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# ON A SPECIAL CLASS OF SEMI-COTANGENT BUNDLE

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**Abstract**. We define a semi-cotangent (pull-back) bundle t*M of cotangent bundle T*M by using projection (submersion) of the tangent bundle TM. The main purpose of this paper is to investigate complete lift of vector and affinor (tensor of type (1,1)) fields for semi-cotangent (pull-back) bundle t*M.

## 1. Introduction

The semi-cotangent (pull-back) bundles of fiber bundles are considered in [8], where it is proved that they admit a degenerate symplectic structure. In [8], a lift problems for a projectable tensor fields are also studied. But the projectable tensor field is not explicit form on any fiber bundle. Therefore, we can not study some properties of fiber components of lifts in the semi-cotangent bundle. For this reasion, by putting the tangent bundle instead of the fiber bundle we introduce a new class of semi-cotangent bundle, which has explicit formulas for a projectable tensor fields. For example, complete lift of tensor fields as a projectable vector fields on the tangent bundle have explicit formulas (see, for details [7]).

Let $M_n$ be an $n$-dimensional differentiable manifold of class $C^\infty$ and $T\left(M_n\right)$ the tangent bundle determined by a natural projection (submersion) $\pi_1 : T\left(M_n\right) \to M_n$. We use the notation $(x^i) = (x^{\overline{\alpha}}, x^\alpha)$, where the indices $i, j, ...$ run from 1 to $2n$, the indices $\alpha, \beta, ...$ from 1 to $n$ and the indices $\overline{\alpha}, \overline{\beta}, ...$ from $n+1$ to $2n$, $x^\alpha$ are coordinates in $M_n$, $x^{\overline{\alpha}} = y^\alpha$ are fibre coordinates of the tangent bundle $T(M_n)$. If $(x^{i'}) = (x^{\overline{\alpha}'}, x^{\alpha'})$ is another system of local adapted coordinates in the tangent bundle $T(M_n)$, then we have

$$\begin{cases} x^{\overline{\alpha}'} = \frac{\partial x^{\alpha'}}{\partial x^\beta} y^\beta, \\ x^{\alpha'} = x^{\alpha'}\left(x^\beta\right). \end{cases} \tag{1.1}$$

The Jacobian of (1.1) has components

$$(A_j^{i'}) = \left(\frac{\partial x^{i'}}{\partial x^j}\right) = \begin{pmatrix} A_\beta^{\alpha'} & A_{\beta\varepsilon}^{\alpha'} y^\varepsilon \\ 0 & A_\beta^{\alpha'} \end{pmatrix},$$

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where $A_{\beta}^{\alpha'}=\frac{\partial x^{\alpha'}}{\partial x^{\beta}}$, $A_{\beta\varepsilon}^{\alpha'}=\frac{\partial^2 x^{\alpha'}}{\partial x^{\beta}\partial x^{\varepsilon}}$. Let $T_x^*(M_n)(x=\pi_1(\widetilde{x}),\widetilde{x}=(x^{\overline{\alpha}},x^{\alpha})\in T(M_n))$ be the cotangent space at a point $x$ of $M_n$. If $p_\alpha$ are components of $p\in T_x^*(M_n)$ with respect to the natural coframe $\{dx^\alpha\}$, i.e. $p=p_i\ dx^i$, then by definition the set $t^*(M_n)$ of all points $\left(x^I\right)=(x^{\overline{\alpha}},x^{\alpha},x^{\overline{\overline{\alpha}}})$, $x^{\overline{\overline{\alpha}}}=p_\alpha$; $I,J,...=1,...,3n$ with projection $\pi_2: t^*(M_n)\to T(M_n)$ (i.e. $\pi_2:(x^{\overline{\alpha}},x^{\alpha},x^{\overline{\overline{\alpha}}})\to(x^{\overline{\alpha}},x^{\alpha})$) is a semi-cotangent (pull-back [8]) bundle of the cotangent bundle by submersion $\pi_1: T(M_n)\to M_n$ (For definition of the pull-back bundle, see for example [1], [2], [4], [6]). It is remarkable fact that the semi-cotangent (pull-back) bundle has a degenerate symplectic structure [8]

$$\omega=(\omega_{AB})=dp=\begin{pmatrix}0&0&0\\0&0&-\delta_\beta^\alpha\\0&\delta_\alpha^\beta&0\end{pmatrix}.$$

It is clear that the pull-back bundle $t^*(M_n)$ of the cotangent bundle $T^*(M_n)$ also has the natural bundle structure over $M_n$, its bundle projection $\pi: t^*(M_n)\to M_n$ being defined by $\pi:(x^{\overline{\alpha}},x^{\alpha},x^{\overline{\overline{\alpha}}})\to(x^\alpha)$, and hence $\pi=\pi_1\circ\pi_2$. Thus $(t^*(M_n),\pi_1\circ\pi_2)$ is the composite bundle [[5], p.9] or step-like bundle [3].

The main purpose of the present paper is to study complete lift of vector fields and tensor fields of type (1,1) from tangent bundle $T(M_n)$ to semi-cotangent (pull-back) bundle $(t^*(M_n),\pi_2)$.

## 2. Complete Lift of Vector Fields

We denote by $\Im_q^p(T(M_n))$ and $\Im_q^p(M_n)$ the modules over $F(T(M_n))$ and $F(M_n)$ of all tensor fields of type $(p,q)$ on $T(M_n)$ and $M_n$ respectively, where $F(T(M_n))$ and $F(M_n)$ denote the rings of real-valued $C^\infty$−functions on $T(M_n)$ and $M_n$, respectively.

To a transformation (1.1) of local coordinates of $T(M_n)$, there corresponds on $t^*(M_n)$ the coordinate transformation

$$\begin{cases} x^{\overline{\alpha}'}=\frac{\partial x^{\alpha'}}{\partial x^{\beta}}y^{\beta},\\ x^{\alpha'}=x^{\alpha'}\left(x^{\beta}\right),\\ x^{\overline{\overline{\alpha}}'}=\frac{\partial x^{\beta}}{\partial x^{\alpha'}}p_{\beta}.\end{cases}\tag{2.1}$$

The Jacobian of (2.1) is given by

$$\overline{A}=(A_J^{I'})=\begin{pmatrix}A_\beta^{\alpha'}&A_{\beta\varepsilon}^{\alpha'}y^\varepsilon&0\\0&A_\beta^{\alpha'}&0\\0&p_\alpha A_\beta^{\beta'}A_{\beta'\alpha'}^{\alpha}&A_{\alpha'}^{\beta}\end{pmatrix},\tag{2.2}$$

where

$$A_\beta^{\alpha'}=\frac{\partial x^{\alpha'}}{\partial x^\beta},\quad A_{\alpha'}^{\beta}=\frac{\partial x^{\beta}}{\partial x^{\alpha'}},\quad A_{\beta\varepsilon}^{\alpha'}=\frac{\partial^2 x^{\alpha'}}{\partial x^\beta\partial x^\varepsilon},\quad A_{\beta'\alpha'}^{\alpha}=\frac{\partial^2 x^{\alpha}}{\partial x^{\beta'}\partial x^{\alpha'}}.$$

It is easily verified that the condition $Det\ \overline{A}\neq 0$ is equivalent to the condition:

$$Det(A_\beta^{\alpha'})\neq 0.$$

Let $X \in \Im_0^1(T(M_n))$, i.e. $X = X^\alpha \partial_\alpha$. The complete lift ${}^cX$ of $X$ to tangent bundle is defined by ${}^cX = X^\alpha \partial_\alpha + \left(y^\beta \partial_\beta X^\alpha\right) \partial_{\overline{\alpha}}$ [[7], p.15]. On putting

$$ {}^{cc}X = ({}^{cc}X^\alpha) = \begin{pmatrix} y^\varepsilon \partial_\varepsilon X^\alpha \\ X^\alpha \\ -p_\varepsilon(\partial_\alpha X^\varepsilon) \end{pmatrix}, \tag{2.3} $$

from (2.2), we easily see that ${}^{cc}X' = \overline{A}\,({}^{cc}X)$. The vector field ${}^{cc}X$ is called the complete lift of ${}^cX \in \Im_0^1(T\,(M_n))$ to $t^*(M_n)$.

Now, consider $\omega \in \Im_1^0(M_n)$ and $F \in \Im_1^1(T(M_n))$, then ${}^{vv}\omega$ (vertical lift) and $\gamma F \in \Im_0^1(t^*(M_n))$ have respectively, components on the semi-cotangent bundle $t^*(M_n)$ [8]

$$ {}^{vv}\omega = \begin{pmatrix} 0 \\ 0 \\ \omega_\alpha \end{pmatrix}, \ \gamma F = (\gamma F^I) = \begin{pmatrix} 0 \\ 0 \\ p_\beta F_\alpha^\beta \end{pmatrix} \tag{2.4} $$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$, where $\omega_\alpha$ and $F_\alpha^\beta$ are local components of $\omega$ and $F$.

