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BOUNDEDNESS OF THE MULTILINEAR FRACTIONAL INTEGRAL OPERATORS WITH ROUGH KERNEL ON MORREY SPACES

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Abstract. In this paper the boundedness of multi-sublinear fractional maximal operator $M_{\Omega,\alpha,m}$ and multilinear fractional integral operator $I_{\Omega,\alpha,m}$ with rough kernels on product Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ are studied. The authors study necessary and sufficient conditions on the parameters of the boundedness on product Morrey spaces for $I_{\Omega,\alpha,m}$ and $M_{\Omega,\alpha,m}$.

1. Introduction

The multilinear theory has been well developed in the past twenty years. Let $m \geq 1$ will denote an integer, θ_j $(j = 1, ..., m)$ will be fixed, distinct, and nonzero real numbers, $0 < \alpha < n$ and we denote $\mathbf{f} = (f_1, ..., f_m)$. In 1992, Grafakos [5] introduced the following multilinear fractional integral which is defined by

$$I_{\alpha,m}\mathbf{f}(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} \prod_{j=1}^{m} f_j(x - \theta_j y)dy.$$

Grafakos [5] proved that $I_{\alpha,m}$ is bounded from product $L^{p_1}(\mathbb{R}^n) \times ... \times L^{p_m}(\mathbb{R}^n)$ spaces to $L^q(\mathbb{R}^n)$ space with $0 < 1/q = 1/p_1 + \ldots + 1/p_m < \alpha/n$, which can be regarded as an extension for the classical fractional integral on the Lebesgue spaces. In [8, 9, 12] was proved a certain O'Neil type inequality for dilated multilinear convolution operators, including permutations of functions. This inequality was used to extend Grafakoss result [5] to more general multi-linear operators of potential type and the relevant maximal operators.

Suppose that Ω is homogeneous of degree zero on $\mathbb{R}^n$ and $\Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s \leq \infty$, where $\mathbb{S}^{n-1}$ denote the unit sphere of $\mathbb{R}^n$. Then the multilinear fractional integral operator $I_{\Omega,\alpha,m}$ with rough kernel Ω on $\mathbb{R}^n$ is given by the formula

$$I_{\Omega,\alpha,m}\mathbf{f}(x) = \int_{\mathbb{R}^n} \frac{\Omega(y)}{|y|^{n-\alpha}} \prod_{j=1}^{m} f_j(x - \theta_j y)dy,$$

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and the multi-sublinear fractional maximal operator $M_{\Omega,\alpha,m}$ with rough kernel Ω

$$M_{\Omega,\alpha,m}\mathbf{f}(x) = \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(0,r)} |\Omega(y)| \prod_{j=1}^{m} |f_j(x-\theta_j y)| dy,$$

where $B(0,r) = \{y \in \mathbb{R}^n : |y| < r\}$. If $\alpha = 0$, then $M_\Omega \equiv M_{\Omega,0}$ is the multi-sublinear maximal operator.

When $m = 1$ and $\Omega \equiv 1$, if let $\theta_1 = 1$, $I_{\Omega,\alpha,m}$ will be the Riesz potential operator I_α [14, 18] given by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-\alpha}} dy.$$

Note that, in [10, 11] Guliyev, Ismayilova was study the boundedness of multi-sublinear fractional maximal operator, multilinear fractional integral operator and multilinear singular integral opertors on product generalized Morrey spaces.

Spanne and Adams obtained two remarkable results on Morrey spaces (see Definition of the Morrey spaces in Section 2) for I_α. Their results can be summarized as follows.

Theorem 1.1. [6, 17] ***(Spanne, but published by Peetre)*** *Let $0 < \alpha < n$, $0 \le \lambda < n - \alpha p$, $1/q = 1/p - \alpha/n$ and $\mu/q = \lambda/p$. Then for $p > 1$, the operator I_α is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ and for $p = 1$, I_α is bounded from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\mu}(\mathbb{R}^n)$.*

Theorem 1.2. [1, 7] *Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $0 \le \lambda < n/\alpha p$.*

(i) *If $p > 1$, then condition $1/p - 1/q = \alpha(n-\lambda)$ is necessary and sufficient for the boundedness of the operator I_α from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*
(ii) *If $p = 1$, then condition $1 - 1/q = \alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of the operator I_α from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\lambda}(\mathbb{R}^n)$.*

If $\lambda = 0$, then the statement of Theorems 1.1 and 1.2 reduces to the well known Hardy-Littlewood-Sobolev inequality.

