

## ON PROPERTIES OF THE GENERALIZED BESOV-MORREY SPACES WITH DOMINANT MIXED DERIVATIVES

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**Abstract.** In this paper, a generalized space with the Besov-Morrey type parameters and dominant mixed derivatives is introduced. Some differential properties of functions from this space are studied.

### 1. Introduction

It is known that spaces with parameters constructed on the base of the Sobolev isotropic spaces  $W_p^{(l)}(G)$ , for some special values of indices, were first studied in the papers of Morrey [7,8]. Further these results were developed and generalized in the papers of V.P. Il'in [3], I. Ross [16], Yu. V. Netrusov [15], A. Mazzucato [6], V.S. Guliyev [2], Y. Sawano [17], L.Tang and J. Xu [18] and [9-14].

The goal of this paper is to investigate differential properties of functions from the space with parameters of the form

$$\bigcap_{i=1}^{2^n} L_{p^i, \theta^i, a, \kappa, \tau}^{<l^i>}(G_h) \tag{1.1}$$

Let  $e_n = \{1, \dots, n\}$ - be the set of natural numbers  $1, \dots, n$ . We denote by  $e$  any fixed subset of  $e_n$ (including the sets  $\emptyset$  and  $e_n$ ). The number of this subsets is equal to  $2^n$ . Assume that all these subsets are numbered, i.e.  $e^i$  ( $i = 1, \dots, 2^n$ )-are subsets (including  $\emptyset$  and  $e_n$ ) of  $e_n$ . With each set  $e^i$ , associate some fixed vector  $l^i = (l_1^i, \dots, l_n^i)$  with the components  $l_j^i \geq 0$  and the support  $e_{l^i} \supseteq e^i$ , i.e.  $l_j^i > 0$  ( $j \in e^i$ ),  $l_j^i \geq 0$  ( $j \in e_{l^i}/e^i$ ). The support of a vector  $l = (l_1, \dots, l_n)$  is the set of indices of nonzero components of the vector  $l$ . It is denoted by  $e_l$ .

### 2. Preliminaries

**Definition 2.1.** A space with parameters of the form (1.1) is defined as a normalized space of functions  $f$  on  $G$  with the finite norm

$$\|f\|_{\bigcap_{i=1}^{2^n} L_{p^i, \theta^i, a, \kappa, \tau}^{<l^i>}(G_h)} = \sum_{i=1}^{2^n} \|f\|_{L_{p^i, \theta^i, a, \kappa, \tau}^{<l^i>}(G_h)}, \tag{2.1}$$

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$$\|f\|_{L_{p^i, \theta^i, a, \varkappa, \tau} \langle l^i \rangle (G_h)} = \left\{ \int_0^{h_1} \dots \int_0^{h_n} \left[ \frac{\|\Delta^{m^i}(h, G_h) D^{k^i} f\|_{p^i, a, \varkappa, \tau}}{\prod_{j \in e_{l^i}} h_j^{l_j^i - k_j^i}} \right]^{\theta^i} \prod_{j \in e_{l^i}} \frac{dh_j}{h_j} \right\}^{\frac{1}{\theta^i}}, \quad (2.2)$$

$$\|f\|_{p^i, a, \varkappa, \tau; G} = \|f\|_{L_{p^i, a, \varkappa, \tau} (G)} = \sup_{x \in G} \left\{ \int_0^\infty \dots \int_0^\infty \left[ \prod_{j \in e_n} [t_j]_1^{-\frac{\varkappa_j a_j}{p^i}} \times \right. \right. \\ \left. \left. \times \|f\|_{p^i, G_t \varkappa(x)}^\tau \right] \prod_{j \in e_n} \frac{dt_j}{t_j} \right\}^{\frac{1}{\tau}}, \quad (2.3)$$

where  $p^i \in [1, \infty)$ ,  $\tau \in [1, \infty]$ ,  $l^i = (l_1^i, \dots, l_n^i)$ ,  $l_j^i > 0$  ( $j \in e^i$ ),  $l_j^i \geq 0$  ( $j \in e_{l^i}/e^i$ );  $m^i = (m_1^i, \dots, m_n^i)$ ,  $m_j^i > 0$  ( $j \in e^i$ ),  $m_j^i \geq 0$  ( $j \in e_{l^i}/e^i$ ) are integers,  $k^i = (k_1^i, \dots, k_n^i)$ ,  $k_j^i$  are non-negative integers. Assume that  $m_j^i \geq l_j^i - k_j^i \geq 0$ ; ( $j \in e_{l^i}/e^i$ ),  $m_j^i > l_j^i - k_j^i > 0$ , ( $j \in e^i$ ), ( $i = 1, \dots, 2^n$ ),

$$\Delta^{m^i}(h, G_h) f(x) = \Delta_1^{m_1^i}(h_1) \dots \Delta_n^{m_n^i}(h_n) f(x); \quad D^{k^i} f = D_1^{k_1^i} \dots D_n^{k_n^i} f(x);$$

$G_h = \{x : x + hI \in G\}$ ;  $G_{t \varkappa}(x) = G \cap I_{t \varkappa} = G \cap \left\{ y : |y_j - x_j| < \frac{1}{2} t_j^{\varkappa_j}, j \in e_n \right\}$ ,  $[t_j]_1 = \min\{1, t_j\}$ ,  $j \in e_n$ ;  $G$  be an open set of  $n$ -dimensional Euclidean space  $R^n$ . The space  $\bigcap_{i=1}^{2^n} L_{p^i, \theta^i, a, \varkappa, \tau} \langle l^i \rangle (G)$  is defined by the same way provided that in the norm (2.2) instead of  $\Delta^{m^i}(h, G_h) f$  it is written  $\Delta^{m^i}(h, G) f$ .

The space (1.1) in the case, when

$$l_j^i = \begin{cases} l_j > 0, & j \in e^i \\ 0, & j \in e_{l^i}/e^i \end{cases}$$

$p^i = p$ ,  $\theta^i = \theta$  ( $i = 1, \dots, 2^n$ ) coincides with the space  $S_{p, \theta, a, \varkappa, \tau}^l B(G_h)$ , introduced and studied in the paper [11]. In the case  $a = (0, \dots, 0)$ ,  $\tau = \infty$  it coincides with the space

$$\bigcap_{i=1}^{2^n} L_{p^i, \theta^i} \langle l^i \rangle (G)$$

which was introduced and studied in A.D. Jabrailov's paper [4].

When  $\tau = \infty$  the space  $L_{p, a, \varkappa, \tau}(G)$  coincides with the space  $L_{p, a, \varkappa}(G)$  introduced and studied in V.P. Il'in's paper [3].

Let us give some properties of the spaces  $L_{p, a, \varkappa, \tau}(G)$  and  $\bigcap_{i=1}^{2^n} L_{p^i, \theta^i, a, \varkappa, \tau} \langle l^i \rangle (G_h)$ .