For $T \in \Im_2^1(M_n)$, we can define an affinor field $\gamma T \in \Im_1^1(t^*(M_n))$ [8]:

$$ \gamma T = ((\gamma T)_J^I) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & p_\varepsilon T_{\beta\alpha}^\varepsilon & 0 \end{pmatrix}, \tag{2.5} $$

where $T_{\beta\alpha}^\varepsilon$ are local components of $T$ in $M_n$.

On the other hand, ${}^{vv}f$ the vertical lift of function $f$ on $t^*(M_n)$ is defined by [8]:

$$ {}^{vv}f = {}^{v}f \circ \pi_2 = f \circ \pi_1 \circ \pi_2 = f \circ \pi. \tag{2.6} $$

**Theorem 2.1.** *For any vector fields $X, Y$ on $T(M_n)$ and $f \in \Im_0^0(M_n)$, we have*

(*i*) ${}^{cc}(X+Y) = {}^{cc}X + {}^{cc}Y$,
(*ii*) ${}^{cc}X {}^{vv}f = {}^{vv}(Xf)$.

*Proof.* (*i*) This immediately follows from (2.3).

(*ii*) Let $X \in \Im_0^1(T(M_n))$. Then we get by (2.3) and (2.6):

$$ \begin{aligned} {}^{cc}X^{vv}f &= {}^{cc}X^I \partial_I^{vv} f \\ &= {}^{cc}X^{\overline{\alpha}} \partial_{\overline{\alpha}}^{vv} f + {}^{cc}X^\alpha \partial_\alpha^{vv} f + {}^{cc}X^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}^{vv} f \\ &= X^\alpha \partial_\alpha^{vv} f = {}^{vv}(Xf), \end{aligned} $$

which gives (*ii*) of Theorem 2.1. □

**Theorem 2.2.** *Let $X, Y \in \Im_0^1(T(M_n))$. For the Lie product, we have*

(*i*) $[{}^{cc}X, {}^{cc}Y] = {}^{cc}[X, Y]$ (*i.e.* $L_{{}^{cc}X}({}^{cc}Y) = {}^{cc}(L_X Y)$),
(*ii*) $[{}^{cc}X, {}^{vv}\omega] = {}^{vv}(L_X \omega)$,
(*iii*) $[{}^{cc}X, \gamma F] = \gamma(L_X F)$

*for any $\omega \in \Im_1^0(M_n)$ and $F \in \Im_1^1(T(M_n))$, where $L_X$ the operator of Lie derivation with respect to $X$.*

*Proof.* $(i)$ If $X, Y \in \Im_0^1(T(M_n))$ and $\begin{pmatrix} [^{cc}X,^{cc}Y]^{\overline{\beta}} \\ [^{cc}X,^{cc}Y]^{\beta} \\ [^{cc}X,^{cc}Y]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[^{cc}X,^{cc}Y]$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t^*(M_n)$, then we have

$$[^{cc}X,^{cc}Y]^J = (^{cc}X)^I \partial_I (^{cc}Y)^J - (^{cc}Y)^I \partial_I (^{cc}X)^J.$$

Firstly, if $J = \overline{\beta}$, we have

$$\begin{aligned}
[^{cc}X,^{cc}Y]^{\overline{\beta}} &= (^{cc}X)^I \partial_I (^{cc}Y)^{\overline{\beta}} - (^{cc}Y)^I \partial_I (^{cc}X)^{\overline{\beta}} \\
&= (^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}Y)^{\overline{\beta}} + (^{cc}X)^{\alpha} \partial_{\alpha} (^{cc}Y)^{\overline{\beta}} + (^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}Y)^{\overline{\beta}} \\
&\quad - (^{cc}Y)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}X)^{\overline{\beta}} - (^{cc}Y)^{\alpha} \partial_{\alpha} (^{cc}X)^{\overline{\beta}} - (^{cc}Y)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}X)^{\overline{\beta}} \\
&= y^{\varepsilon} \partial_{\varepsilon} X^{\alpha} \partial_{\overline{\alpha}} y^{\sigma} \partial_{\sigma} Y^{\beta} + X^{\alpha} \partial_{\alpha} (y^{\varepsilon} \partial_{\varepsilon} Y^{\beta}) \\
&\quad - y^{\varepsilon} \partial_{\varepsilon} Y^{\alpha} \partial_{\overline{\alpha}} y^{\sigma} \partial_{\sigma} X^{\beta} - Y^{\alpha} \partial_{\alpha} (y^{\varepsilon} \partial_{\varepsilon} X^{\beta}) \\
&= y^{\varepsilon} (\partial_{\varepsilon} X^{\sigma}) \left(\partial_{\sigma} Y^{\beta}\right) + y^{\varepsilon} X^{\alpha} \partial_{\alpha} \partial_{\varepsilon} Y^{\beta} \\
&\quad - y^{\varepsilon} (\partial_{\varepsilon} Y^{\sigma}) \left(\partial_{\sigma} X^{\beta}\right) - y^{\varepsilon} Y^{\alpha} \partial_{\alpha} \partial_{\varepsilon} X^{\beta} \\
&= y^{\varepsilon} \partial_{\varepsilon} [X, Y]^{\beta}
\end{aligned}$$

by virtue of (2.3). Secondly, if $J = \beta$, we have

$$\begin{aligned}
[^{cc}X,^{cc}Y]^{\beta} &= (^{cc}X)^I \partial_I (^{cc}Y)^{\beta} - (^{cc}Y)^I \partial_I (^{cc}X)^{\beta} \\
&= (^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}Y)^{\beta} + (^{cc}X)^{\alpha} \partial_{\alpha} (^{cc}Y)^{\beta} + (^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}Y)^{\beta} \\
&\quad - (^{cc}Y)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}X)^{\beta} - (^{cc}Y)^{\alpha} \partial_{\alpha} (^{cc}X)^{\beta} - (^{cc}Y)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}X)^{\beta} \\
&= (^{cc}X)^{\alpha} \partial_{\alpha} (^{cc}Y)^{\beta} - (^{cc}Y)^{\alpha} \partial_{\alpha} (^{cc}X)^{\beta} \\
&= X^{\alpha} \partial_{\alpha} Y^{\beta} - Y^{\alpha} \partial_{\alpha} X^{\beta} \\
&= [X, Y]^{\beta}
\end{aligned}$$

by virtue of (2.3). Thirdly, if $J = \overline{\overline{\beta}}$, then we have

$$\begin{aligned}
[^{cc}X,^{cc}Y]^{\overline{\overline{\beta}}} &= (^{cc}X)^I \partial_I (^{cc}Y)^{\overline{\overline{\beta}}} - (^{cc}Y)^I \partial_I (^{cc}X)^{\overline{\overline{\beta}}} \\
&= (^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}Y)^{\overline{\overline{\beta}}} + (^{cc}X)^{\alpha} \partial_{\alpha} (^{cc}Y)^{\overline{\overline{\beta}}} + (^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}Y)^{\overline{\overline{\beta}}} \\
&\quad - (^{cc}Y)^{\overline{\alpha}} \partial_{\overline{\alpha}} (^{cc}X)^{\overline{\overline{\beta}}} - (^{cc}Y)^{\alpha} \partial_{\alpha} (^{cc}X)^{\overline{\overline{\beta}}} - (^{cc}Y)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (^{cc}X)^{\overline{\overline{\beta}}} \\
&= -X^{\alpha} \partial_{\alpha} p_{\varepsilon} (\partial_{\beta} Y^{\varepsilon}) + p_{\varepsilon} \partial_{\alpha} X^{\varepsilon} (\partial_{\beta} Y^{\alpha}) \\
&\quad + Y^{\alpha} \partial_{\alpha} p_{\varepsilon} (\partial_{\beta} X^{\varepsilon}) - p_{\varepsilon} \partial_{\alpha} Y^{\varepsilon} (\partial_{\beta} X^{\alpha}) \\
&= p_{\varepsilon} (-X^{\alpha} \partial_{\alpha} \partial_{\beta} Y^{\varepsilon} + \partial_{\beta} Y^{\alpha} \partial_{\alpha} X^{\varepsilon} \\
&\quad + Y^{\alpha} \partial_{\alpha} \partial_{\beta} X^{\varepsilon} - \partial_{\beta} X^{\alpha} \partial_{\alpha} Y^{\varepsilon}) \\
&= -p_{\varepsilon} (\partial_{\beta} (X^{\alpha} \partial_{\alpha} Y^{\varepsilon} - Y^{\alpha} \partial_{\alpha} X^{\varepsilon})) \\
&= -p_{\varepsilon} (\partial_{\beta} [X, Y]^{\varepsilon})
\end{aligned}$$

by virtue of (2.3). On the other hand, we know that ${}^{cc}[X,Y]$ have components

$$ {}^{cc}[X,Y]=\begin{pmatrix} y^{\varepsilon}\partial_{\varepsilon}[X,Y]^{\beta} \\ [X,Y]^{\beta} \\ -p_{\varepsilon}(\partial_{\beta}[X,Y]^{\varepsilon}) \end{pmatrix} $$

with respect to the coordinates $(x^{\overline{\beta}},x^{\beta},x^{\overline{\overline{\beta}}})$ on $t^{*}(M_n)$. Thus, we have $(i)$ of Theorem 2.2.