Let $0 < \alpha < n$, $\Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s \le \infty$, and $1/p - 1/q = \alpha/(n-\lambda)$. In this work, we prove the boundedness of the multi-sublinear fractional maximal operator $M_{\Omega,\alpha,m}$ and multilinear fractional integral operator $I_{\Omega,\alpha,m}$ with rough kernels on product Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to Morrey space $L^{q,\lambda}(\mathbb{R}^n)$, if $p > s'$, $1 < p_1, \ldots, p_m < \infty$, $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/(n-\lambda)$ and from the space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak space $WL^{q,\lambda}(\mathbb{R}^n)$, if $p = s'$, $1 \le p_1, \ldots, p_m < \infty$, $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/(n-\lambda)$ and at least one exponent p_i, $1 \le i \le m$ equals one.

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Morrey spaces

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r. Suppose that S^{n-1} is the unit sphere in $\mathbb{R}^n$ $(n \ge 2)$ equipped with the normalized Lebesgue measure $d\sigma$.

Morrey spaces $L^{p,\lambda}$, named after C. Morrey, were introduced by him in 1938 in [15] and defined as follows: For $\lambda \in \mathbb{R}, 0 < p \le \infty, f \in L^{p,\lambda}$ if $f \in L^p_{loc}$ and

$$\|f\|_{L^{p,\lambda}} \equiv \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda} \|f\|_{Lp(B(x,r))} < \infty,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at the point $x \in \mathbb{R}^n$ of radius $r > 0$.

In other words $f \in L^{p,\lambda}$ if $f \in L^p_{loc}(\mathbb{R}^n)$ and there exists $c > 0$ (depending on f) such that for all $x \in \mathbb{R}^n$ and for all $r > 0$

$$\|f\|_{L^p(B(x,r))} \le cr^{\lambda}.$$

The minimal value of c in this inequality is $\|f\|_{L^{p,\lambda}}$.

If $\lambda = 0$, then

$$L^{p,0}(\mathbb{R}^n) = L^p(\mathbb{R}^n).$$

If $\lambda = \frac{n}{p}$, then

$$L^{p,\frac{n}{p}}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n).$$

If $\lambda > \frac{n}{p}$ or $\lambda < 0$, then

$$L^{p,\lambda} = \Theta,$$

where $\Theta \equiv \Theta(\mathbb{R}^n)$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$. So the admissible range of the parameters is

$$0 < p \le \infty \quad and \quad 0 \le \lambda \le \frac{n}{p}.$$

(If $p = \infty$ then the inequality for λ holds only if $\lambda = 0$ and $L^{\infty,0} = L^\infty$.)

Under these assumptions, which will always be assumed in the sequel, the space $L^{p,\lambda}$ is a Banach space for $1 \le p \le \infty$ and a quasi-Banach space for $0 < p < 1$. Also the space $L^{p,\lambda}$ does not coincide with a Lebesgue space, if and only if

$$0 < p < \infty \quad and \quad 0 < \lambda < \frac{n}{p}.$$

Furthermore,

$$L^\infty \cap L^p \subset L^{p,\lambda}.$$

If $f \in L^p$, then $f \in L^{p,\lambda}$ if and only if

$$\sup_{x\in\mathbb{R}^n, 0<r\le 1} r^{-\lambda} \|f\|_{L^p(B(x,r))} < \infty,$$

hence under this assumption only local properties of f are of importance.

Also by $WL^{p,\lambda}$ we denote the weak Morrey space, the space the space of all functions $f \in WL^p_{loc}$ with finite quasi-norm

$$\|f\|_{WL^{p,\lambda}} \equiv \|f\|_{WL^{p,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} r^{-\lambda} \|f\|_{WL^p(B(x,r))}.$$

Recall the definition of $M_{\Omega,m}$, as a special case when $m = 1$, $\Omega \equiv 1$ and $\theta_1 = 1$, $M_{\Omega,m}$ is the Hardy-Littlewood maximal operator M. In [2] Chiarenza and Frasca obtained the boundedness of M on Morrey spaces.

