_2\mu_n(1-t)}\right)+\frac{i}{\mu_n}\sum_{j=1,j\neq 2}^{3}\overline{i^2\omega_j^3}\left(f',\int_0^t M\left(u_n\left(\xi\right)\right)e^{-i\omega_j\mu_n(\xi-t)}d\xi\right)+$$

$$\frac{i}{\mu_n}\overline{\omega_2^3}\left(f',\int_t^1 M\left(u_n\left(\xi\right)\right)e^{-i\omega_2\mu_n(\xi-t)}d\xi\right). \tag{17}$$

Estimate each term in this equality. Obviously

$$\overline{\left(f',X_j^+\left(0\right)e^{i\omega_j\mu_nt}\right)}=X_j^+\left(0\right)\int_0^1\overline{f'\left(t\right)}e^{i\omega_j\mu_nt}dt,\quad j=1,3.$$

Taking into account the inequality

$$\left|X_j^+\left(0\right)\right|\leq const\left\|u_n\right\|_\infty\leq const,\quad j=1,3, \tag{18}$$

that follows from estimation (13), uniform boundedness of the system $\{u_n\left(x\right)\}_{n=1}^\infty$ and expression $\left|X_j^+\left(0\right)\right|$, using the estimations (see [2], [3])

$$\left|\int_0^1\overline{f'\left(t\right)}e^{i\omega_j\mu_nt}dt\right|\leq const\left\{\omega_1\left(f',\frac{1}{\mu_n}\right)+\frac{1}{\mu_n}\left\|f'\right\|_1\right\},\quad j=1,3,$$

we find

$$\left|\left(f',X_j^+\left(0\right)e^{i\omega_j\mu_nt}\right)\right|\leq const\left\{\omega_1\left(f',\frac{1}{\mu_n}\right)+\frac{1}{\mu_n}\left\|f'\right\|_1\right\}. \tag{19}$$

Estimate B_2^+. For that we write formula (10) for $\ell=0$. For B_2^+ from this formula we find

$$\left|B_2^+\right|\leq c\left\|\exp\left(-i\omega_2\mu_n\left(1-\cdot\right)\right)\right\|_\infty^{-1}\left\{\left\|u_n\right\|_\infty+\right.$$

$$\sum_{j=1,j\neq 2}^{3}\left|X_j^{+}(0)\right|+\sum_{j=1,j\neq 2}^{3}\left\|\int_0^{\cdot}|M(u_n(\xi))|\,d\xi\right\|_\infty+\left\|\int_{\cdot}^{1}|M(u_n(\xi))|\,d\xi\right\|_\infty\Bigg\}.$$

Taking into account estimations (18) and

$$|M(u_n(\xi))|\leq\frac{1}{3\mu_n^2}\sum_{r=1}^{3}|P_r(\xi)|\left|u_n^{(3-r)}(\xi)\right|\leq\frac{1}{3\mu_n^2}\sum_{r=1}^{3}|P_r(\xi)|\left\|u_n^{(3-r)}\right\|_\infty\leq$$

$$\frac{const}{\mu_n}\left[\sum_{r=1}^{3}|P_r(\xi)|\,\mu_n^{2-r}\right]\|u_n\|_\infty\leq\frac{const}{\mu_n}\left[\sum_{r=1}^{3}|P_r(\xi)|\,\mu_n^{2-r}\right]\tag{20}$$

we find $B_2^{+}\leq const$ for any $n\in N$.