$(ii)$ If $\omega\in\Im^{0}_{1}(M_n)$, $X\in\Im^{1}_{0}(T(M_n))$ and $\begin{pmatrix} [{}^{cc}X,{}^{vv}\omega]^{\overline{\beta}} \\ [{}^{cc}X,{}^{vv}\omega]^{\beta} \\ [{}^{cc}X,{}^{vv}\omega]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[{}^{cc}X,{}^{vv}\omega]$ with respect to the coordinates $(x^{\overline{\beta}},x^{\beta},x^{\overline{\overline{\beta}}})$ on $t^{*}(M_n)$, then we have

$$ [{}^{cc}X,{}^{vv}\omega]^{J}=({}^{cc}X)^{I}\partial_{I}({}^{vv}\omega)^{J}-({}^{vv}\omega)^{I}\partial_{I}({}^{cc}X)^{J}. $$

Firstly, if $J=\overline{\beta}$, we have

$$ \begin{aligned} [{}^{cc}X,{}^{vv}\omega]^{\overline{\beta}} &= ({}^{cc}X)^{I}\partial_{I}({}^{vv}\omega)^{\overline{\beta}}-({}^{vv}\omega)^{I}\partial_{I}({}^{cc}X)^{\overline{\beta}} \\ &= -({}^{vv}\omega)^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}X)^{\overline{\beta}}-({}^{vv}\omega)^{\alpha}\partial_{\alpha}({}^{cc}X)^{\overline{\beta}}-({}^{vv}\omega)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\beta}} \\ &= -({}^{vv}\omega)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}y^{\varepsilon}\partial_{\varepsilon}X^{\beta} \\ &= 0 \end{aligned} $$

by virtue of (2.3) and (2.4). Secondly, if $J=\beta$, we have

$$ \begin{aligned} [{}^{cc}X,{}^{vv}\omega]^{\beta} &= ({}^{cc}X)^{I}\partial_{I}({}^{vv}\omega)^{\beta}-({}^{vv}\omega)^{I}\partial_{I}({}^{cc}X)^{\beta} \\ &= -({}^{vv}\omega)^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}X)^{\beta}-({}^{vv}\omega)^{\alpha}\partial_{\alpha}({}^{cc}X)^{\beta}-({}^{vv}\omega)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\beta} \\ &= -({}^{vv}\omega)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}X^{\beta} \\ &= 0 \end{aligned} $$

by virtue of (2.3) and (2.4). Thirdly, if $J=\overline{\overline{\beta}}$, then we have

$$ \begin{aligned} [{}^{cc}X,{}^{vv}\omega]^{\overline{\overline{\beta}}} &= ({}^{cc}X)^{I}\partial_{I}({}^{vv}\omega)^{\overline{\overline{\beta}}}-({}^{vv}\omega)^{I}\partial_{I}({}^{cc}X)^{\overline{\overline{\beta}}} \\ &= ({}^{cc}X)^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{vv}\omega)^{\overline{\overline{\beta}}}+({}^{cc}X)^{\alpha}\partial_{\alpha}({}^{vv}\omega)^{\overline{\overline{\beta}}}+({}^{cc}X)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{vv}\omega)^{\overline{\overline{\beta}}} \\ &\quad -({}^{vv}\omega)^{\overline{\alpha}}\partial_{\overline{\alpha}}({}^{cc}X)^{\overline{\overline{\beta}}}-({}^{vv}\omega)^{\alpha}\partial_{\alpha}({}^{cc}X)^{\overline{\overline{\beta}}}-({}^{vv}\omega)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\overline{\beta}}} \\ &= ({}^{cc}X)^{\alpha}\partial_{\alpha}({}^{vv}\omega)^{\overline{\overline{\beta}}}-({}^{vv}\omega)^{\overline{\overline{\alpha}}}\partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\overline{\beta}}} \\ &= X^{\alpha}\partial_{\alpha}\omega_{\beta}+\omega_{\alpha}\partial_{\overline{\overline{\alpha}}}p_{\varepsilon}(\partial_{\beta}X^{\varepsilon}) \\ &= X^{\alpha}\partial_{\alpha}\omega_{\beta}+(\partial_{\beta}X^{\alpha})\omega_{\alpha} \\ &= (L_X\omega)_{\beta} \end{aligned} $$

by virtue of (2.3) and (2.4). On the other hand, we know that the vertical lift ${}^{vv}(L_X\omega)$ of $(L_X\omega)$ has components of the form

$$ {}^{vv}(L_X\omega)=\begin{pmatrix} 0 \\ 0 \\ (L_X\omega)_{\beta} \end{pmatrix} $$

with respect to the coordinates $(x^{\overline{\beta}},x^{\beta},x^{\overline{\overline{\beta}}})$ on $t^{*}(M_n)$. Thus, we have $(ii)$ of Theorem 2.2.

$(iii)$ We shall prove the last equation. If $F \in \Im_1^1(T(M_n))$, $X \in \Im_0^1(T(M_n))$ and $\begin{pmatrix} [{}^{cc}X, \gamma F]^{\overline{\beta}} \\ [{}^{cc}X, \gamma F]^{\beta} \\ [{}^{cc}X, \gamma F]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[{}^{cc}X, \gamma F]$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t^*(M_n)$, then we have

$$[{}^{cc}X, \gamma F]^J = ({}^{cc}X)^I \partial_I (\gamma F)^J - (\gamma F)^I \partial_I ({}^{cc}X)^J.$$

Firstly, if $J = \overline{\beta}$, we have

$$\begin{aligned} [{}^{cc}X, \gamma F]^{\overline{\beta}} &= ({}^{cc}X)^I \partial_I (\gamma F)^{\overline{\beta}} - (\gamma F)^I \partial_I ({}^{cc}X)^{\overline{\beta}} \\ &= -(\gamma F)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}X)^{\overline{\beta}} - (\gamma F)^{\alpha} \partial_{\alpha} ({}^{cc}X)^{\overline{\beta}} - (\gamma F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} ({}^{cc}X)^{\overline{\beta}} \\ &= 0 \end{aligned}$$

by virtue of (2.3) and (2.4). Secondly, if $J = \beta$, we have

$$\begin{aligned} [{}^{cc}X, \gamma F]^{\beta} &= ({}^{cc}X)^I \partial_I (\gamma F)^{\beta} - (\gamma F)^I \partial_I ({}^{cc}X)^{\beta} \\ &= -(\gamma F)^{\overline{\alpha}} \partial_{\overline{\alpha}} X^{\beta} - (\gamma F)^{\alpha} \partial_{\alpha} X^{\beta} - (\gamma F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} X^{\beta} \\ &= 0 \end{aligned}$$

by virtue of (2.3) and (2.4). Thirdly, if $J = \overline{\overline{\beta}}$, then we have

$$\begin{aligned} [{}^{cc}X, \gamma F]^{\overline{\overline{\beta}}} &= ({}^{cc}X)^I \partial_I (\gamma F)^{\overline{\overline{\beta}}} - (\gamma F)^I \partial_I ({}^{cc}X)^{\overline{\overline{\beta}}} \\ &= ({}^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\gamma F)^{\overline{\overline{\beta}}} + ({}^{cc}X)^{\alpha} \partial_{\alpha} (\gamma F)^{\overline{\overline{\beta}}} + ({}^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\gamma F)^{\overline{\overline{\beta}}} \\ &\quad -(\gamma F)^{\overline{\alpha}} \partial_{\overline{\alpha}} ({}^{cc}X)^{\overline{\overline{\beta}}} - (\gamma F)^{\alpha} \partial_{\alpha} ({}^{cc}X)^{\overline{\overline{\beta}}} - (\gamma F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} ({}^{cc}X)^{\overline{\overline{\beta}}} \\ &= y^{\varepsilon} \partial_{\varepsilon} X^{\alpha} \partial_{\overline{\alpha}} p_{\varepsilon} F_{\beta}^{\varepsilon} + X^{\alpha} \partial_{\alpha} p_{\varepsilon} F_{\beta}^{\varepsilon} \\ &\quad -p_{\varepsilon} (\partial_{\alpha} X^{\varepsilon}) \partial_{\overline{\overline{\alpha}}} p_{\varepsilon} F_{\beta}^{\varepsilon} + p_{\varepsilon} F_{\alpha}^{\varepsilon} \partial_{\overline{\overline{\alpha}}} p_{\varepsilon} (\partial_{\beta} X^{\varepsilon}) \\ &= X^{\alpha} \partial_{\alpha} p_{\varepsilon} F_{\beta}^{\varepsilon} - p_{\varepsilon} (\partial_{\alpha} X^{\varepsilon}) F_{\beta}^{\alpha} + p_{\varepsilon} F_{\alpha}^{\varepsilon} (\partial_{\beta} X^{\alpha}) \\ &= p_{\varepsilon} (X^{\alpha} \partial_{\alpha} F_{\beta}^{\varepsilon} - \partial_{\alpha} X^{\varepsilon} F_{\beta}^{\alpha} + \partial_{\beta} X^{\alpha} F_{\alpha}^{\varepsilon}) \\ &= p_{\varepsilon} (L_X F)_{\beta}^{\varepsilon} \end{aligned}$$

by virtue of (2.3) and (2.4). On the other hand, we know that $\gamma(L_X F)$ have components

$$\gamma(L_X F) = \begin{pmatrix} 0 \\ 0 \\ p_{\varepsilon} (L_X F)_{\beta}^{\varepsilon} \end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t^*(M_n)$. Thus, we have $(iii)$ of Theorem 2.2. □