Lemma 2.1. [2] *Let $1 \le p < \infty$ and $0 \le \lambda < n$.Then for $p > 1$, M is bounded from $L^{p,\lambda}$ to $L^{p,\lambda}$ and for $p = 1$, to M is bounded from $L^{1,\lambda}$ to $WL^{1,\lambda}$.*

The following theorem was proved by Ding, Lu in [3].

Theorem 2.1. *Let p be the harmonic mean of $p_1, p_2, \dots, p_m > 1$. Then we have the following conclusions.*

(i) If $p > 1$, $\Omega \in L^s\left(\mathbb{S}^{n-1}\right)$, $s \geq 1$, then $M_{\Omega,m}(\mathbf{f})$ maps $L^{p_1} \times L^{p_2} \times \ldots \times L^{p_m}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$.

(ii) If $p = 1$, $\Omega \in L\log^+ L\left(\mathbb{S}^{n-1}\right)$, then $M_{\Omega,m}(\mathbf{f})$ maps $L^{p_1} \times L^{p_2} \times \cdots \times L^{p_m}(\mathbb{R}^n)$ into $L^{1,\infty}(\mathbb{R}^n)$.

When $m \geq 2$ and $\Omega \in L^s(\mathbb{S}^{n-1})$, we find out $M_{\Omega,m}$ also have the same properties by providing the following multi-version of the Lemma 2.1. The following theorem was proved in [13].

Theorem 2.2. *Let $\Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s \leq \infty$, $0 \leq \lambda < n$, p be the harmonic mean of $p_1, ..., p_m > 1$, $p \geq s'$ and satisfy*

$$\frac{\lambda}{p} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j} \quad for\ 0 \leq \lambda_j < n. \tag{2.1}$$

(i) *If $p > s'$, there exists a positive constant C such that*

$$\|M_{\Omega,m}\boldsymbol{f}\|_{L^{p,\lambda}} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) *If $p = s'$, there exists a positive constant C such that*

$$\|M_{\Omega,m}\boldsymbol{f}\|_{WL^{p,\lambda}} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

3. Boundedness of $I_{\Omega,\alpha,m}$ and $M_{\Omega,\alpha,m}$

When $m \geq 2$ and $\Omega \equiv 1$, Grafakos [5] studied Lebesgue boundedness of $I_{1,\alpha,m}$ Recently, Gunawan [4] extended Grafakos's result to Morrey spaces and provided a multi-version for the sufficiency of conclusion (i) in Theorem 1.2.

Theorem 3.1. [4] *Let $0 < \alpha < n$, p be the harmonic mean of $p_1, ..., p_m > 1$, $1 < p < n/\alpha$, $0 \leq \lambda < n - \alpha p$, $1/p - 1/q = \alpha/(n-\lambda)$, then the operator $I_{1,\alpha,m}$ is bounded from $L^{p_1,\lambda}(\mathbb{R}^n) \times ... \times L^{p_m,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*

When $m \geq 2$ and $\Omega \in L^s(\mathbb{S}^{n-1})$, Ding and Lu [3] studied the $L^{p_1} \times ... \times L^{p_m}$ boundedness for $I_{\Omega,\alpha,m}$. After these works above, a natural question is: what properties does the operator $I_{\Omega,\alpha,m}$ have on Morrey spaces. We give answers as follows:

Theorem 3.2. *Let $0 < \alpha < n$, $\Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s \leq \infty$, p be the harmonic mean of $p_1, ..., p_m > 1$, $0 \leq \lambda < n - \alpha p$, $1 \leq p < n/\alpha$ and satisfy the condition (2.1).*

(i) *If $p > s'$, then the condition $1/p - 1/q = \alpha(n - \lambda)$ is necessary and sufficient for the boundedness of the operator $I_{\Omega,\alpha,m}$ from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times ... \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.*

(ii) *If $p = s'$, then the condition $1/s' - 1/q = \alpha(n - \lambda)$ is necessary and sufficient for the boundedness of the operator $I_{\Omega,\alpha,m}$ from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times ... \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $WL^{q,\lambda}(\mathbb{R}^n)$.*

Moreover, similar conclusions hold for $M_{\Omega,\alpha,m}$.