1) For any  $\varkappa_j > 0$ ,  $0 \leq a_j \leq 1$ ,  $j = 1, \dots, n$  the following embeddings hold:

$$\|f\|_{p; G} \leq \|f\|_{p, a, \varkappa; G} \leq c \|f\|_{p, a, \varkappa, \tau; G}, \\ \|f\|_{\bigcap_{i=1}^{2^n} L_p \langle l^i \rangle (G_h)} \leq \|f\|_{\bigcap_{i=1}^{2^n} L_{p^i, a, \varkappa} \langle l^i \rangle (G_h)} \leq c \|f\|_{\bigcap_{i=1}^{2^n} L_{p^i, a, \varkappa, \tau} \langle l^i \rangle (G_h)}, \quad (2.4)$$

2) For any  $c > 0$

$$\|f\|_{p,a,c\kappa,\tau;G} \approx \|f\|_{p,a,\kappa,\tau;G}, \|f\|_{\bigcap_{i=1}^{2n} L_{p^i,\theta^i,a,c\kappa,\tau}^{<I^i>}(G)} \approx \|f\|_{\bigcap_{i=1}^{2n} L_{p^i,\theta^i,a,\kappa,\tau}^{<I^i>}(G)} \quad (2.5)$$

3) For any  $\kappa_j > 0, j = 1, \dots, n$  the following relations are valid:

$$\text{a) } \|f\|_{p,0,\kappa,\infty;G} = \|f\|_{p;G}, \|f\|_{\bigcap_{i=1}^{2n} L_{p^i,\theta^i,0,\kappa,\infty}^{<I^i>}(G_h)} = \|f\|_{\bigcap_{i=1}^{2n} L_{p^i,\theta^i}^{<I^i>}(G_h)}$$

$$\text{b) } \|f\|_{p,1,\kappa,\tau;G} \geq \|f\|_{\infty;G}, \|f\|_{\bigcap_{i=1}^{2n} L_{p^i,\theta^i,1,\kappa,\tau}^{<I^i>}(G_h)} \geq \|f\|_{\bigcap_{i=1}^{2n} L_{\infty,\theta^i}^{<I^i>}(G_h)}.$$

Let  $\psi_{e^i}(\cdot, y, z) \in C_0^\infty(R^n)$  be such that  $S(\psi_{e^i}) = \text{supp}\psi_{e^i}$ ,  $S(\psi_{e^i}) \subset I_1 = \{x : |x_j| < \frac{1}{2}, j \in e_n\}$ ,  $0 < T_j \leq 1, j \in e_n$ . Set  $V = \bigcup_{0 \leq t_j \leq T_j, j \in e_n} \{y : (\frac{y}{t}) \in S(\psi_{e^i})\}$ .

It is clear that  $V \subset I_T, U$  is an open set contained in the domain  $G$ . We'll also assume that  $U + V \subset G$ . Let

$$G_{T^\kappa}(U) = \bigcup_{x \in G} G_{T^\kappa}(x) = (U + I_{T^\kappa}(x)) \cap G.$$

Note that if  $0 < \kappa_j \leq 1, 0 < T_j \leq 1, j \in e_n$  and  $I_T \subset I_{T^\kappa}$ , then

$$U + V \subset G_{T^\kappa e^i}(U) = Z.$$

**Lemma 2.1.** Let  $1 \leq p^i \leq p \leq r \leq \infty, 0 < \kappa_j \leq 1, 0 < t_j, \eta_j \leq T_j \leq 1, 0 < \rho_j < \infty (j \in e_n), 1 \leq \tau \leq \infty, \nu = (\nu_1, \dots, \nu_n), \nu_j \geq 0$  be integers ( $j \in e_n$ ),

$$\varepsilon_j^i = l_j^i - \nu_j - (1 - \kappa_j a_j) \left( \frac{1}{p^i} - \frac{1}{p} \right)$$

and

$$B_\eta^i(x) = \prod_{j \in e_n \setminus e^i} T_j^{-1-\nu_j} \int_0^{\eta^{e^i}} \prod_{0^{e^i} j \in e^i} t_j^{-3-\nu_i} \varphi_i(x, t) dt^{e^i} \quad (2.6)$$

$$B_{\eta,T}^i(x) = \prod_{j \in e_n \setminus e^i} T_j^{-1-\nu_j} \int_{\eta^{e^i}}^{T^{e^i}} \prod_{\eta^{e^i} j \in e^i} t_j^{-3-\nu_i} \varphi_i(x, t) dt^{e^i} \quad (2.7)$$

where

$$\begin{aligned} \varphi_i(x, t) &= \int_{R^n - \infty}^\infty \int_{-\infty}^\infty \dots \int_{-\infty}^\infty \psi_{e^i} \left( \frac{y}{t^{e^i} + T^{e_n \setminus e^i}}, \frac{\rho(t^{e^i} + T^{e_n \setminus e^i}, x)}{t^{e^i} + T^{e_n \setminus e^i}} \right) \times \\ &\times \zeta^i \left( \frac{u}{t}, \frac{\rho(t, x)}{2t}, \frac{1}{2} \rho'(t, x) \right) \Delta^{m^i}(\delta u) f(x + y + u) dudy, \end{aligned} \quad (2.8)$$

Then the following inequalities hold

$$\begin{aligned} \sup_{\bar{x} \in U} \|B_\eta^i(x)\|_{p, U, \rho^\kappa(\bar{x})} &\leq C_1 \left\| \prod_{j \in e^i} t_j^{-l_j^i} \Delta^{m^i}(t; Z) f \right\|_{p^i, a, \kappa, \tau} \prod_{j \in e_n} [\rho_j]_1^{\frac{\kappa_j a_j}{p}} \times \\ &\times \prod_{j \in e^i} \eta_j^{\varepsilon_j^i} \quad (\varepsilon_j^i > 0) \end{aligned} \quad (2.9)$$

$$\begin{aligned} \sup_{\bar{x} \in U} \|B_{\eta, T}^i(x)\|_{p, U_{\rho^\kappa}(\bar{x})} &\leq C_1 \left\| \prod_{j \in e^i} t_j^{-l_j^i} \Delta^{m^i}(t; Z) f \right\|_{p^i, a, \kappa, \tau} \times \\ &\times \prod_{j \in e_n} [\rho_j]_1^{-\frac{\kappa_j a_j}{p}} \begin{cases} \prod_{j \in e^i} T_j^{\varepsilon_j^i}, \text{ if } \varepsilon_j^i > 0 \\ \prod_{j \in e^i} \ln \frac{T_j}{\eta_j}, \text{ if } \varepsilon_j^i = 0 \\ \prod_{j \in e^i} \eta_j^{\varepsilon_j^i}, \text{ if } \varepsilon_j^i < 0 \end{cases} \end{aligned} \quad (2.10)$$

Here  $U_{\rho^\kappa}(\bar{x}) = \left\{ x : |x_j - \bar{x}_j| < \frac{\rho_j^{\kappa_j}}{2}, j \in e_n \right\}$ ,  $C_1$  and  $C_2$  are constants independent of  $\Phi, \rho, \eta$  and  $T$ .