## 3. Complete Lift of Tensor Fields of Type (1,1)

Suppose now that $F \in \Im_1^1(T(M_n))$ and $F$ has local components $F_{\beta}^{\alpha}$ in a neighborhood $U$ of $M_n$, $F = F_{\beta}^{\alpha} \partial_{\alpha} \otimes dx^{\beta}$. If we take account of (2.2), we can prove that ${}^{cc}F^{I'}_{J'} = A^{I'}_{I} A^{J}_{J'} \; {}^{cc}F^{I}_{J}$, where ${}^{cc}F$ is an affinor field defined by

$$ {}^{cc}F = ( {}^{cc}F^{I}_{J}) = \begin{pmatrix} F^{\alpha}_{\beta} & y^{\varepsilon}\partial_{\varepsilon}F^{\alpha}_{\beta} & 0 \\ 0 & F^{\alpha}_{\beta} & 0 \\ 0 & p_{\sigma}(\partial_{\beta}F^{\sigma}_{\alpha} - \partial_{\alpha}F^{\sigma}_{\beta}) & F^{\beta}_{\alpha} \end{pmatrix}, \tag{3.1} $$

with respect to the coordinates $(x^{\overline{\alpha}}, x^{\alpha}, x^{\overline{\overline{\alpha}}})$ on $t^{*}(M_n)$. We call ${}^{cc}F$ the complete lift of the tensor field $F$ of type (1,1) to $t^{*}(M_n)$.

*Proof.* For simplicity, we put $I' = \overline{\alpha}', J' = \beta'$ in ${}^{cc}F^{I'}_{J'}$ and take account of (2.2), we obtain

$$ \begin{aligned} {}^{cc}F^{\overline{\alpha'}}_{\beta'} &= A^{\overline{\alpha'}}_{\overline{\alpha}} A^{\overline{\beta}}_{\beta'} \; {}^{cc}F^{\overline{\alpha}}_{\overline{\beta}} + A^{\overline{\alpha'}}_{\overline{\alpha}} A^{\beta}_{\beta'} \; {}^{cc}F^{\overline{\alpha}}_{\beta} + A^{\overline{\alpha'}}_{\alpha} A^{\beta}_{\beta'} \; {}^{cc}F^{\alpha}_{\beta} \\ &= A^{\alpha'}_{\alpha} A^{\beta}_{\beta'\sigma'} y^{\sigma'} F^{\alpha}_{\beta} + A^{\alpha'}_{\alpha} A^{\beta}_{\beta'} y^{\sigma'} \partial_{\sigma'} F^{\alpha}_{\beta} + A^{\alpha'}_{\alpha\;\sigma'} y^{\sigma'} A^{\beta}_{\beta'} F^{\alpha}_{\beta} \\ &= -y^{\sigma'} (\partial_{\sigma'} A^{\alpha'}_{\alpha}) A^{\beta}_{\beta'} F^{\alpha}_{\beta} + A^{\alpha'}_{\alpha} A^{\beta}_{\beta'} y^{\sigma'} \left(\partial_{\sigma'} F^{\alpha}_{\beta}\right) + y^{\sigma'} \left(\partial_{\sigma'} A^{\alpha'}_{\alpha}\right) A^{\beta}_{\beta'} F^{\alpha}_{\beta} \\ &= y^{\sigma'} A^{\alpha'}_{\alpha} \left(\partial_{\sigma'} A^{\beta}_{\beta'}\right) F^{\alpha}_{\beta} + y^{\sigma'} A^{\alpha'}_{\alpha} A^{\beta}_{\beta'} \left(\partial_{\sigma'} F^{\alpha}_{\beta}\right) + y^{\sigma'} \left(\partial_{\sigma'} A^{\alpha'}_{\alpha}\right) A^{\beta}_{\beta'} F^{\alpha}_{\beta} \\ &= y^{\sigma'} \partial_{\sigma'} \left(A^{\alpha'}_{\alpha} A^{\beta}_{\beta'} F^{\alpha}_{\beta}\right) \\ &= y^{\varepsilon'} \partial_{\varepsilon'} F^{\alpha'}_{\beta'}. \end{aligned} $$

Similarly, we can easily find another components of ${}^{cc}F^{I}_{J}$. □

**Theorem 3.1.** *If* $X \in \Im^{1}_{0}(T(M_n))$, $\omega \in \Im^{0}_{1}(M_n)$ *and* $F \in \Im^{1}_{1}(T(M_n))$ *then*

(*i*) ${}^{cc}F\,{}^{cc}X = {}^{cc}(FX) + \gamma(L_X F)$,

(*ii*) ${}^{cc}F\,{}^{vv}\omega = {}^{vv}(\omega \circ F)$,

*where* $L_X$ *the operator of Lie derivation with respect to* $X$.

*Proof.* (*i*) Let $F, X$ be elements of $\Im^{1}_{1}(T(M_n))$ and $\Im^{1}_{0}(T(M_n))$, respectively. Then we get by (2.3), (2.4) and (3.1):

$$ \begin{aligned} {}^{cc}F\;{}^{cc}X &= \begin{pmatrix} F^{\alpha}_{\beta} & y^{\varepsilon}\partial_{\varepsilon}F^{\alpha}_{\beta} & 0 \\ 0 & F^{\alpha}_{\beta} & 0 \\ 0 & p_{\sigma}(\partial_{\beta}F^{\sigma}_{\alpha} - \partial_{\alpha}F^{\sigma}_{\beta}) & F^{\beta}_{\alpha} \end{pmatrix} \begin{pmatrix} y^{\varepsilon}\partial_{\varepsilon}X^{\beta} \\ X^{\beta} \\ -p_{\varepsilon}(\partial_{\beta}X^{\varepsilon}) \end{pmatrix} \\ &= \begin{pmatrix} F^{\alpha}_{\beta} y^{\varepsilon}\partial_{\varepsilon}X^{\beta} + y^{\varepsilon}\partial_{\varepsilon}F^{\alpha}_{\beta}X^{\beta} \\ F^{\alpha}_{\beta}X^{\beta} \\ p_{\sigma}(\partial_{\beta}F^{\sigma}_{\alpha} - \partial_{\alpha}F^{\sigma}_{\beta})X^{\beta} - p_{\varepsilon}(\partial_{\beta}X^{\varepsilon})F^{\beta}_{\alpha} \end{pmatrix} \\ &= \begin{pmatrix} y^{\varepsilon}\partial_{\varepsilon}F^{\alpha}_{\beta}X^{\beta} + F^{\alpha}_{\beta} y^{\varepsilon}\partial_{\varepsilon}X^{\beta} \\ (FX)^{\alpha} \\ p_{\sigma}(\partial_{\beta}F^{\sigma}_{\alpha} - \partial_{\alpha}F^{\sigma}_{\beta})X^{\beta} - p_{\varepsilon}(\partial_{\beta}X^{\varepsilon})F^{\beta}_{\alpha} \end{pmatrix} \\ &= \begin{pmatrix} y^{\varepsilon}\partial_{\varepsilon}\,(FX)^{\alpha} \\ (FX)^{\alpha} \\ -p_{\sigma}\partial_{\alpha}(FX)^{\sigma} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ p_{\sigma}(X^{\beta}\partial_{\beta}F^{\sigma}_{\alpha} - (\partial_{\alpha}X^{\beta})F^{\sigma}_{\beta} - (\partial_{\beta}X^{\sigma})F^{\beta}_{\alpha}) \end{pmatrix} \\ &= \begin{pmatrix} y^{\varepsilon}\partial_{\varepsilon}\,(FX)^{\alpha} \\ (FX)^{\alpha} \\ -p_{\sigma}\partial_{\alpha}(FX)^{\sigma} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ p_{\sigma}(L_X F)^{\sigma}_{\alpha} \end{pmatrix} = {}^{cc}(FX) + \gamma(L_X F). \end{aligned} $$

Thus, we have (*i*) of Theorem 3.1.