Remark 3.1. Note that Theorem 3.2 cover Theorem 1.2. Also, the case $\lambda = \lambda_1 = ... = \lambda_m$ and $\Omega \equiv 1$ reduces to Theorem 3.1; the case $\lambda = \lambda_1 = ... = \lambda_m = 0$ gives the result of Ding and Lu [3] on Lebesgue spaces.

We observe that, in Theorem 3.2, the boundedness in the limiting case $p = (n-\lambda)/\alpha$ remains open. In fact, when $p = n/\alpha$ (i.e. $\lambda = 0$), Ding and Lu [3] found $M_{\Omega,\alpha,m}$ is bounded from $L^{p_1} \times ... \times L^{p_m}$ to L^∞, but this corresponding result for $I_{\Omega,\alpha,m}$ in this case does not hold. Our next goal is to extend Ding and Lu's result to the case $0 \le \lambda < n - \alpha$, as the continuation of Theorem 3.2.

Theorem 3.3. *Let $0 < \alpha < n$, $\Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s \le \infty$, p be the harmonic mean of $p_1, ..., p_m > 1$ and satisfy (2.1).*

If $p = (n-\lambda)/\alpha \ge s'$, then the operator $M_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times ... \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

In this part, we will prove Theorem 3.2. Let us begin with a requisite Hedberg's type estimates, which plays a key role during proving Theorem 3.2.

Lemma 3.1. *Let $0 < \alpha < n$, $\Omega \in L^s(\mathbb{S}^{n-1})$ with $1 < s \le \infty$, p be the harmonic mean of $p_1 ..., p_m > 1$, $0 \le \lambda < n - \alpha p$, $s' \le p < n/\alpha$ and satisfy (2.1), then there exists a positive constant C such that*

$$|I_{\Omega,\alpha,m}\boldsymbol{f}(x)| \le C[M_\Omega \boldsymbol{f}(x)]^{1-p\alpha/(n-\lambda)} \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}^{p\alpha/(n-\lambda)}.$$

Proof. For any $\delta > 0$, we split the integral into two parts:

$$\begin{aligned} I_{\Omega,\alpha,m}\mathbf{f}(x) &= \Big(\int_{B(0,\delta)} + \int_{\mathbb{R}^n \setminus B(0,\delta)} \Big) \frac{\Omega(y)}{|y|^{n-\alpha}} \prod_{j=1}^m f_j(x - \theta_j y) dy \\ &=: A(x,\delta) + B(x,\delta). \end{aligned}$$

For $A(x,\delta)$, we have

$$\begin{aligned} |A(x,\delta)| &\le \int_{B(0,\delta)} \frac{\Omega(y)}{|y|^{n-\alpha}} \prod_{j=1}^m |f_j(x-\theta_j y)| dy \\ &\le \sum_{i=0}^\infty \int_{B(0,2^{-i}\delta) \setminus B(0,2^{-i-1}\delta)} \frac{\Omega(y)}{|y|^{n-\alpha}} \prod_{j=1}^m |f_j(x-\theta_j y)| dy \\ &\le \sum_{i=0}^\infty \big(2^{-i-1}\delta\big)^{\alpha-n} \int_{B(0,2^{-i}\delta)} |\Omega(y)| \prod_{j=1}^m |f_j(x-\theta_j y)| dy \\ &\le \sum_{i=0}^\infty \big(2^{-i-1}\delta\big)^{\alpha-n} \big(2^{-i}\delta\big)^n M_{\Omega,m}\mathbf{f}(x) \\ &\le 2^{n-\alpha}\delta^\alpha M_{\Omega,m}\mathbf{f}(x) \sum_{i=0}^\infty 2^{-i\alpha} \le C\delta^\alpha M_{\Omega,m}\mathbf{f}(x). \end{aligned}$$

Recalling the conditions of Lemma 3.1, we can see $s' \le p < (n-\lambda)/\alpha$, which implies $\alpha < (n-\lambda)/p \le (n-\lambda)/s'$, then we get

$$n - \alpha s' > n - (n-\lambda)s'/p \ge n - (n-\lambda) = \lambda.$$

In order to estimate $B(x,\delta)$, we choose a real number σ such that

$$n - \alpha s' > \sigma > n - (n-\lambda)s'/p \ge \lambda.$$

One can then see from the chose of σ that

$$n - (n - \alpha - \sigma/s')s < 0 \tag{3.1}$$

and

$$(n-\sigma)/s' - (n-\lambda)/p < 0. \tag{3.2}$$

Then, using the Hölder inequality, we obtain

$$|B(x,\delta)| \le \int_{\mathbb{R}^n \setminus B(0,\delta)} \frac{|\Omega(y)|}{|y|^{n-\alpha-\sigma/s'}} \frac{1}{|y|^{\sigma/s'}} \prod_{j=1}^m |f_j(x-\theta_j y)| dy$$