Now apply the last estimation in the second term of equality (17). As a result we have

$$\left|\frac{i}{\mu_n}\overline{(i\omega_2)^2B_2^{+}}\left(f',e^{-i\omega_2\mu_n(1-t)}\right)\right|\leq\frac{const}{\mu_n}\left\{\omega_1\left(f',\frac{1}{\mu_n}\right)+\frac{1}{\mu_n}\left\|f'\right\|_1\right\}.\tag{21}$$

The third and fourth terms in equality (17) are estimated by the same scheme. Therefore we estimate the third term. For that we use the representation

$$M(u_n(\xi))=\frac{1}{3\mu_n^2}P_1(\xi)\,u_u^{(2)}(\xi)+\frac{1}{3\mu_n^2}\sum_{r=2}^{3}P_r(\xi)\,u^{(3-r)}(\xi).$$

Then

$$\left|\frac{i}{\mu_n}\sum_{j=1,j\neq 2}^{3}i^2\omega_j^3\left(f',\int_0^t M(u_n(\xi))\,e^{-i\omega_j\mu_n(\xi-t)}d\xi\right)\right|\leq$$

$$\frac{i}{3\mu_n^3}\sum_{j=1,j\neq 2}^{3}\left|\left(f',\int_0^t P_1(\xi)\,u_n^{(2)}(\xi)\,e^{-i\omega_j\mu_n(\xi-t)}d\xi\right)\right|+$$

$$\frac{const}{\mu_n^2}\sum_{r=2}^{3}\|P_r\|_1\,\mu_n^{2-r}\left\|f'\right\|_1.\tag{22}$$

After changing the integration order in the second term, we get that it doesn't exceed the quantity

$$\frac{const}{\mu_n}\sum_{j=1,j\neq 2}^{3}\int_0^1|P_1(\xi)|\left|\int_\xi^t\overline{f'(t)}e^{-i\omega_j\mu_n(\xi-t)}dt\right|d\xi.\tag{23}$$

Taking into account the following chain of inequalities (see [2], [7])

$$\left|\int_\xi^t\overline{f'(t)}e^{-i\omega_j\mu_n(\xi-t)}dt\right|\leq const\left\{\omega_1\left(g_\xi,\mu_n^{-1}\right)+\frac{1}{\mu_n}\|g_\xi\|_1\right\}\leq$$

$$const\left\{\omega_1\left(f',\mu_n^{-1}\right)+\mu_n^{-1}\left\|f'\right\|_1+\frac{\|f'\|_1}{\mu_n}\right\}\leq$$

$$const\left\{\omega_1\left(f',\mu_n^{-1}\right)+\mu_n^{-1}\left\|f'\right\|_1\right\},$$

where

$$g_{\xi}(z) = \begin{cases} \overline{f'}(\xi+z), & 0 \le z \le 1-\xi, \\ 0, & 1-\xi < z \le 1, \end{cases} \quad \xi \in [0,1],$$

we prove that expression (23) is bounded from above by the quantity

$$\frac{c}{\mu_n} \|P_1\|_1 \left\{ \omega_1\left(f', \mu_n^{-1}\right) + \frac{\|f'\|_1}{\mu_n} \right\}.$$

Consequently, the left side of (22) doesn't exceed the quantity

$$\frac{c}{\mu_n} \|P_1\|_1 \left\{ \omega_1\left(f', \mu_n^{-1}\right) + \frac{\|f'\|_1}{\mu_n} \right\}.$$

Hence and from estimations (19), (21) and (17) we find

$$\frac{1}{|\lambda_n|} \left| \left(f', u_n^{(2)}\right) \right| \le \frac{const}{\mu_n} \times$$

$$\left\{ (1+\|P_1\|) \left[\omega_1\left(f', \frac{1}{\mu_n}\right) + \frac{\|f'\|_1}{\mu_n} \right] + \frac{\|f'\|_1}{\mu_n} \sum_{r=2}^{3} \|P_r\|_1 \mu_n^{2-r} \right\}. \quad (24)$$

Estimate now the term $\frac{1}{\overline{\lambda}_n}\left(f, P_1(x)\, u_n^{(2)}\right)$ in equality (12).