(*ii*) If $\omega \in \Im^{0}_{1}(M_n)$ and $F \in \Im^{1}_{1}(T(M_n))$, from (2.4) and (3.1), we have

$$
{}^{cc}F\ {}^{vv}\omega = \begin{pmatrix} F_\beta^\alpha & y^\varepsilon \partial_\varepsilon F_\beta^\alpha & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & F_\alpha^\beta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega_\beta \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 \\ 0 \\ \omega_\beta F_\alpha^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (\omega \circ F)_\alpha \end{pmatrix} =^{vv} (\omega \circ F).
$$

Thus, we have $(ii)$ of Theorem 3.1. □

**Theorem 3.2.** *For any $F \in \Im_1^1(T(M_n))$, $F^2 = -I$*

$$
\left({}^{cc}F\right)^2 = -I - \gamma(N_F).
$$

*Proof.* Let $F \in \Im_1^1(T(M_n))$. Then we have by (2.5) and (3.1):

$$
\left({}^{cc}F\right)^2 = \begin{pmatrix} F_\beta^\alpha & y^\varepsilon \left(\partial_\varepsilon F_\beta^\alpha\right) & 0 \\ 0 & F_\beta^\alpha & 0 \\ 0 & p_\sigma(\partial_\beta F_\alpha^\sigma - \partial_\alpha F_\beta^\sigma) & F_\alpha^\beta \end{pmatrix} \begin{pmatrix} F_\theta^\beta & y^\varepsilon \left(\partial_\varepsilon F_\theta^\beta\right) & 0 \\ 0 & F_\theta^\beta & 0 \\ 0 & p_\sigma(\partial_\theta F_\beta^\sigma - \partial_\beta F_\theta^\sigma) & F_\beta^\theta \end{pmatrix}
$$

$$
= \begin{pmatrix} -\delta_\theta^\alpha & 0 & 0 \\ 0 & -\delta_\theta^\alpha & 0 \\ 0 & -p_\sigma (N_F)_{\theta\,\alpha}^\sigma & -\delta_\alpha^\theta \end{pmatrix}
$$

$$
= \begin{pmatrix} -\delta_\theta^\alpha & 0 & 0 \\ 0 & -\delta_\theta^\alpha & 0 \\ 0 & 0 & -\delta_\alpha^\theta \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -p_\sigma (N_F)_{\theta\,\alpha}^\sigma & 0 \end{pmatrix} = -I - \gamma(N_F),
$$

where $N_F \in \Im_2^1(M_n)$ is the Nijenhuis tensor of $F \in \Im_1^1(T(M_n))$. Thus, proof is complete. □

**Theorem 3.3.** *Let $N_F$ and $N_{\ {}^{cc}F}$ be respectively the Nijenhuis tensors of $F \in \Im_1^1(T(M_n))$ and of the complete lift ${}^{cc}F$ of $F$. Then*

$$
N_{\ {}^{cc}F} = 0
$$

*if and only if $N_F = 0$.*

*Proof.* For simplicity we take only $(N_{\ {}^{cc}F})_{\overline{\beta}\theta}^{\overline{\alpha}}$. In fact,

$$
\begin{aligned}
(N_{\ {}^{cc}F})_{\overline{\beta}\theta}^{\overline{\alpha}} &= {}^{cc}F_{\overline{\beta}}^{\overline{\gamma}} \partial_{\overline{\gamma}}\ {}^{cc}F_\theta^{\overline{\alpha}} - {}^{cc}F_\theta^\gamma \partial_\gamma\ {}^{cc}F_{\overline{\beta}}^{\overline{\alpha}} - {}^{cc}F_{\overline{\gamma}}^{\overline{\alpha}} \partial_{\overline{\beta}}\ {}^{cc}F_\theta^{\overline{\gamma}} + {}^{cc}F_{\overline{\gamma}}^{\overline{\alpha}} \partial_\theta\ {}^{cc}F_{\overline{\beta}}^{\overline{\gamma}} \\
&= F_\beta^\gamma \partial_{\overline{\gamma}} y^\varepsilon \partial_\varepsilon F_\beta^\alpha - F_\theta^\gamma \partial_\gamma F_\beta^\alpha - F_\gamma^\alpha \partial_{\overline{\beta}} y^\varepsilon \partial_\varepsilon F_\theta^\gamma + F_\gamma^\alpha \partial_\theta F_\beta^\gamma \\
&= F_\beta^\gamma \partial_\gamma F_\theta^\alpha - F_\theta^\gamma \partial_\gamma F_\beta^\alpha - F_\gamma^\alpha \partial_\beta F_\theta^\gamma + F_\gamma^\alpha \partial_\theta F_\beta^\gamma \\
&= F_\beta^\gamma \partial_\gamma F_\theta^\alpha - F_\theta^\gamma \partial_\gamma F_\beta^\alpha - F_\gamma^\alpha \left(\partial_\beta F_\theta^\gamma + \partial_\theta F_\beta^\gamma\right) \\
&= (N_F)_{\beta\,\theta}^{\alpha} = 0.
\end{aligned}
$$

Similarly, from (3.1), we can easily find another vanishing components of $N_{\ {}^{cc}F} : (N_{\ {}^{cc}F})_{BC}^{A}$, where $A = (\overline{\alpha}, \alpha, \overline{\overline{\alpha}})$, $B = \left(\overline{\beta}, \beta, \overline{\overline{\beta}}\right)$, $C = \left(\overline{\theta}, \theta, \overline{\overline{\theta}}\right)$. □

**Theorem 3.4.** *Let $X \in \Im_0^1(T(M_n))$ and $F \in \Im_1^1(T(M_n))$. Then*

$$
L_{\ {}^{cc}X}\ {}^{cc}F = 0
$$

*if $L_X F = 0$.*

*Proof.* If $X,Y,Z \in \Im_0^1(T(M_n))$ and $F \in \Im_1^1(T(M_n))$, from $(i)$ of Theorem 2.2 and $(i)$ of Theorem 3.1, we have

$$\begin{aligned}\left(L_{\,^{cc}X}\,^{cc}F\right)\,^{cc}Y &= L_{\,^{cc}X}\,^{cc}F\,^{cc}Y - \,^{cc}F\left(L_{\,^{cc}X}\,^{cc}Y\right) \\ &= L_{\,^{cc}X}\left(\,^{cc}(FY)+\gamma(L_YF)\right) - \,^{cc}F\,^{cc}(L_XY) \\ &= \,^{cc}L_X\,^{cc}(FY) + L_{\,^{cc}X}\gamma(L_YF) - \,^{cc}(F(L_XY)) - \gamma_{[X,Y]}F \\ &= \,^{cc}(L_X(FY)) - \,^{cc}(F(L_XY)) + L_{\,^{cc}X}\gamma(L_YF) - \gamma_{[X,Y]}F \\ &= \,^{cc}((L_XF)Y) + L_{\,^{cc}X}[\,^{cc}Y,\gamma F] - \gamma_{[X,Y]}F. \end{aligned} \tag{3.2}$$

On the other hand, using the $([X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0)$ Jacobi identity, from $(i)$ and $(iii)$ of Theorem 2.2, we find

$$\begin{aligned}[\,^{cc}X,[\,^{cc}Y,\gamma F]] + [\,^{cc}Y,[\gamma F,\,^{cc}X]] + [\gamma F,[\,^{cc}X,\,^{cc}Y]] &= 0, \\ [\,^{cc}X,[\,^{cc}Y,\gamma F]] + [\,^{cc}Y,[\gamma F,\,^{cc}X]] - [[\,^{cc}X,\,^{cc}Y],\gamma F] &= 0, \\ [\,^{cc}X,[\,^{cc}Y,\gamma F]] + [\,^{cc}Y,[\gamma F,\,^{cc}X]] - [\,^{cc}[X,Y],\gamma F] &= 0, \\ L_{\,^{cc}X}[\,^{cc}Y,\gamma F] + [\,^{cc}Y,[\gamma F,\,^{cc}X]] - \gamma_{[X,Y]}F &= 0,\end{aligned}$$

$$L_{\,^{cc}X}[\,^{cc}Y,\gamma F] - \gamma_{[X,Y]}F = -[\,^{cc}Y,[\gamma F,\,^{cc}X]]. \tag{3.3}$$

Substituting (3.3) into (3.2), from $(iii)$ of Theorem 2.2, we obtain

$$\begin{aligned}&= \,^{cc}((L_XF)Y) - [\,^{cc}Y,[\gamma F,\,^{cc}X]] \\ &= \,^{cc}((L_XF)Y) + [\,^{cc}Y,[\,^{cc}X,\gamma F]] \\ &= \,^{cc}((L_XF)Y) + [\,^{cc}Y,\gamma(L_XF)] \\ &= \,^{cc}((L_XF)Y) + \gamma(L_YL_XF).\end{aligned}$$