$$\le \Big(\int_{\mathbb{R}^n \setminus B(0,\delta)} \frac{|\Omega(y)|}{|y|^{(n-\alpha-\sigma/s')s}} dy \Big)^{1/s} \Big(\int_{\mathbb{R}^n \setminus B(0,\delta)} \frac{1}{|y|^{\sigma}} \prod_{j=1}^m |f_j(x-\theta_j y)|^{s'} dy \Big)^{1/s'}$$

$$=: E_\sigma(\delta) \times F_\sigma(x,\delta).$$

For $E_\sigma(\delta)$, by the fact (3.1), we obtain

$$E_\sigma(\delta) = \Big(\int_\delta^\infty \int_{\mathbb{S}^{n-1}} |\Omega(\xi)^s| r^{n-(n-\alpha-\sigma/s')^{s-1}} d\xi dr \Big)^{1/s} = C\delta^{\alpha-(n-\sigma)/s'}.$$

For $F_\sigma(x,\delta)$, we have

$$\begin{aligned} F_\sigma(x,\delta) &\le \Big(\sum_{i=0}^\infty \int_{B(0,2^{i+1}\delta)\setminus B(0,2^i\delta)} \frac{1}{|y|^\sigma} \prod_{j=1}^m |f_j(x-\theta_j y)|^{s'} dy \Big)^{1/s'} \\ &\le \sum_{i=0}^\infty (2^{-i}\delta)^{-\sigma/s'} \Big(\int_{B(0,2^{i+1}\delta)} \prod_{j=1}^m |f_j(x-\theta_j y)|^{s'} dy \Big)^{1/s'} \end{aligned}$$

If $p > s'$, applying Hölder's inequality and the fact (3.2), we have

$$F_\sigma(x,\delta) \le \sum_{i=0}^\infty (2^{-i}\delta)^{-\sigma/s'} \Big(\int_{B(0,2^{i+1}\delta)} dy \Big)^{1/s'-1/p}$$

$$\times \Big(\int_{B(0,2^{i+1}\delta)} \prod_{j=1}^m |f_j(x-\theta_j y)|^p dy \Big)^{1/p}$$

$$\le C \sum_{i=0}^\infty (2^{-i}\delta)^{(n-\sigma)/s'-n/p} \Big(\int_{B(0,2^{i+1}\delta)} \prod_{j=1}^m |f_j(x-\theta_j y)|^p dy \Big)^{1/p}$$

$$\le C \sum_{i=0}^\infty (2^{-i}\delta)^{(n-\sigma)/s'-(n-\lambda)/p} \Big(\frac{1}{(2^{i+1}\delta)^\lambda} \int_{B(0,2^{i+1}\delta)} \prod_{j=1}^m |f_j(x-\theta_j y)|^p dy \Big)^{1/p}$$

$$\le C \sum_{i=0}^\infty (2^{-i}\delta)^{(n-\sigma)/s'-(n-\lambda)/p} \prod_{j=1}^m \Big(\frac{1}{(2^{i+1}\delta)^{\lambda_j}} \int_{B(0,2^{i+1}\delta)} |f_j(x-\theta_j y)|^p dy \Big)^{1/p_j}$$

$$\leq C\sum_{i=0}^{\infty}(2^{-i}\delta)^{(n-\sigma)/s'-(n-\lambda)/p}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}$$

$$\leq C\delta^{(n-\sigma)/s'-(n-\lambda)/p}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}.$$