Obviously

$$\left| \frac{1}{\overline{\lambda}_n} \left(f, P_1(x)\, u_n^{(2)}\right) \right| = \left| \frac{1}{\lambda_n} \right| \left| \left(f\overline{P}_1, u_n^{(2)}\right) \right|.$$

Since the function $f\overline{P}_1$ belongs to the class $L_1(G)$, we can apply estimation (24) with substitution of $f\overline{P}_1$ for f'. As a result, we have

$$\left| \frac{1}{\overline{\lambda}_n} \left(f, P_1(x)\, u_n^{(2)}\right) \right| \le$$

$$\frac{const}{\mu_n} \left\{ (1+\|P_1\|_1) \left[\omega_1\left(f\overline{P}_1, \frac{1}{\mu_n}\right) + \frac{\|f\overline{P}_1\|_1}{\mu_n} \right] + \frac{\|f\overline{P}_1\|_1}{\mu_n} \sum_{r=2}^{3} \|P_r\|_1 \mu_n^{2-r} \right\}. \quad (25)$$

Consequently, by (14)-(16), (24) and (25) from equality (12) we get

$$|f_n| \le const \left\{ C(f)\, \mu_n^{\alpha-3} + \mu_n^{-1} (1+\|P_1\|_1) \left[\omega_1\left(f', \mu_n^{-1}\right) + \omega_1\left(f\overline{P_1}, \mu_n^{-1}\right) + \right.\right.$$

$$\left.\left. \mu_n^{-1} \left\|f'\right\|_1 + \mu_n^{-1} \left\|f\overline{P_1}\right\|_1 \right] + \mu_n^{-2} \left(\left\|f'\right\|_1 + \left\|f\overline{P_1}\right\| + \|f\|_\infty \right) \sum_{r=2}^{3} \|P_r\|_1 \mu_n^{2-r} \right\}.$$

The case $\operatorname{Im}\lambda_n > 0$ is considered in the same way. The lemma 2 is proved.

Proof of the theorem. Prove that the series $\sum_{n=1}^{\infty} |f_n|\, |u_n(x)|$ converges uniformly on $\overline{G} = [0,1]$. For that we represent it in the form

$$\sum_{n=1}^{\infty} |f_n|\, |u_n(x)| = \sum_{0 \le \mu_n < 4\pi} |f_n|\, |u_n(x)| + \sum_{\mu_n \ge 4\pi} |f_n|\, |u_n(x)|.$$

By the orthonormality of the system $\{u_n(x)\}_{n=1}^{\infty}$ in $L_2(G)$ the condition of the "sum of units" (see [4]) is fulfilled

$$\sum_{\tau \le \mu_n < \tau+1} 1 \le const, \quad \forall \tau \ge 0. \quad (26)$$

Taking into account this and uniform boundedness of the system $\{u_n(x)\}_{n=1}^{\infty}$, we get

$$\sum_{0\le\mu_n<4\pi}|f_n|\,|u_n(x)|\le const\sum_{0\le\mu_n<4\pi}|f_n|\le const\,\|f\|_1\sum_{\mu_n\ge4\pi}1\le const\,\|f\|_1\,.$$

For estimating $\sum\limits_{\mu_n\ge4\pi}|f_n|\,|u_n(x)|$ we use the statement of the lemma 2, the condition of the "sum of units" (26) and condition (2). As a result we get

$$\sum_{\mu_n\ge4\pi}|f_n|\,|u_n(x)|\le const\sum_{\mu_n\ge4\pi}|f_n|\le$$

$$const\sum_{\mu_n\ge4\pi}\Big\{C(f)\,\mu_n^{\alpha-3}+\mu_n^{-1}\,(1+\|P_1\|_1)\times$$

$$\left[\omega_1\left(f',\mu_n^{-1}\right)+\omega_1\left(f\overline{P}_1,\mu_n^{-1}\right)+\mu_n^{-1}\left\|f'\right\|_1+\mu_n^{-1}\left\|f\overline{P}_1\right\|_1\right]+$$

$$\left.\mu_n^{-2}\left(\left\|f'\right\|_1+\left\|f\overline{P}_1\right\|_1+\|f\|_\infty\right)\sum_{r=2}^{3}\|P_r\|_1\,\mu_n^{2-r}\right\}\le const\times$$