**Proof.** Apply the generalized Minkowski inequality for  $B_\eta^i(x)$  determined by equality (2.6) Then we get

$$\|B_\eta^i\|_{p, U_{\rho^\kappa}(\bar{x})} \leq \prod_{j \in e_n \setminus e^i} T_j^{-1-\nu_j} \int \prod_{0^{e^i} j \in e^i} t_j^{-3-\nu_i} \|\varphi_i(x, t)\|_{p, U_{\rho^\kappa}(\bar{x})} dt^{e^i}. \quad (2.11)$$

Estimate the norm  $\|\varphi_i(\cdot, t)\|_{p, U_{\rho^\kappa}(\bar{x})}$ . From the Holder inequality ( $p \leq r$ ) we have

$$\|\varphi_i(\cdot, t)\|_{p, U_{\rho^\kappa}(\bar{x})} \leq \|\varphi_i(\cdot, t)\|_{p, U_{\rho^\kappa}(\bar{x})} \prod_{j \in e_n} \rho_j^{\kappa_j \left( \frac{1}{p} - \frac{1}{r} \right)}. \quad (2.12)$$

Let  $X$  be a characteristic function of the set  $S(\psi_{e^i})$ . Considering that  $1 \leq p \leq r \leq \infty$ ,  $s \leq r$  ( $\frac{1}{s} = 1 - \frac{1}{p^i} + \frac{1}{r}$ ), represent the subintegrand function in (2.8) in the form

$$\begin{aligned} \left| \int_{-\infty^{e^i}}^{\infty^{e^i}} \psi_{e^i} \zeta_{e^i} \Delta^{m^i} f du \right| &= \left( \left| \int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i} \Delta^{m^i} f du \right|^{p^i} |\psi_{e^i}|^s \right)^{\frac{1}{r}} \times \\ &\times \left( \left| \int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i} \Delta^{m^i} f du \right|^{p^i} X \left( \frac{y}{t^{e^i} + T e_n \setminus e^i} \right) \right)^{\frac{1}{p^i} - \frac{1}{r}} (|\psi_{e^i}|^s)^{\frac{1}{p^i} - \frac{1}{r}}. \end{aligned}$$

We can apply the Holder inequality again and get

$$\begin{aligned} \|\varphi_i(\cdot, t)\|_{r, U_{\rho^\kappa}(\bar{x})} &\leq \\ &\leq \sup_{x \in U} \left( \int_{R^n} \left| \int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i} \left( \frac{u}{t}, \frac{\rho(t, x)}{t}, \frac{1}{2} \rho'(t, x) \right) \times \right. \right. \\ &\times \Delta^{m^i}(\delta u) f(x + y + u^{e^i}) du \left. \right|^{p^i} X \left( \frac{y}{t} \right) dy \right)^{\frac{1}{p^i} - \frac{1}{r}} \end{aligned}$$

$$\begin{aligned}
& \times \sup_{y \in V} \left( \int_{U_{\rho^\varkappa}(\bar{x})} \left| \int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i} \left( \frac{u}{t}, \frac{\rho(t, x)}{t}, \frac{1}{2} \rho'(t, x) \right) \times \right. \right. \\
& \quad \left. \left. \times \Delta^{m^i} (\delta u) f \left( x + y + u^{e^i} \right) du \right|^{p^i} X \left( \frac{y}{t} \right) dy \right)^{\frac{1}{r}} \\
& \times \left( \int_{R^n} \left| \psi_{e^i} \left( \frac{y}{t^{e^i} + T^{e_n \setminus e^i}}, \frac{\rho \left( t^{e^i} + T^{e_n \setminus e^i}, x \right)}{t^{e^i} + T^{e_n \setminus e^i}} \right) \right|^s dy \right)^{\frac{1}{s}}. \quad (2.13)
\end{aligned}$$

Taking into account that  $G_{t^{e^i} + T^{e_n \setminus e^i}}(x) \subset G_{(t^\varkappa)^{e^i} + (T^\varkappa)^{e_n \setminus e^i}}(x)$  for  $0 < t \leq 1$ ,  $\varkappa_j \leq 1$  ( $j \in e_n$ ) for any  $x \in U$  we have

$$\begin{aligned}
& \int_{R^n} \left| \int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i} \left( \frac{u}{t}, \frac{\rho(t, x)}{t}, \frac{1}{2} \rho'(t, x) \right) \times \right. \\
& \quad \left. \times \Delta^{m^i} (\delta u) f \left( x + y + u^{e^i} \right) du \right|^{p^i} X \left( \frac{y}{t^{e^i} + T^{e_n \setminus e^i}} \right) dy \leq \\
& \leq \int_{(U+V)_{(t^\varkappa)^{e^i} + (T^\varkappa)^{e_n \setminus e^i}}(x)} \left| \int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i} \left( \frac{u}{t}, \frac{\rho(t, x)}{t}, \frac{1}{2} \rho'(t, x) \right) \times \right. \\
& \quad \left. \times \Delta^{m^i} (\delta u) f \left( y + u^{e^i} \right) du \right|^{p^i} dy \leq \\
& \leq \prod_{j \in e_{l^i}} t^{p^i l_j^i} \left\| \int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i} \left( \frac{u}{t}, \frac{\rho(t, x)}{t}, \frac{1}{2} \rho'(t, x) \right) \times \right. \\
& \quad \left. \times \prod_{j \in e_{l^i}} t_j^{-l_j^i} \Delta^{m^i} (\delta u^{e^i}) f du^{e^i} \right\|_{p^i, G_{(t^\varkappa)^{e^i} + (T^\varkappa)^{e_n \setminus e^i}}(x)}^{p^i} \leq \\
& \leq \prod_{j \in e_{l^i}} t^{p^i + p^i l_j^i} \left\| \prod_{j \in e_{l^i}} t_j^{-l_j^i} \Delta^{m^i} (t, Z_t) f \right\|_{p^i, a, \varkappa}^{p^i} \prod_{j \in e_n \setminus e^i} T_j^{\varkappa_j a_j}, \quad (2.14)
\end{aligned}$$