If $p=s'$, using the Hölder inequality and the fact $\lambda<\sigma$, we get

$$F_\sigma(x,\delta)\leq\sum_{i=0}^{\infty}(2^{-i}\delta)^{-\sigma/s'}\Big(\int_{B(0,2^{i+1}\delta)}\prod_{j=1}^{m}|f_j(x-\theta_j y)|^{s'}dy\Big)^{1/s'}$$

$$\leq C\sum_{i=0}^{\infty}(2^{-i}\delta)^{-\sigma/s'+\lambda/s'}\Big(\frac{1}{(2^{i+1}\delta)^{\lambda}}\int_{B(0,2^{i+1}\delta)}\prod_{j=1}^{m}|f_j(x-\theta_j y)|^{s'}dy\Big)^{1/s'}$$

$$\leq C\sum_{i=0}^{\infty}(2^{-i}\delta)^{(\lambda-\sigma)/s'}\prod_{j=1}^{m}\Big(\frac{1}{(2^{i+1}\delta)^{\lambda}}\int_{B(0,2^{i+1}\delta)}|f_j(x-\theta_j y)|^{p_j}dy\Big)^{1/p_j}$$

$$\leq C\sum_{i=0}^{\infty}(2^{-i}\delta)^{(\lambda-\sigma)/s'}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}$$

$$\leq C\delta^{(\lambda-\sigma)/s'}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}\leq C\delta^{(n-\sigma)/s'-(n-\lambda)/p}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}.$$

Hence, for every $p\geq s'$, we have

$$|S_\sigma(x,\delta)|\leq C\delta^{\sigma-(n-\lambda)/p}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}.$$

Thus

$$|I_{\Omega,\alpha,m}\mathbf{f}(x)|\leq C\Big(\delta^{\sigma}M_{\Omega,\alpha,m}\mathbf{f}(x)+\delta^{\sigma-(n-\lambda)/p}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}\Big),\ \delta>0.$$

Now take

$$\delta=\Big[(M_{\Omega,\alpha,m}\mathbf{f}(x))^{-1}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}\Big]^{p/(n-\lambda)}$$

and then we get the conclusion of Lemma 3.1. □

Now we are ready to the proof of Theorem 3.2.

Proof of Theorem 3.2. Firstly, we will devote to the proof of (i). Sufficiency. By Lemma 3.1 and the conclusion (i) of Theorem 2.2, we have

$$\left(\frac{1}{t^{\lambda}}\int_{B(x,t)}|I_{\Omega,\alpha,m}\mathbf{f}(y)|^{q}dy\right)^{1/q}\leq C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}^{1-p/q}\left(\frac{1}{t^{\lambda}}\int_{B(x,t)}(M_{\Omega}\mathbf{f}(y))^{p}dy\right)^{1/q}$$

$$\leq C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}^{1-p/q}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}^{p/q}\leq C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}.$$

Taking the supremum for $x \in \mathbb{R}^n$ and $t > 0$, we will get the desired conclusion. Necessity. Suppose that $I_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1} \times ... \times L^{p_m,\lambda_m}$ to $L^{q,\lambda}$. Define $\mathbf{f}_\varepsilon(x) = (f_1(\varepsilon x), ..., f_m(\varepsilon x))$ for $\varepsilon > 0$. Then it is easy to show that

$$I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon(y) = \varepsilon^{-\alpha} I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y). \tag{3.3}$$

Thus

$$\begin{aligned}
\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon(y)\|_{L^{q,\lambda}} &= \varepsilon^{-\alpha} \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda}\int_{B(x,t)} |I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)|^q dy\right)^{1/q} \\
&= \varepsilon^{-\alpha-n/q} \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda}\int_{B(\varepsilon x,\varepsilon t)} |I_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\right)^{1/q} \\
&= \varepsilon^{-\alpha-n/q+\lambda/q} \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{(\varepsilon t)^\lambda}\int_{B(\varepsilon x,\varepsilon t)} |I_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\right)^{1/q} \\
&= \varepsilon^{-\alpha-(n-\lambda)/q} \|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{L^{q,\lambda}}.
\end{aligned}$$

Since $I_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1} \times ... \times L^{p_m,\lambda_m}$ to $L^{q,\lambda}$, we have

$$\begin{aligned}
\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} &= \varepsilon^{\alpha+(n-\lambda)/q}\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{L^{q,\lambda}} \\
&\le C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \|f_j(\varepsilon\cdot)\|_{L^{p_j,\lambda_j}} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^{\lambda_j}}\int_{B(x,t)} |f_j(\varepsilon y)|^{p_j} dy\right)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \varepsilon^{-n/p_j} \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^{\lambda_j}}\int_{B(\varepsilon x,\varepsilon t)} |f_j(y)|^{p_j} dy\right)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^m \varepsilon^{(\lambda_j-n)/p_j} \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{(\varepsilon t)^{\lambda_j}}\int_{B(\varepsilon x,\varepsilon t)} |f_j(y)|^{p_j} dy\right)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q-(n-\lambda)/p} \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}},
\end{aligned}$$

where C is independent of ε.