$$\left\{C(f)\sum_{k=[4\pi]}^{\infty}\left(\sum_{k\le\mu_n\le k+1}\mu_n^{\alpha-3}\right)+(1+\|P_1\|_1)\left[\sum_{k=[4\pi]}^{\infty}\left(\sum_{k\le\mu_n\le k+1}\mu_n^{-1}\omega_1\left(f',\mu_n^{-1}\right)\right)\right.\right.$$

$$\left.+\sum_{k=[4\pi]}^{\infty}\left(\sum_{k\le\mu_n\le k+1}\mu_n^{-1}\omega_1\left(f\overline{P}_1,\mu_n^{-1}\right)\right)+\left(\left\|f'\right\|_1+\left\|f\overline{P}_1\right\|_1\right)\sum_{k=[4\pi]}^{\infty}\left(\sum_{k\le\mu_n\le k+1}\mu_n^{-2}\right)\right]$$

$$\left.+\left(\|f\|_\infty+\left\|f'\right\|_1+\left\|f\overline{P}_1\right\|_1\right)\sum_{r=2}^{3}\|P_r\|\left(\sum_{k=[4\pi]}^{\infty}\left(\sum_{k\le\mu_n\le k+1}\mu_n^{-\ell}\right)\right)\right\}\le const\times$$

$$\left\{C(f)\sum_{k=[4\pi]}^{\infty}k^{\alpha-3}\left(\sum_{k\le\mu_n\le k+1}1\right)+(1+\|P_1\|_1)\left[\sum_{k=[4\pi]}^{\infty}k^{-1}\omega_1\left(f',k^{-1}\right)\left(\sum_{k\le\mu_n\le k+1}1\right)\right.\right.$$

$$\left.+\sum_{k=[4\pi]}^{\infty}k^{-1}\omega_1\left(f\overline{P}_1,k^{-1}\right)\left(\sum_{k\le\mu_n\le k+1}1\right)+\left(\left\|f'\right\|_1+\left\|f\overline{P}_1\right\|_1\right)\sum_{k=[4\pi]}^{\infty}k^{-2}\left(\sum_{k\le\mu_n\le k+1}1\right)\right]$$

$$\left.+\left(\|f\|_\infty+\left\|f'\right\|_1+\left\|f\overline{P}_1\right\|_1\right)\sum_{\ell=2}^{3}\|P_\ell\|_1\sum_{k=[4\pi]}^{\infty}k^{-\ell}\left(\sum_{k\le\mu_n\le k+1}1\right)\right\}\le$$

$$const\left\{C(f)\,[4\pi]^{\alpha-2}+(1+\|P_1\|_1)\left[\sum_{k=[4\pi]}^{\infty}k^{-1}\omega_1\left(f',k^{-1}\right)+\right.\right.$$

$$\left.\sum_{k=[4\pi]}^{\infty}k^{-1}\omega_1\left(f\overline{P}_1,k^{-1}\right)+\left(\left\|f'\right\|_1+\left\|f\overline{P}_1\right\|_1\right)[4\pi]^{-1}\right]+$$

$$\left.\left(\|f\|_\infty+\left\|f'\right\|_1+\left\|f\overline{P}_1\right\|_1\right)\sum_{\ell=2}^{3}\|P_\ell\|_1\,[4\pi]^{1-\ell}\right\}<\infty.$$

Thus, the expansion $\sum\limits_{n=1}^{\infty} f_n u_n(x)$ converges absolutely and uniformly on $\overline{G}$.