for  $y \in V$

$$\begin{aligned}
& \int_{U_{\rho^\varkappa}(\bar{x})} \left| \int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i} \left( \frac{u}{t}, \frac{\rho(t, x)}{t}, \frac{1}{2} \rho'(t, x) \right) \Delta^{m^i} (\delta u^{e^i}) f \left( x + y + u^{e^i} \right) du^{e^i} \right|^{p^i} dx \leq \\
& \leq \int_{Z_{\rho^\varkappa}(\bar{x}+y)} \left| \int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_{e^i} \left( \frac{u}{t}, \frac{\rho(t, x)}{t}, \frac{1}{2} \rho'(t, x) \right) \Delta^{m^i} (\delta u^{e^i}) f \left( x + u^{e^i} \right) du^{e^i} \right|^{p^i} dx \leq
\end{aligned}$$

$$\begin{aligned}
&\leq \prod_{j \in e_{l_i}} t^{p^i l_j^i} \left\| \int_{-\infty^{e^i}}^{\infty^{e^i}} \zeta_i \left( \frac{u}{t}, \frac{\rho(t, x)}{t}, \frac{1}{2} \rho'(t, x) \right) \times \right. \\
&\quad \left. \times \Delta^{m^i} \left( \delta u^{e^i} \right) f \left( x + u^{e^i} \right) du^{e^i} \right\|_{p^i, Z_{\rho^i}(\bar{x}+y)}^{p^i} \leq \\
&\leq \prod_{j \in e_{l_i}} t^{p^i + p^i l_j^i} \left\| \prod_{j \in e_{l_i}} t_j^{-l_j^i} \Delta^{m^i} (t, Z_t) f \right\|_{p^i, a, \varkappa}^{p^i} \prod_{j \in e_n} [\rho]_1^{\varkappa_j a_j}, \quad (2.15) \\
&\int_{\mathbb{R}^n} \left| \psi_{e^i}^{(\nu)} \left( \frac{y}{t^{e^i} + T_{e_n \setminus e^i}}, \frac{\rho(t^{e^i} + T_{e_n \setminus e^i}, x)}{t^{e^i} + T_{e_n \setminus e^i}} \right) \right|^s dy = \prod_{j \in e^i} t_j \prod_{j \in e_n \setminus e^i} T_j^{-1-\nu_j} \left\| \psi_{e^i}^{(\nu)} \right\|^s. \quad (2.16)
\end{aligned}$$

From inequalities (2.12) – (2.16), we have

$$\begin{aligned}
\|\varphi_i(\cdot, t)\|_{r, U_{\rho^i}(\bar{x})} &\leq C \left\| \prod_{j \in e_{l_i}} t_j^{-l_j^i} \Delta^{m^i} (t, Z_t) f \right\|_{p^i, a, \varkappa} \prod_{j \in e_n \setminus e^i} T_j^{1 - \left(\frac{1}{p^i} - \frac{1}{r}\right)(1 - \varkappa_j a_j)} \times \\
&\quad \times \prod_{j \in e_{l_i}} t_j^{2 + l_j^i - \left(\frac{1}{p^i} - \frac{1}{r}\right)(1 - \varkappa_j a_j)} \prod_{j \in e_n} [\rho_j]_1^{\varkappa_j a_j} \prod_{j \in e_n} \rho_j^{\left(\frac{1}{p^i} - \frac{1}{r}\right)\varkappa_j}. \quad (2.17)
\end{aligned}$$

For any  $\tau, 1 \leq \tau \leq \infty$  taking into account the inequality  $\|\cdot\|_{p, a, \varkappa} \leq c \|\cdot\|_{p, a, \varkappa, \tau}$  and substituting inequality (2.17) in (2.11) for  $r = p$  we get inequality (2.9). Inequality (2.10) is proved by the same way.

**Corollary 2.1.** *For  $1 \leq \tau_1 \leq \tau_2 \leq \infty$  the following inequality is valid:*

$$\|\varphi_i(\cdot, t)\|_{p, b, \varkappa, \tau_2, U} \leq C^1 \left\| \prod_{j \in e_{l_i}} t_j^{-l_j^i} \Delta^{m^i} (t, Z_t) f \right\|_{p^i, a, \varkappa, \tau_1}. \quad (2.18)$$

where  $b \in [0, 1]^n$ ,  $C^1$  is a constant dependent on  $t$  and  $M_i$ , but independent of  $f$ .

**Lemma 2.2.** *Let  $1 \leq p^i \leq p \leq \infty, 0 < \varkappa_j \leq 1, 0 < T_j \leq 1, (j \in e_n), \nu = (\nu_1, \dots, \nu_n), \nu_j \geq 0$  be integers ( $j = 1, \dots, n$ );  $1 \leq \tau_1 \leq \tau_2 \leq \infty, \varepsilon_j^i > 0$  and*

$$\varepsilon_j^{i, 0} = l_j^i - \nu_j - (1 - \varkappa_j a_j) \frac{1}{p^i}.$$

*Then for the function  $B_T^i$  determined by equality (2.6) the following inequality is valid:*

$$\|B_T^i\|_{p, b, \varkappa, \tau_2, U} \leq C^2 \left\| \prod_{j \in e_{l_i}} t_j^{-l_j^i} \Delta^{m^i} (t, Z_t) f \right\|_{p^i, a, \varkappa, \tau_1}. \quad (2.19)$$

where  $b = (b_1, \dots, b_n), b_j$  are arbitrary numbers satisfying the inequalities:

$$\begin{aligned}
0 \leq b_j \leq 1, & \quad \text{if } \varepsilon_j^{i, 0} > 0 \quad (j \in e^i) \\
0 \leq b_j < 1, & \quad \text{if } \varepsilon_j^{i, 0} = 0 \quad (j \in e^i) \\
0 \leq b_j < a_j + \frac{\varepsilon_j^{i, 0} p(1-a_j)}{1-\varkappa_j a_j} = a_j + \frac{\varepsilon_j^i p(1-a_j)}{1-\varkappa_j a_j}, & \quad \text{if } \varepsilon_j^{i, 0} < 0 \quad (j \in e^i)
\end{aligned} \quad (2.20)$$

$j \in e^i, 0 \leq b_j \leq a_j, j \in e_n \setminus e^i$  and  $C^2$  is a constant independent of  $f$ .