Using $L_XF=0$, from last equations we have $L_{\,^{cc}X}\,^{cc}F=0$. This completes the proof. □

Let $(T(M_n),F)$, $n=2k$ be a complex manifold, i.e. $N_F=0$. A vector field $X$ on complex manifold $(T(M_n),F)$ is called a holomorphic vector field if $L_XF=0$. Thus, from Theorem 3.2, Theorem 3.3 and Theorem 3.4, we have

**Theorem 3.5.** *A complete lift $^{cc}X$ is holomorphic with respect to complex structure $^{cc}F$ if $X$ is holomorphic with respect to $F$.*

## 4. $\widetilde{\gamma}-$Operator

In this section, we will now define new $\widetilde{\gamma}-$operators on $t^*(M_n)$. Let $X$ be a vector field on $T(M_n)$. We define a function $\widetilde{\gamma}X$ on $t^*(M_n)$ by

$$\widetilde{\gamma}X = p_\beta X^\beta. \tag{4.1}$$

For any $F \in \Im_1^1(T(M_n))$, if we take account of (2.2), we can prove that $(\widetilde{\gamma}F)' = \overline{(A)}(\widetilde{\gamma}F)$, where $\widetilde{\gamma}F$ is a vector field defined by

$$\widetilde{\gamma}F = (\widetilde{\gamma}F^A) = \begin{pmatrix} y^\varepsilon F_\varepsilon^\alpha \\ 0 \\ -p_\sigma F_\alpha^\sigma \end{pmatrix} \tag{4.2}$$

with respect to the coordinates $(x^{\overline{\alpha}}, x^\alpha, x^{\overline{\overline{\alpha}}})$.

Let $T \in \Im_2^1(M_n)$. On putting

$$\widetilde{\gamma}T = (\widetilde{\gamma}T_B^A) = \begin{pmatrix} 0 & y^\varepsilon T_{\varepsilon\ \beta}^{\alpha} & 0 \\ 0 & 0 & 0 \\ 0 & -p_\sigma T_{\beta\ \alpha}^{\sigma} & 0 \end{pmatrix}, \tag{4.3}$$

from (2.2), we easily see that $\widetilde{\gamma}T_{B'}^{A'} = A_A^{A'} A_{B'}^B \widetilde{\gamma}T_B^A$, where $\overline{(A)}^{-1} = (A_{B'}^B)$ is the inverse matrix of $\overline{A}$.

**Theorem 4.1.** *Let $X, Z \in \Im_0^1(T(M_n))$. If $f \in \Im_0^0(M_n)$, $\omega \in \Im_1^0(M_n)$ and $F \in \Im_1^1(T(M_n))$, then*

*(i)* ${}^{vv}\omega(\widetilde{\gamma}Z) = {}^{vv}(\omega(Z))$,
*(ii)* $(\widetilde{\gamma}F)({}^{vv}f) = 0$,
*(iii)* $(\widetilde{\gamma}F)\widetilde{\gamma}Z = -\widetilde{\gamma}(FZ)$,
*(iv)* ${}^{cc}X(\widetilde{\gamma}Z) = \widetilde{\gamma}[X, Z]$.

*Proof.* $(i)$ If $\omega \in \Im_1^0(M_n)$ and $Z \in \Im_0^1(T(M_n))$, then we have by (2.4) and (4.1):

$$\begin{aligned} {}^{vv}\omega(\widetilde{\gamma}Z) &= {}^{vv}\omega^I \partial_I(\widetilde{\gamma}Z) \\ &= {}^{vv}\omega^{\overline{\alpha}} \partial_{\overline{\alpha}}(p_\beta Z^\beta) + {}^{vv}\omega^\alpha \partial_\alpha(p_\beta Z^\beta) + {}^{vv}\omega^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}(p_\beta Z^\beta) \\ &= \omega_\alpha Z^\alpha = {}^{vv}(\omega(Z)). \end{aligned}$$

Thus we have ${}^{vv}\omega(\widetilde{\gamma}Z) = {}^{vv}(\omega(Z))$.
$(ii)$ If $F \in \Im_1^1(T(M_n))$,then we have by (2.6) and (4.2):

$$\begin{aligned} (\widetilde{\gamma}F)({}^{vv}f) &= (\widetilde{\gamma}F)^I \partial_I({}^{vv}f) \\ &= (\widetilde{\gamma}F)^{\overline{\alpha}} \partial_{\overline{\alpha}}({}^{vv}f) + (\widetilde{\gamma}F)^\alpha \partial_\alpha({}^{vv}f) + (\widetilde{\gamma}F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}({}^{vv}f) \\ &= y^\varepsilon F_\varepsilon^\alpha \partial_{\overline{\alpha}} f - p_\sigma F_\alpha^\sigma \partial_{\overline{\overline{\alpha}}} f \\ &= 0. \end{aligned}$$

Thus we have $(\widetilde{\gamma}F)({}^{vv}f) = 0$, so that $\widetilde{\gamma}F$ is a vertical vector field on $t^*(M_n)$.
$(iii)$ If $F \in \Im_1^1(T(M_n))$ and $Z \in \Im_0^1(T(M_n))$, then we have by (4.1) and (4.2):

$$\begin{aligned} (\widetilde{\gamma}F)\widetilde{\gamma}Z &= (\widetilde{\gamma}F)^I \partial_I(\widetilde{\gamma}Z) \\ &= (\widetilde{\gamma}F)^{\overline{\alpha}} \partial_{\overline{\alpha}}(p_\beta Z^\beta) + (\widetilde{\gamma}F)^\alpha \partial_\alpha(p_\beta Z^\beta) + (\widetilde{\gamma}F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}(p_\beta Z^\beta) \\ &= y^\varepsilon F_\varepsilon^\alpha \partial_{\overline{\alpha}}(p_\beta Z^\beta) - p_\sigma F_\alpha^\sigma \partial_{\overline{\overline{\alpha}}}(p_\beta Z^\beta) \\ &= -p_\sigma F_\alpha^\sigma Z^\alpha = -p_\sigma (FZ)^\sigma = -\widetilde{\gamma}(FZ), \end{aligned}$$

which proves $(iii)$ of Theorem 4.1.
$(iv)$ If $X, Z \in \Im_0^1(T(M_n))$, then taking account of (2.3) and (4.1), we have

$$\begin{aligned} {}^{cc}X\widetilde{\gamma}(Z) &= {}^{cc}X^I \partial_I(\widetilde{\gamma}Z) \\ &= {}^{cc}X^{\overline{\alpha}} \partial_{\overline{\alpha}}(p_\beta Z^\beta) + {}^{cc}X^\alpha \partial_\alpha(p_\beta Z^\beta) + {}^{cc}X^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}(p_\beta Z^\beta) \\ &= X^\alpha \partial_\alpha(p_\beta Z^\beta) - p_\beta(\partial_\alpha X^\beta) Z^\alpha \\ &= p_\beta(X^\alpha \partial_\alpha Z^\beta - Z^\alpha \partial_\alpha X^\beta) \\ &= p_\beta [X, Z]^\beta = \widetilde{\gamma}[X, Z], \end{aligned}$$

which gives equation $(iv)$ of Theorem 4.1. □

**Theorem 4.2.** *Let $X \in \Im_0^1(T(M_n))$. For the Lie product, we have*

(*i*) $[\widetilde{\gamma}F, \widetilde{\gamma}G] = \widetilde{\gamma}[F, G] + 2\gamma[F, G]$,
(*ii*) $[{}^{cc}X, \widetilde{\gamma}F] = \widetilde{\gamma}(L_X F)$

*for any* $F, G \in \Im_1^1(T(M_n))$, *where* $L_X$ *the operator of Lie derivation with respect to* $X$.