From the completeness of the system $\{u_n(x)\}_{n=1}^{\infty}$ in $L_2(G)$ the given expansion uniformly converges exactly to the function $f(x)$. Consequently,

$$f(x) = \sum_{n=1}^{\infty} f_n u_n(x), \quad x \in \overline{G}. \tag{26}$$

Estimate now difference $R_\nu(x,f)$. For that we use equality (27), the uniform boundedness of the system $\{u_n(x)\}_{n=1}^{\infty}$, conditions (2), (26) and the lemma 2:

$$\sup_{x\in\overline{G}} |R_\nu(x,f)| = \sup_{x\in\overline{G}} |\sigma_\nu(x,f) - f(x)| = \sup_{x\in\overline{G}} \left| \sigma_\nu(x,f) - \sum_{n=1}^{\infty} f_n u_n(x) \right| =$$

$$\sup_{x\in\overline{G}} \left| \sum_{\mu_n>\nu} f_n u_n(x) \right| \le \sum_{\mu_n\ge\nu} |f_n| \, \|u_n\|_\infty \le C \sum_{\mu_n\ge\nu} |f_n| \le C \sum_{k=[\nu]}^{\infty} \left(\sum_{k\le\mu_n\le k+1} |f_n| \right) \le$$

$$\sum_{k=[\nu]}^{\infty} \left(\sum_{k\le\mu_n\le k+1} \left\{ C(f)\,\mu_n^{\alpha-3} + \left(1+\|P_1\|_1\right) \left[\mu_n^{-1}\omega_1\left(f',\mu_n^{-1}\right) + \mu_n^{-1}\omega_1\left(f\overline{P}_1,\mu_n^{-1}\right) \right.\right.\right.$$

$$\left.\left.\left. +\mu_n^{-2}\left(\left\|f'\right\|_1 + \left\|f\overline{P}_1\right\|_1\right)\right] + \left(\|f\|_\infty + \left\|f'\right\|_1 + \left\|f\overline{P}_1\right\|_1\right) \sum_{\ell=2}^{3} \mu_n^{-\ell} \|P_\ell\|_1 \right\} \right) \le$$

$$const \left\{ C(f)\,\nu^{\alpha-2} + \left(1+\|P_1\|_1\right) \left[\sum_{k=[\nu]}^{\infty} k^{-1}\omega_1\left(f\overline{P}_1,k^{-1}\right) + \sum_{k=[\nu]}^{\infty} \omega_1\left(f',k^{-1}\right) k^{-1} \right.\right.$$

$$\left.\left. +\nu^{-1}\left(\left\|f'\right\|_1 + \left\|f\overline{P}_1\right\|_1\right)\right] + \left(\left\|f'\right\|_1 + \left\|f\overline{P}_1\right\|_1 + \|f\|_\infty\right) \sum_{r=2}^{3} \nu^{1-r} \|P_r\|_1 \right\}.$$

Consequently, estimation (3) is established. The theorem is proved.

A number of corollaries follow from the proved theorem.

Corollary 1. *If $P_1(x) \equiv 0$, the system $\{u_n(x)\}_{n=1}^{\infty}$ is uniformly bounded, $f(x) \in W_1^1(G)$, $f(0) = f(1) = 0$ and $f'(x) \in H_1^\alpha(G)$, $0 < \alpha < 1$ ($H_1^\alpha(G)$ is the Nikolskii class), then*

$$\sup_{x\in\overline{G}} |R_\nu(x,f)| \le const\,\nu^{-\alpha} \left\|f'\right\|_1^\alpha,$$

where

$$\|g\|_1^\alpha = \|g\|_1 + \sup_{\delta>0} \frac{\omega_1(g,\delta)}{\delta^\alpha}.$$

Corollary 2. *If $P_1(x) \equiv 0$, the system $\{u_n(x)\}_{n=1}^{\infty}$ is uniformly bounded, $f(x) \in W_1^1(G)$, $f(0) = f(1) = 0$ and for some $\beta > 0$ the following estimation is fulfilled:*

$$\omega_1(f,\delta) = O\left(\ln^{-(1+\beta)} \frac{1}{\delta} \right), \quad \delta \to +0,$$

then

$$\sup_{x\in\overline{G}} |R_\nu(x,f)| = O\left(\ln^{-\beta} \nu \right), \quad \nu \to \infty.$$

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