**Proof.** Estimate  $\|B_T^i\|_{p, U_{\rho^\varkappa}(\bar{x})}, \bar{x} \in U, 0 < \rho_j < \infty (j \in e_n)$ . First let  $0 < \rho_j < T_j (j \in e_n)$ . Then

$$\|B_T^i\|_{p, U_{\rho^\varkappa}(\bar{x})} \leq \|B_\rho^i\|_{p, U_{\rho^\varkappa}(\bar{x})} + \|B_{\rho T}^i\|_{p, U_{\rho^\varkappa}(\bar{x})}. \quad (2.21)$$

From inequality (2.9) ( $\eta_j = \rho_j, j \in e_n$ ) for  $\tau = \infty$  we have

$$\|B_\rho^i\|_{p, U_{\rho^\varkappa}(\bar{x})} \leq C_1 \left\| \prod_{j \in e^i} t_j^{-l_j^i} \Delta^{m^i}(t, Z_t) f \right\|_{p^i, a, \varkappa} \prod_{j \in e_n} \rho_j^{\varepsilon_j^i + \frac{\varkappa_j a_j}{p^i}}, \quad (2.22)$$

where  $C$  is independent of  $f$  and  $\rho$ . Further, by the generalized Minkowski inequality and inequality (2.17) we have:

$$\begin{aligned} \|B_{\rho T}^i\|_{p, U_{\rho^\varkappa}(\bar{x})} &\leq \int_{\rho^{e^i}}^{T^{e^i}} \prod_{j \in e^i} t_j^{-3-\nu_j} \|\varphi_i(x, t)\|_{p, U_{\rho^\varkappa}(\bar{x})} dt^{e^i} \leq \\ &\leq C_2 \left\| \prod_{j \in e^i} t_j^{-l_j^i} \Delta^{m^i}(t, Z_t) f \right\|_{p^i, a, \varkappa} \psi(\rho, T, r), \end{aligned} \quad (2.23)$$

where  $\psi(\rho, T, r) = \prod_{j \in e_n} \rho_j^{\delta_j(r)} \int_{\rho^{e^i}}^{T^{e^i}} \prod_{j \in e^i} t_j^{\varepsilon_j^i(r)-1} dt^{e^i}$ ,

$$\varepsilon_j^i(r) = l_j^i - \nu_j - (1 - \varkappa_j a_j) \left( \frac{1}{p^i} - \frac{1}{p} \right), \delta_j(r) = \frac{\varkappa_j}{p} - \frac{\varkappa_j}{r} (1 - a_j)$$

and  $C_2$  is a constant independent of  $f$  and  $\rho$ . Estimation (2.23) is valid for any  $r, (p \leq r \leq \infty)$ . Choose  $r$  from the all possible numbers such that the index of power of  $\rho_j$  in this estimation is maximal. For this, note that  $\delta(r)$  monotonically increases,  $\varepsilon_j^i(r)$  monotonically decreases on  $[p, \infty]$  and  $\varepsilon_j^i(p) = \varepsilon_j^i, \varepsilon_j^i(\infty) = \varepsilon_j^{i,0}$ . We consider two cases:  $\varepsilon_j^{i,0} \geq 0$  and  $\varepsilon_j^{i,0} < 0$ . If  $\varepsilon_j^{i,0} \geq 0$ , then maximum index of  $\rho_j$  in this estimation we get for  $r = \infty$ . Considering that  $\delta_j(\infty) = \frac{\varkappa_j}{p}$ , we have

$$\psi(\rho, T, r) = \begin{cases} \frac{1}{\varepsilon_j^{i,0}} \prod_{j \in e_n} \rho_j^{\frac{\varkappa_j}{p}} \left( \prod_{j \in e_n} T_j^{\varepsilon_j^{i,0}} - \prod_{j \in e_n} \rho_j^{\varepsilon_j^{i,0}} \right), & \text{if } \varepsilon_j^{i,0} > 0 \\ \prod_{j \in e_n} \rho_j^{\frac{\varkappa_j}{p}} \ln \prod_{j \in e_n} \frac{T_j}{\rho_j}, & \text{if } \varepsilon_j^{i,0} = 0 \end{cases}$$

Let  $\varepsilon_j^{i,0} < 0$ . Since  $\varepsilon_j^i(p) = \varepsilon_j^i > 0$ , and  $\varepsilon_j^i(\infty) = \varepsilon_j^{i,0} < 0$ , then at some  $r_0, (p \leq r_0 \leq \infty)$ ,  $\varepsilon_j^i(r_0) = 0$ . It is easy to see that the best estimation in this case is obtained if in  $\psi(\rho, T, r)$  we put  $r = r_0$ . Then  $\psi(\rho, T, r_0) = \prod_{j \in e_n} \rho_j^{\delta_j(r_0)} \ln \prod_{j \in e^i} \frac{T_j}{\rho_j}$ ,

where

$$\delta_j(r_0) = \frac{\varkappa_j}{p} \left( 1 + \frac{\varepsilon_j^{i,0} p (1 - a_j)}{1 - \varkappa_j a_j} \right) = \frac{\varkappa_j}{p} \left( a + \frac{\varepsilon_j^{i,0} p (1 - a_j)}{1 - \varkappa_j a_j} \right),$$

$(1 - \varkappa_j a_j, \text{ since } \varepsilon_j^i > 0 \text{ and } \varepsilon_j^{i,0} < 0)$ . Improving, on the base of it, the estimation (2.23) and taking into account that

$$\varepsilon_j^i + \frac{\varkappa_j a_j}{p} \geq \frac{\varkappa_j}{p} \quad \text{for } \varepsilon_j^{i,0} \geq 0$$

$$\varepsilon_j^i + \frac{\varkappa_j a_j}{p} \geq \frac{\varkappa_j}{p} \left( a + \frac{\varepsilon_j^{i,0} p (1 - a_j)}{1 - \varkappa_j a_j} \right) = \delta_j(r_0) \quad \text{for } \varepsilon_j^{i,0} < 0$$

and inequalities (2.21) – (2.23), we get  $(\tau = \infty)$

$$\begin{aligned} \|B_T^i\|_{p, U_{\rho^\varkappa}(\bar{x})} &\leq C_3 \left\| \prod_{j \in e_{li}} t_j^{-l_j^i} \Delta^{m^i}(t, Z_t) f \right\|_{p^i, a, \varkappa} \times \\ &\times \psi(\rho, T, r) = \begin{cases} \frac{1}{\varepsilon_j^{i,0}} \prod_{j \in e_n} \rho_j^{\frac{\varkappa_j}{p}} \left( \prod_{j \in e_n} T_j^{\varepsilon_j^{i,0}} - \prod_{j \in e_n} \rho_j^{\varepsilon_j^{i,0}} \right), & \text{if } \varepsilon_j^{i,0} > 0 \\ \prod_{j \in e_n} \rho_j^{\frac{\varkappa_j}{p}} \ln \prod_{j \in e_n} \frac{T_j}{\rho_j}, & \text{if } \varepsilon_j^{i,0} = 0 \end{cases} \times \\ &\times \begin{cases} \prod_{j \in e_n} \rho_j^{\frac{\varkappa_j}{p}} \prod_{j \in e_n} T_j^{\varepsilon_j^{i,0}}, & \text{if } \varepsilon_j^{i,0} > 0 \\ \prod_{j \in e_n} \rho_j^{\frac{\varkappa_j}{p}} \ln \prod_{j \in e_n} \frac{T_j}{\rho_j}, & \text{if } \varepsilon_j^{i,0} = 0 \\ \prod_{j \in e_n} \rho_j^{\delta_j(r_0)} \ln \prod_{j \in e_{li}} \frac{T_j}{\rho_j}, & \text{if } \varepsilon_j^{i,0} > 0 \end{cases} \end{aligned} \quad (2.24)$$