*Proof.* (*i*) If $F, G \in \Im_1^1(T(M_n))$ and $\begin{pmatrix} [\widetilde{\gamma}F, \widetilde{\gamma}G]^{\overline{\beta}} \\ [\widetilde{\gamma}F, \widetilde{\gamma}G]^{\beta} \\ [\widetilde{\gamma}F, \widetilde{\gamma}G]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[\widetilde{\gamma}F, \widetilde{\gamma}G]$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t^*(M_n)$, then we have by (4.2)

$$\begin{aligned}
[\widetilde{\gamma}F, \widetilde{\gamma}G]^J &= (\widetilde{\gamma}F)^I \partial_I (\widetilde{\gamma}G)^J - (\widetilde{\gamma}G)^I \partial_I (\widetilde{\gamma}F)^J \\
&= (\widetilde{\gamma}F)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\widetilde{\gamma}G)^J + (\widetilde{\gamma}F)^{\alpha} \partial_{\alpha} (\widetilde{\gamma}G)^J + (\widetilde{\gamma}F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\widetilde{\gamma}G)^J \\
&\quad - (\widetilde{\gamma}G)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\widetilde{\gamma}F)^J - (\widetilde{\gamma}G)^{\alpha} \partial_{\alpha} (\widetilde{\gamma}F)^J - (\widetilde{\gamma}G)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\widetilde{\gamma}F)^J \\
&= (\widetilde{\gamma}F)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\widetilde{\gamma}G)^J + (\widetilde{\gamma}F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\widetilde{\gamma}G)^J - (\widetilde{\gamma}G)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\widetilde{\gamma}F)^J - (\widetilde{\gamma}G)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\widetilde{\gamma}F)^J.
\end{aligned}$$

Firstly, if $J = \overline{\beta}$, we have

$$\begin{aligned}
[\widetilde{\gamma}F, \widetilde{\gamma}G]^{\overline{\beta}} &= (\widetilde{\gamma}F)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\widetilde{\gamma}G)^{\overline{\beta}} + (\widetilde{\gamma}F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\widetilde{\gamma}G)^{\overline{\beta}} - (\widetilde{\gamma}G)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\widetilde{\gamma}F)^{\overline{\beta}} - (\widetilde{\gamma}G)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\widetilde{\gamma}F)^{\overline{\beta}} \\
&= (y^{\varepsilon} F_{\varepsilon}^{\alpha})\, \partial_{\overline{\alpha}} y^{\varepsilon} G_{\varepsilon}^{\beta} - p_{\sigma} F_{\alpha}^{\sigma} \partial_{\overline{\overline{\alpha}}} y^{\varepsilon} G_{\varepsilon}^{\beta} - y^{\varepsilon} G_{\varepsilon}^{\alpha} \partial_{\overline{\alpha}} y^{\varepsilon} F_{\varepsilon}^{\beta} + p_{\sigma} G_{\alpha}^{\sigma} \partial_{\overline{\overline{\alpha}}} y^{\varepsilon} F_{\varepsilon}^{\beta} \\
&= y^{\varepsilon} F_{\varepsilon}^{\alpha} G_{\alpha}^{\beta} - y^{\varepsilon} G_{\varepsilon}^{\alpha} F_{\alpha}^{\beta} \\
&= y^{\varepsilon} \left(F \circ G - G \circ F\right)_{\varepsilon}^{\beta} \\
&= y^{\varepsilon} \left[FG\right]_{\varepsilon}^{\beta}
\end{aligned}$$

by virtue of (4.2). Secondly, if $J = \beta$, we have

$$[\widetilde{\gamma}F, \widetilde{\gamma}G]^{\beta} = (\widetilde{\gamma}F)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\widetilde{\gamma}G)^{\beta} + (\widetilde{\gamma}F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\widetilde{\gamma}G)^{\beta} - (\widetilde{\gamma}G)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\widetilde{\gamma}F)^{\beta} - (\widetilde{\gamma}G)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\widetilde{\gamma}F)^{\beta} = 0$$

by virtue of (4.2). Thirdly, let $J = \overline{\overline{\beta}}$. Then we have

$$\begin{aligned}
[\widetilde{\gamma}F, \widetilde{\gamma}G]^{\overline{\overline{\beta}}} &= (\widetilde{\gamma}F)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\widetilde{\gamma}G)^{\overline{\overline{\beta}}} + (\widetilde{\gamma}F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\widetilde{\gamma}G)^{\overline{\overline{\beta}}} \\
&\quad - (\widetilde{\gamma}G)^{\overline{\alpha}} \partial_{\overline{\alpha}} (\widetilde{\gamma}F)^{\overline{\overline{\beta}}} - (\widetilde{\gamma}G)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}} (\widetilde{\gamma}F)^{\overline{\overline{\beta}}} \\
&= -y^{\varepsilon} F_{\varepsilon}^{\alpha} \partial_{\overline{\alpha}} p_{\sigma} G_{\beta}^{\sigma} + p_{\sigma} F_{\alpha}^{\sigma} \partial_{\overline{\overline{\alpha}}} p_{\sigma} G_{\beta}^{\sigma} \\
&\quad + y^{\varepsilon} G_{\varepsilon}^{\alpha} \partial_{\overline{\alpha}} p_{\sigma} F_{\beta}^{\sigma} - p_{\sigma} G_{\alpha}^{\sigma} \partial_{\overline{\overline{\alpha}}} p_{\sigma} F_{\beta}^{\sigma} \\
&= p_{\varepsilon} F_{\alpha}^{\sigma} G_{\beta}^{\alpha} - p_{\varepsilon} G_{\alpha}^{\sigma} F_{\beta}^{\alpha} \\
&= p_{\sigma} (F_{\alpha}^{\sigma} G_{\beta}^{\alpha} - G_{\alpha}^{\sigma} F_{\beta}^{\alpha}) \\
&= p_{\sigma} [F, G]_{\beta}^{\sigma}
\end{aligned}$$

by virtue of (4.2). We know that $\widetilde{\gamma}[F, G]$ and $\gamma[F, G]$ have respectively, components on $t^*(M_n)$

$$\widetilde{\gamma}[F, G] = \begin{pmatrix} y^{\varepsilon}\, [F, G]_{\varepsilon}^{\beta} \\ 0 \\ -p_{\sigma} [F, G]_{\beta}^{\sigma} \end{pmatrix}, \ \gamma[F, G] = \begin{pmatrix} 0 \\ 0 \\ p_{\sigma} [F, G]_{\beta}^{\sigma} \end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ by (2.4) and (4.2). Then, we have

$$\begin{aligned}
[\widetilde{\gamma}F, \widetilde{\gamma}G] &= \begin{pmatrix} y^{\varepsilon}\,[F,G]^{\beta}_{\varepsilon} \\ 0 \\ p_{\sigma}[F,G]^{\sigma}_{\beta} \end{pmatrix} = \begin{pmatrix} y^{\varepsilon}\,[F,G]^{\beta}_{\varepsilon} \\ 0 \\ -p_{\sigma}[F,G]^{\sigma}_{\beta} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 2p_{\sigma}[F,G]^{\sigma}_{\beta} \end{pmatrix} \\
&= \widetilde{\gamma}[F,G] + 2\gamma[F,G].
\end{aligned}$$

Thus, we have $(i)$ of Theorem 4.2.

$(ii)$ If $F \in \Im^1_1(T(M_n))$, $X \in \Im^1_0(T(M_n))$ and $\begin{pmatrix} [{}^{cc}X, \widetilde{\gamma}F]^{\overline{\beta}} \\ [{}^{cc}X, \widetilde{\gamma}F]^{\beta} \\ [{}^{cc}X, \widetilde{\gamma}F]^{\overline{\overline{\beta}}} \end{pmatrix}$ are components of $[{}^{cc}X, \widetilde{\gamma}F]$ with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t^*(M_n)$, then we have

$$[{}^{cc}X, \widetilde{\gamma}F]^J = ({}^{cc}X)^I \partial_I(\widetilde{\gamma}F)^J - (\widetilde{\gamma}F)^I \partial_I({}^{cc}X)^J.$$

Firstly, if $J = \overline{\beta}$, we have

$$\begin{aligned}
[{}^{cc}X, \widetilde{\gamma}F]^{\overline{\beta}} &= ({}^{cc}X)^I \partial_I(\widetilde{\gamma}F)^{\overline{\beta}} - (\widetilde{\gamma}F)^I \partial_I({}^{cc}X)^{\overline{\beta}} \\
&= ({}^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}}(\widetilde{\gamma}F)^{\overline{\beta}} + ({}^{cc}X)^{\alpha} \partial_{\alpha}(\widetilde{\gamma}F)^{\overline{\beta}} + ({}^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}(\widetilde{\gamma}F)^{\overline{\beta}} \\
&\quad -(\widetilde{\gamma}F)^{\overline{\alpha}} \partial_{\overline{\alpha}}({}^{cc}X)^{\overline{\beta}} - (\widetilde{\gamma}F)^{\alpha} \partial_{\alpha}({}^{cc}X)^{\overline{\beta}} - (\widetilde{\gamma}F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\beta}} \\
&= y^{\varepsilon}\partial_{\varepsilon}X^{\alpha}\partial_{\overline{\alpha}}y^{\varepsilon}F^{\beta}_{\varepsilon} + X^{\alpha}\partial_{\alpha}y^{\varepsilon}F^{\beta}_{\varepsilon} - y^{\varepsilon}F^{\alpha}_{\varepsilon}\partial_{\overline{\alpha}}y^{\varepsilon}\partial_{\varepsilon}X^{\beta} \\
&= y^{\varepsilon}\partial_{\varepsilon}X^{\alpha}F^{\beta}_{\alpha} + y^{\varepsilon}X^{\alpha}\partial_{\alpha}F^{\beta}_{\varepsilon} - y^{\varepsilon}F^{\alpha}_{\varepsilon}\partial_{\alpha}X^{\beta} \\
&= y^{\varepsilon}\left(\partial_{\varepsilon}X^{\alpha}F^{\beta}_{\alpha} + X^{\alpha}\partial_{\alpha}F^{\beta}_{\varepsilon} - F^{\alpha}_{\varepsilon}\partial_{\alpha}X^{\beta}\right) \\
&= y^{\varepsilon}\,(L_X F)^{\beta}_{\varepsilon}
\end{aligned}$$