If $1/p < 1/q + \alpha/(n-\lambda)$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times ... \times L^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = 0$ as $\varepsilon \to 0$.

If $1/p > 1/q + \alpha/(n-\lambda)$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times ... \times L^{p_m,\lambda_m}$, we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = 0$ as $\varepsilon \to 0$.

Therefore we get $1/p = 1/q + \alpha/(n-\lambda)$.

We proceed to prove (ii). Suffciency. For any $\beta > 0$, applying Lemma 3.1 and the conclusion (ii) of Theorem 2.2, we get

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} = \sup_{\beta>0}\beta \sup_{x\in\mathbb{R}^n, t>0} \left(\frac{1}{t^\lambda}|\{y \in B(x,t) : |I_{\Omega,\alpha,m}\mathbf{f}(y)| > \beta\}|\right)^{1/q}$$

$$\leq \sup_{\beta>0}\beta \sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^\lambda}|\{y\in B(x,t):|(CM_\Omega\mathbf{f}(y))^{s'/q}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}^{1-s'/q}>\beta\}|\right)^{1/q}$$

$$\leq \sup_{\beta>0}\beta \sup_{x\in\mathbb{R}^n,t>0}\left[\frac{1}{t^\lambda}\left|\left\{y\in B(x,t):M_\Omega\mathbf{f}(y)>\left(\frac{\beta}{C\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}^{1-s'/q}}\right)^{q/s'}\right\}\right|\right]^{1/q}$$

$$\leq \sup_{\beta>0}\beta \sup_{x\in\mathbb{R}^n,t>0}\left[\frac{1}{t^\lambda}\left|\left\{y\in B(x,t):M_\Omega\mathbf{f}(y)>\left(\frac{\beta}{C\Pi_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}^{1-s'/q}}\right)^{q/s'}\right\}\right|\right]^{\frac{1}{s'}\times\frac{s'}{q}}$$

$$\leq C\sup_{\beta>0}\beta\Big[\left(\frac{\Pi_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}^{1-s'/q}}{\beta}\right)^{q/s'}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}\Big]^{s'/q}$$

$$\leq C\sup_{\beta>0}\beta\Big[\frac{\Pi_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}^{1-s'/q}}{\beta}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}\Big]^{s'/q}\leq\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}.$$

Thus, we complete the sufficiency of (ii).

Necessity. Let $I_{\Omega,\alpha,m}$ is bounded from $f\in L^{p_1,\lambda_1}\times...\times L^{p_m,\lambda_m}$ to $WL^{q,\lambda}$. Because we have (3.3) for $\mathbf{f}_\varepsilon(x)=(f_1(\varepsilon x),...,f_m(\varepsilon x))$ with $\varepsilon>0$, then we obtain

$$\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{WL^{q,\lambda}}=\sup_{r>0}r\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^\lambda}\int_{\{y\in B(x,t):|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon(y)|>r\}}dy\right)^{\frac{1}{q}}$$

$$=\sup_{r>0}r\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^\lambda}\int_{\{y\in B(x,t):|I_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)|>r\varepsilon^\alpha\}}dy\right)^{\frac{1}{q}}$$

$$=\varepsilon^{-n/q}\sup_{r>0}r\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{t^\lambda}\int_{\{y\in B(\varepsilon x,\varepsilon t):|I_{\Omega,\alpha,m}\mathbf{f}(y)|>r\varepsilon^\alpha\}}dy\right)^{\frac{1}{q}}$$

$$=\varepsilon^{-\alpha-n/q+\lambda/q}\sup_{r>0}r\varepsilon^\alpha\sup_{x\in\mathbb{R}^n,t>0}\left(\frac{1}{(\varepsilon t)^\lambda}\int_{\{y\in B(\varepsilon x,\varepsilon t):|I_{\Omega,\alpha,m}\mathbf{f}(y)|>r\varepsilon^\alpha\}}dy\right)^{\frac{1}{q}}$$

$$=\varepsilon^{-\alpha-n/q+\lambda/q}\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}}.$$