Let  $\rho_j \geq T_j (j \in e_n)$ . Again applying estimation (2.9) ( $\eta_j \leq T_j, j \in e_n, \tau = \infty$ ) we have

$$\|B_T^i\|_{p, U_{\rho^\varkappa}(\bar{x})} \leq C_4 \left\| \prod_{j \in e_{li}} t_j^{-l_j^i} \Delta^{m^i}(t, Z_t) f \right\|_{p^i, a, \varkappa} \begin{cases} \prod_{j \in e_n} \rho_j^{\varepsilon_j^i + \frac{\varkappa_j a_j}{p}}, & \text{if } T_j \leq \rho_j \leq 1 \\ \prod_{j \in e_n} T_j^{\varepsilon_j^i}, & \text{if } 1 < \rho_j < \infty. \end{cases} \quad (2.25)$$

It follows from inequalities (2.24) and (2.25) that for any  $\bar{x} \in U$  and any  $\rho, 0 < \rho < \infty$

$$\|B_T^i\|_{p, U_{\rho^\varkappa}(\bar{x})} \leq C_5 \left\| \prod_{j \in e_{li}} t_j^{-l_j^i} \Delta^{m^i}(t, Z_t) f \right\|_{p^i, a, \varkappa} \prod_{j \in e_n} [\rho_j]_1^{\frac{\varkappa_j b_j}{p}},$$

where  $b_j (j = 1, \dots, n)$  are the numbers satisfying inequalities (2.20), and  $C_5$  is a constant independent of  $f, \rho$  and  $\bar{x}$ . Considering the relation  $1 \leq \tau_1 \leq \tau_2 \leq \infty$ , from the last inequality we get inequality (2.19)

### 3. Main results

In this section, we prove two theorems on some properties of functions from the space  $\bigcap_{i=1}^{2^n} L_{p^i, \theta^i, a, \varkappa, \tau}^{< l^i >}(G_h)$ .

**Theorem 3.1.** *Let the open set  $G \subset R^n$  satisfy the condition of flexible  $(A_1)$  cone ([10]),  $1 \leq p^i \leq p \leq \infty$ ,  $(i = 0, 1, 2, \dots, 2^n)$ ;  $\bar{\varkappa} = c\varkappa$ ,  $\frac{1}{c} = \max_{j \in e_n} l_j \varkappa_j$ ,*

*$\nu = (\nu_1, \dots, \nu_n)$ ,  $\nu_j \geq 0$  be integers  $j \in e_n$ ;  $\nu_j \geq l_j^i$  for  $j \in e_i \setminus e^i$ ,  $1 \leq \tau_1 \leq \tau_2 \leq \infty$ ;*

$$\varepsilon_j^i = l_j^i - \nu_j - (1 - \varkappa_j a_j) \left( \frac{1}{p^i} - \frac{1}{p} \right) > 0, \left( j \in e^i, i = 1, \dots, 2^n \right)$$

*and let  $f \in \bigcap_{i=1}^{2^n} L_{p^i, a, \varkappa, \tau}^{<l^i>}(G_h)$ . Then  $D^\nu : \bigcap_{i=1}^{2^n} L_{p^i, \theta^i, a, \varkappa, \tau_1}^{<l^i>}(G_h) \hookrightarrow L_{p, b, \varkappa, \tau_2}(G)$ . More exactly, for  $f \in \bigcap_{i=1}^{2^n} L_{p^i, a, \varkappa, \tau}^{<l^i>}(G_h)$  in domain  $G$  there exists a generalized derivative  $D^\nu f$ , for which the following inequalities are valid:*

$$\|D^\nu f\|_{p, G} \leq C_1 \sum_{i=1}^{2^n} \prod_{j \in e_n} T_j^{\varepsilon_j^i} \|f\|_{L_{p^i, a, \varkappa, \tau_1}^{<l^i>}(G_t)}, \quad (3.1)$$

and

$$\|D^\nu f\|_{p, b, \varkappa, \tau_2; G} \leq C_2 \|f\|_{\bigcap_{i=1}^{2^n} L_{p^i, \theta^i, a, \varkappa, \tau_1}^{<l^i>}(G_t)}, \quad (p^i \leq p < \infty, i = 1, 2, \dots, 2^n). \quad (3.2)$$

In particular if,

$$\varepsilon_{j,0}^i = l_j^i - \nu_j - (1 - \bar{\varkappa}_j a_j) \frac{1}{p^i} > 0, \quad (j \in e^i, j = 1, \dots, 2^n)$$

then  $D^\nu f$  is continuous on  $G$  and

$$\sup_{x \in G} |D^\nu f(x)| \leq C_1 \sum_{i=1}^{2^n} \prod_{j \in e_n} T_j^{\varepsilon_{j,0}^i} \|f\|_{L_{p^i, a, \varkappa, \tau_1}^{<l^i>}(G_t)}, \quad (3.3)$$

where  $T_j \in (0, \min(1, T_{j,0})]$  ( $j \in e_n$ ),  $C_1, C_2$  are the constants independent of  $f$ ,  $C_1$  is independent also on  $T$  and  $b = (b_1, \dots, b_n)$ ,  $b_j$  satisfying

$$\begin{aligned} 0 &\leq b_j \leq 1, \text{ if } \varepsilon_{j,0}^i > 0; \\ 0 &\leq b_j < 1, \text{ if } \varepsilon_{j,0}^i = 0; \\ 0 &\leq b_j < a + \frac{\varepsilon_{j,0}^i p (1 - a_j)}{1 - \varkappa_j a_j}, \text{ if } \varepsilon_{j,0}^i < 0; \end{aligned}$$

$j \in e^j, 0 \leq b_j < a_j, j \in e_n \setminus e^i$  but with substitution  $\bar{\varkappa}$  for  $\varkappa$ .

**Proof.** First of all note that since  $\bar{\varkappa} = c\varkappa, c > 0$ , then  $f \in \bigcap_{i=1}^{2^n} L_{p^i, \theta^i, a, \bar{\varkappa}, \tau_1}^{<l^i>}(G)$  and everywhere in inequalities (3.1) – (3.3) we can replace  $\varkappa$  by  $\bar{\varkappa}$ . Passage from  $\varkappa$  to  $\bar{\varkappa}$ , is explained with the following properties of the vector  $\bar{\varkappa}$ :

1)  $\bar{\varkappa}_j \leq 1, j \in e_n$ ; 2) among the vectors of the form  $d\varkappa : d > 0$ , satisfying the inequality  $d\varkappa \leq 1$ , the vector  $\bar{\varkappa}$  has the greatest absolute value.