by virtue of (2.3) and (4.2). Secondly, if $J = \beta$, we have

$$\begin{aligned}
[{}^{cc}X, \widetilde{\gamma}F]^{\beta} &= ({}^{cc}X)^I \partial_I(\widetilde{\gamma}F)^{\beta} - (\widetilde{\gamma}F)^I \partial_I({}^{cc}X)^{\beta} \\
&= ({}^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}}(\widetilde{\gamma}F)^{\beta} + ({}^{cc}X)^{\alpha} \partial_{\alpha}(\widetilde{\gamma}F)^{\beta} + ({}^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}(\widetilde{\gamma}F)^{\beta} \\
&\quad -(\widetilde{\gamma}F)^{\overline{\alpha}} \partial_{\overline{\alpha}}({}^{cc}X)^{\beta} - (\widetilde{\gamma}F)^{\alpha} \partial_{\alpha}({}^{cc}X)^{\beta} - (\widetilde{\gamma}F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\beta} \\
&= 0
\end{aligned}$$

by virtue of (2.3) and (4.2). Thirdly, if $J = \overline{\overline{\beta}}$, then we have

$$\begin{aligned}
[{}^{cc}X, \widetilde{\gamma}F]^{\overline{\overline{\beta}}} &= ({}^{cc}X)^I \partial_I(\widetilde{\gamma}F)^{\overline{\overline{\beta}}} - (\widetilde{\gamma}F)^I \partial_I({}^{cc}X)^{\overline{\overline{\beta}}} \\
&= ({}^{cc}X)^{\overline{\alpha}} \partial_{\overline{\alpha}}(\widetilde{\gamma}F)^{\overline{\overline{\beta}}} + ({}^{cc}X)^{\alpha} \partial_{\alpha}(\widetilde{\gamma}F)^{\overline{\overline{\beta}}} + ({}^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}(\widetilde{\gamma}F)^{\overline{\overline{\beta}}} \\
&\quad -(\widetilde{\gamma}F)^{\overline{\alpha}} \partial_{\overline{\alpha}}({}^{cc}X)^{\overline{\overline{\beta}}} - (\widetilde{\gamma}F)^{\alpha} \partial_{\alpha}({}^{cc}X)^{\overline{\overline{\beta}}} - (\widetilde{\gamma}F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\overline{\beta}}} \\
&= ({}^{cc}X)^{\alpha} \partial_{\alpha}(\widetilde{\gamma}F)^{\overline{\overline{\beta}}} + ({}^{cc}X)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}(\widetilde{\gamma}F)^{\overline{\overline{\beta}}} - (\widetilde{\gamma}F)^{\overline{\overline{\alpha}}} \partial_{\overline{\overline{\alpha}}}({}^{cc}X)^{\overline{\overline{\beta}}} \\
&= -X^{\alpha}\partial_{\alpha}p_{\sigma}F^{\sigma}_{\beta} + p_{\sigma}(\partial_{\alpha}X^{\sigma})\partial_{\overline{\overline{\alpha}}}p_{\sigma}F^{\sigma}_{\beta} - p_{\sigma}F^{\sigma}_{\alpha}\partial_{\overline{\overline{\alpha}}}p_{\sigma}(\partial_{\beta}X^{\sigma}) \\
&= -X^{\alpha}\partial_{\alpha}p_{\sigma}F^{\sigma}_{\beta} + p_{\sigma}(\partial_{\alpha}X^{\sigma})F^{\alpha}_{\beta} - p_{\sigma}F^{\sigma}_{\alpha}(\partial_{\beta}X^{\alpha}) \\
&= -p_{\sigma}(X^{\alpha}\partial_{\alpha}F^{\sigma}_{\beta} - \partial_{\alpha}X^{\sigma}F^{\alpha}_{\beta} + \partial_{\beta}X^{\alpha}F^{\sigma}_{\alpha}) \\
&= -p_{\sigma}(L_X F)^{\sigma}_{\beta}
\end{aligned}$$

by virtue of (2.3) and (4.2). On the other hand, we know that $\widetilde{\gamma}(L_X F)$ have components

$$\widetilde{\gamma}(L_X F) = \begin{pmatrix} y^{\varepsilon}\,(L_X F)^{\beta}_{\varepsilon} \\ 0 \\ -p_{\sigma}(L_X F)^{\sigma}_{\beta} \end{pmatrix}$$

with respect to the coordinates $(x^{\overline{\beta}}, x^{\beta}, x^{\overline{\overline{\beta}}})$ on $t^*(M_n)$. Thus, we have $(ii)$ of Theorem 4.2. □

**Theorem 4.3.** *Let* $X \in \Im^1_0(T(M_n))$. *If* $\omega \in \Im^0_1(M_n)$, $F \in \Im^1_1(T(M_n))$ *and* $S, T \in \Im^1_2(M_n)$, *then*

$(i)$ $(\widetilde{\gamma}S)^{cc}X = \widetilde{\gamma}(S_X)$,
$(ii)$ $(\widetilde{\gamma}S)({}^{vv}\omega) = 0$,
$(iii)$ $(\widetilde{\gamma}S)(\widetilde{\gamma}F) = 0$,
$(iv)$ $(\widetilde{\gamma}S)(\widetilde{\gamma}T) = 0$,

*where* $S_X$ *is tensor field of type (1,1) on* $T(M_n)$ *defined by* $S_X(Z) = S(X, Z)$ *for any* $Z \in \Im^1_0(T(M_n))$.

*Proof.* $(i)$ Using (2.3) and (4.3), we have

$$\begin{aligned} (\widetilde{\gamma}S)^{cc}X &= \begin{pmatrix} 0 & y^{\varepsilon}S^{\alpha}_{\varepsilon\,\beta} & 0 \\ 0 & 0 & 0 \\ 0 & -p_{\sigma}S^{\sigma}_{\beta\,\alpha} & 0 \end{pmatrix} \begin{pmatrix} y^{\varepsilon}\partial_{\varepsilon}X^{\beta} \\ X^{\alpha} \\ -p_{\sigma}(\partial_{\beta}X^{\sigma}) \end{pmatrix} \\ &= \begin{pmatrix} y^{\varepsilon}S^{\alpha}_{\varepsilon\,\beta}X^{\beta} \\ 0 \\ -p_{\sigma}S^{\sigma}_{\beta\,\alpha}X^{\alpha} \end{pmatrix} = \begin{pmatrix} y^{\varepsilon}(S_X)^{\alpha}_{\varepsilon} \\ 0 \\ -p_{\sigma}(S_X)^{\sigma}_{\beta} \end{pmatrix} = \widetilde{\gamma}(S_X). \end{aligned}$$

Similarly, we have

$$(\widetilde{\gamma}S)({}^{vv}\omega) = 0, \quad (\widetilde{\gamma}S)(\widetilde{\gamma}F) = 0, \quad (\widetilde{\gamma}S)(\widetilde{\gamma}T) = 0.$$

□

**Theorem 4.4.** *For any affinor fields* $F, G$ *on* $T(M_n)$,

$${}^{cc}F(\widetilde{\gamma}G) = \widetilde{\gamma}(G \circ F).$$

*Proof.* If $F$ and $G$ are affinor fields on $T(M_n)$, then we have by (3.1) and (4.2):

$$\begin{aligned} {}^{cc}F(\widetilde{\gamma}G) &= \begin{pmatrix} F^{\alpha}_{\beta} & y^{\varepsilon}\partial_{\varepsilon}F^{\alpha}_{\beta} & 0 \\ 0 & F^{\alpha}_{\beta} & 0 \\ 0 & p_{\sigma}(\partial_{\beta}F^{\sigma}_{\alpha} - \partial_{\alpha}F^{\sigma}_{\beta}) & F^{\beta}_{\alpha} \end{pmatrix} \begin{pmatrix} y^{\varepsilon}G^{\beta}_{\varepsilon} \\ 0 \\ -p_{\sigma}G^{\sigma}_{\beta} \end{pmatrix} \\ &= \begin{pmatrix} y^{\varepsilon}G^{\beta}_{\varepsilon}F^{\alpha}_{\beta} \\ 0 \\ -p_{\sigma}G^{\sigma}_{\beta}F^{\beta}_{\alpha} \end{pmatrix} = \begin{pmatrix} y^{\varepsilon}\,(G \circ F)^{\alpha}_{\varepsilon} \\ 0 \\ -p_{\sigma}\,(G \circ F)^{\sigma}_{\alpha} \end{pmatrix} = \widetilde{\gamma}(G \circ F). \end{aligned}$$

Thus we have ${}^{cc}F(\widetilde{\gamma}G) = \widetilde{\gamma}(G \circ F)$. □

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