The importance of the first property will be seen from the proof of the theorem, and the significance of the second property follows from the theorem statement (the more  $\varkappa_j$ , the more  $\varepsilon_j^i$ ). Under the conditions of our theorem there exists a generalized derivative  $D^\nu f$ . Indeed, if  $\varepsilon_j^i > 0$  ( $j \in e^i, i = 1, \dots, 2^n$ ), from  $p^i \leq$

$p, \bar{\alpha}_j \leq 1, (j \in e_n), a \in [0, 1]^n$  it follows that  $l_j - \nu_j > 0, j \in e^i (j \in e^i, i = 1, \dots, 2^n)$ . This means that for  $f \in \bigcap_{i=1}^n L_{p^i, \theta^i, a, \bar{\alpha}, \tau_1}^{<l^i>}(G) \hookrightarrow \bigcap_{i=1}^n L_{p^i, \theta}^{<l^i>}(G)$  there exists  $D^\nu f$  and  $D^\nu f \in L_{p^i}(G)$ . Then for almost each point  $x \in G$  the following equality holds

$$\begin{aligned} D^\nu f(x) &= \sum_{i=1}^{2^n} (-1)^{|\omega^{e^i}| + |\nu|} \prod_{j \in e_n \setminus e_i} T_j^{-1 - \nu_j} \int_{0^{e^i}}^{T^{e^i}} \frac{dt^{e^i}}{\prod_{j \in e^i} t_j^{3 + \nu_j}} \times \\ &\times \int_{R^n} \int_{-\infty^{e^i}}^{\infty^{e^i}} \psi_{e^i}^{(\nu)} \left( \frac{y}{t^{e^i} + T^{e_n \setminus e^i}}, \frac{\rho(t^{e^i} + T^{e_n \setminus e^i}, x)}{t^{e^i} + T^{e_n \setminus e^i}} \right) \times \\ &\times \zeta_{e^i} \left( \frac{u}{t}, \frac{\rho(t, x)}{t}, \frac{1}{2} \rho'(t, x) \right) \Delta^{m^i}(\delta u) f(x + y + u^{e^i}) du^{e^i} dy. \end{aligned} \quad (3.4)$$

Note that  $\psi_{e^i}^{(\nu)}(\cdot, z) \in C_0^\infty(R^n), \zeta_i \in C_0^\infty(R)$ , their supports are contained in  $I_1$  and are such that the supports of representations (3.4) are contained in the flexible cone  $x + V \subset G$ . The parameter of the representation  $\delta > 0$  is rather small, owing to which the replacement of  $\Delta^{m^i}(\delta u, G) f (i = 1, \dots, 2^n)$  in the right side (3.4) by  $\Delta^{m^i}(\delta u, G_{\delta u})$  doesn't influence on the value of the right side. On the base of Minkowski inequality we have

$$\|D^\nu f\|_{p, G} = \sum_{i=1}^{2^n} \prod_{j \in e_n \setminus e_i} T_j^{-1 - \nu_j} \|B_T^i\|_{p, G}, \quad (3.5)$$

and by means of inequality (2.9) for  $U \in G, \psi_i^{(\nu)} = M_i, \rho \rightarrow \infty$ , we get

$$\|B_T^i\|_{p, G} \leq C_2 \prod_{j \in e_n} T_j^{\varepsilon_j^i} \left\| \prod_{j \in e_{i^i}} t_j^{-l_j^i} \Delta^{m^i}(t, Z_t) f \right\|_{p^i, a, \bar{\alpha}, \tau_1}. \quad (3.6)$$

By means of inequalities (3.5)–(3.6) under condition  $1 \leq \theta^i \leq \infty (i = 1, \dots, 2^n)$  we get inequality (3.1). By the similar way, based on inequalities (2.19), estimation (3.2) can be established.

Now let  $\varepsilon_j^{i,0} > 0 (j \in e^i, i = 1, \dots, 2^n)$ . Let us show that  $D^\nu f$  is continuous on  $G$ . On the base of identities (3.4), from inequality (3.5) for  $p = \infty, \varepsilon_j^i = \varepsilon_j^{i,0} > 0 (j \in e^i), 1 \leq \theta^i \leq \infty (i = 1, 2, \dots, 2^n)$ , we obtain that

$$\|D^\nu f - \bar{f}_T^{(\nu)}\|_{\infty, G} \leq \sum_{e^i \neq 0} \prod_{j \in e_n} T_j^{\varepsilon_j^{i,0}} \left\{ \int_0^{t_0} \left\| \prod_{j \in e_{i^i}} t_j^{-l_j^i} \Delta^{m^i}(t, Z_t) f \right\|_{p^i, a, \bar{\alpha}, \tau_1}^{\theta^i} \frac{dt}{t} \right\}^{\frac{1}{\theta^i}},$$

Hence the left hand side of the inequality tends to zero as  $T_j \rightarrow 0 (j \in e_n)$ . Since  $\bar{f}_T^{(\nu)}$  is continuous on  $G$ , the convergence in  $L_\infty(G)$  coincides at the present case with the uniform one, and consequently the limit function  $D^\nu f$  is continuous on  $G$ . Theorem 3.1 is proved.

Now let  $\gamma$  be  $n$ -dimensional vector.

**Theorem 3.2.** *Let domain  $G$ , parameters  $p^i, p, \tau$  and vectors  $\varkappa, a, \varepsilon_j^i, \varepsilon_{j,0}^i, \nu$  satisfy the conditions of theorem 3.1. Then for  $\varepsilon_j^i > 0, (j \in e^i, i = 1, \dots, 2^n)$  the derivative  $D^\nu f$  satisfies on  $G$  the Holder condition with the exponent  $\beta_j$ , more exactly*

$$\|\Delta(\gamma, G) D^\nu f\|_{p,G} \leq C \|f\|_{\bigcap_{i=1}^{2^n} L_{p^i, \theta^i, a, \varkappa, \tau}(G^i)}^{2^n} \prod_{j \in e_n} |\gamma_j|^{\beta_j}, \quad (3.7)$$

here  $\beta = (\beta_1, \dots, \beta_n), \beta_j$  is any number satisfying the inequalities:

$$\begin{aligned} 0 &\leq \beta_j \leq 1, \text{ if } \varepsilon_j^i > 1, \\ 0 &\leq \beta_j < 1, \text{ if } \varepsilon_j^i = 1, \\ 0 &\leq \beta_j < \varepsilon_j^i, \text{ if } \varepsilon_j^i < 1. \end{aligned} \quad (3.8)$$

where  $(j \in e_n, i = 1, \dots, 2^n), C$  is a constant independent of  $f$  and  $\gamma$   
In particular, if  $\varepsilon_j^{i,0} > 0$ , then

$$\sup_{x \in G} |\Delta(\gamma, G) D^\nu f(x)| \leq C \|f\|_{\bigcap_{i=1}^{2^n} L_{p^i, \theta^i, a, \varkappa, \tau}(G^i)}^{2^n} \prod_{j \in e_n} |\gamma_j|^{\beta_j^0}, \quad (3.9)$$

where  $\beta_j^0$  satisfies the same conditions as  $\sigma_j$ , but with replacement of  $\varepsilon_j^i$  by  $\varepsilon_j^{i,0}$ .

**Proof.** According to lemma 8.6 in [1], there exists the domain  $G_\sigma \subset G$  ( $\sigma = (\sigma_1, \dots, \sigma_n), \sigma_j = \xi_j r(x), \xi_j > 0, r(x) = \rho(x, \partial G), x \in G$ ). Assume that  $|\gamma_j| < \sigma_j (j \in e_n)$ , then for any  $x \in G_\sigma$  the interval connecting the points  $x, x + \gamma$  is contained in  $G$  and for all the points of this interval, identities (3.4) are valid with the same kernels. After some transformations we have

$$\begin{aligned} &|\Delta(\gamma, G) D^\nu f(x)| \leq \\ &\leq C_1 \sum_{i=1}^{2^n} \prod_{j \in e_n \setminus e^i} T_j^{-1-\nu_j} \int_0^{|\gamma|^{e^i}} \frac{dt^{e^i}}{\prod_{j \in e^i} t_j^{3+\nu_j}} \times \\ &\times \int_{R^n} \int_{-\infty}^{\infty} \left| \zeta_{e^i} \left( \frac{u}{t}, \frac{\rho(t, x)}{t}, \frac{1}{2} \rho'(t, x) \right) \right| \left| \psi_{e^i}^{(\nu)} \left( \frac{y}{t^{e^i} + T^{e_n \setminus e^i}}, \frac{\rho(t^{e^i} + T^{e_n \setminus e^i}, x)}{t^{e^i} + T^{e_n \setminus e^i}} \right) \right| \times \\ &\times \left| \Delta(\gamma, G) \Delta^{m^i}(\delta u) f(x + y + u^{e^i}) \right| du^{e^i} dy \leq \\ &\leq C_2 \sum_{i=1}^{2^n} \prod_{j \in e_n \setminus e^i} T_j^{-1-\nu_j} \int_{|\gamma|^{e^i}}^{T^{e^i}} \frac{dt^{e^i}}{\prod_{j \in e^i} t_j^{3+\nu_j}} \times \\ &\times \int_{R^n} \int_{-\infty}^{\infty} \left| \psi_{e^i}^{(\nu+1)} \left( \frac{y}{t^{e^i} + T^{e_n \setminus e^i}}, \frac{\rho(t^{e^i} + T^{e_n \setminus e^i}, x)}{t^{e^i} + T^{e_n \setminus e^i}} \right) \right| \left| \zeta_{e^i} \left( \frac{u}{t}, \frac{\rho(t, x)}{t}, \frac{1}{2} \rho'(t, x) \right) \right| \times \\ &\times \int_0^1 \dots \int_0^1 \left| \Delta^{m^i}(\delta u) \right| f(x + y + u + v_1 \gamma_1 + \dots + v_n \gamma_n) dv_1 \dots dv_n dy = \end{aligned}$$

$$= C_1 \sum_{i=1}^{2^n} \prod_{j \in e_n \setminus e^i} T_j^{-1-\nu_j} (B_\gamma^i(x, \gamma) + B_{\gamma T}^i(x, \gamma)). \quad (3.10)$$

here  $|\gamma|^{e^i} = (|\gamma_1|^{e^i}, \dots, |\gamma_n|^{e^i})$ ,  $|\gamma_j|^{e^i} = |\gamma|$  for  $j \in e^i$ ;  $|\gamma_j|^{e^i} = 0$  for  $j \in e_n \setminus e^i$ , ( $i = 1, \dots, 2^n$ ),  $0 < T_j \leq \min(1, T_{j,0})$ ,  $j \in e_n$ . We also assume that  $|\gamma_j| < T_j$  ( $j \in e_n$ ) and consequently,  $|\gamma| < \min(\sigma_j, T_j)$ ,  $j \in e_n$ . If  $x \in G \setminus G_\sigma$ , then  $\Delta(\gamma, G) D^\nu f(x) = 0$ . Therefore,

$$\begin{aligned} & \|\Delta(\gamma, G) D^\nu f(x)\|_{p,G} \leq \\ & \leq C_1 \sum_{i=1}^{2^n} \prod_{j \in e_n \setminus e^i} T_j^{-1-\nu_j} \left( \|B_\gamma^i(\cdot, \gamma)\|_{p,G_\sigma} + \|B_{\gamma T}^i(\cdot, \gamma)\|_{p,G_\sigma} \right). \end{aligned} \quad (3.11)$$

By means of inequality (2.9) for  $\eta = |\gamma|$ ,  $j \in e_n$ ,  $U = G$ ,  $\rho \rightarrow \infty$  we get

$$\|B_\gamma^i(\cdot, \gamma)\|_{p,G_\sigma} \leq C_2 \prod_{j \in e_n} |\gamma_j|^{e_j^i} \left\| \prod_{j \in e_{i^i}} t_j^{-l_j^i} \Delta^{m^i}(t, Z_t) f \right\|_{p^i, a, \varkappa, \tau}, \quad (3.12)$$

and by means of inequality (2.10) for  $U = G$ ,  $\eta_j = |\gamma_j|$ ,  $j \in e_n$ ,  $\rho \rightarrow \infty$  we get

$$\|B_{\gamma T}^i(\cdot, \gamma)\|_{p,G_\sigma} \leq C_3 \prod_{j \in e_n} |\gamma_j|^{\beta_j} \left\| \prod_{j \in e_{i^i}} t_j^{-l_j^i} \Delta^{m^i}(t, Z_t) f \right\|_{p^i, a, \varkappa, \tau}, \quad (3.13)$$

where  $\beta_j$  ( $j \in e_n$ ) is a number satisfying inequalities (3.8).

Then by means of inequalities (3.12) – (3.13) from inequality (3.11) under the condition  $1 \leq \theta^i \leq \infty$  ( $i = 1, \dots, 2^n$ ), we get inequality (3.7).

Now suppose that  $|\gamma_j| \geq \min(\sigma_j, T_j)$ , ( $j \in e_n$ ). Then

$$\|\Delta(\gamma, G) D^\nu f(x)\|_{p,G} \leq C(\sigma, T) \prod_{j \in e_n} |\gamma_j|^{\beta_j} \|D^\nu f(x)\|_{p,G}.$$

Estimating  $\|D^\nu f(x)\|_{p,G}$  by means of inequality (3.1), we get the required estimation. Theorem 3.2 is proved.

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