Since $I_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1}\times...\times L^{p_m,\lambda_m}$ to $WL^{q,\lambda}$, we have

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}}=\varepsilon^{\alpha+(n-\lambda)/q}\|I_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{WL^{q,\lambda}}$$

$$\leq C\varepsilon^{-\alpha-n/q+\lambda/q}\prod_{j=1}^m\|f_j(\varepsilon\cdot)\|_{L^{p_j,\lambda_j}}\leq\varepsilon^{-\alpha-n/q+\lambda/q-(n-\lambda)/p}\prod_{j=1}^m\|f_j\|_{L^{p_j,\lambda_j}}$$

where C is independent of ε.

If $1/p<1/q+\alpha/(n-\lambda)$, then for all $\mathbf{f}\in L^{p_1,\lambda_1}\times...\times L^{p_m,\lambda_m}$ we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}}=0$ as $\varepsilon\to0$.

If $1/p>1/q+\alpha/(n-\lambda)$, then for all $\mathbf{f}\in L^{p_1,\lambda_1}\times...\times L^{p_m,\lambda_m}$ we have $\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}}=0$ as $\varepsilon\to0$.

Consequently, we get $1/p=1/q+\alpha/(n-\lambda)$.

Next, we prove conclusions (i) and (ii) hold for $M_{\Omega,\alpha,m}$. By the same arguments as above we get the necessity part and the sufficiency part follows from the conclusion of $I_{\Omega,\alpha,m}$ and following lemma.

Lemma 3.2. [3] *Suppose that* $0<\alpha<n$, $\Omega\in L^s(\mathbb{S}^{n-1})$ *with* $1<s\leq\infty$. *Then*

$$M_{\Omega,\alpha,m}(\boldsymbol{f})(x)\leq C_{\alpha,n}I_{|\Omega|,\alpha,m}(|\boldsymbol{f}|)(x),$$

where $|\boldsymbol{f}|=(|f_1|,...,|f_m|)$.

Then the proof of Theorem 3.2 is completed.

As a application of Theorem 3.2, we get Spanne type estimates, which can be seen a multi-version of Theorem 1.1.

Corollary 3.1. *Let* $\alpha,\Omega,s,p_j,\lambda_j,p$ *and* λ *are as in Theorem 3.2,* $1/q=1/p-\alpha/n,\mu/q=\lambda/p$
(i) If $p>s$, *then* $I_{\Omega,\alpha,m}$ *is bounded from* $L^{p_1,\lambda_1}\times...\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ *to* $L^{q,\mu}(\mathbb{R}^n)$.
(ii) If $p=s'$, *then* $I_{\Omega,\alpha,m}$ *is bounded from* $L^{p_1,\lambda_1}\times...\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ *to* $WL^{q,\mu}(\mathbb{R}^n)$.
Moreover, similar estimates hold for $M_{\Omega,\alpha,m}$.

Proof. From Lemma 3.2, we only need to show the boundedness of $I_{\Omega,\alpha,m}$
Firstly, we choose t satisfy $(n-\mu)/q=(n-\lambda)/t$, then we get

$$1/t=(n-\mu)/(q(n-\lambda))=1/p-\alpha/(n-\lambda)<1/p-\alpha/n=1/q.$$

Then Hölder's inequality implies $L^{t,\lambda}(\mathbb{R}^n)\subset L^{q,\mu}(\mathbb{R}^n)$ and $WL^{t,\lambda}(\mathbb{R}^n)\subset WL^{q,\mu}(\mathbb{R}^n)$. In fact, there exists a constant $C>0$ such that

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,u}}\leq C\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{t,\lambda}}$$

and

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,u}}\leq C\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{t,\lambda}}$$

Then, by Theorem 3.2, we have

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,u}}\leq C\|I_{\Omega,\alpha,m}\mathbf{f}\|_{L^{t,\mu}}\leq C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}\ for\ p>s'$$

$$\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,u}}\leq C\|I_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{t,\mu}}\leq C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}\ for\ p=s'.$$

Thus, the proof of Corollary 3.1 is completed. □

As an another application, by Hölder's inequality, we obtain an Olsen's inequality as in the following corollary, which is a multi-version of the results in considered by Olsen in [16] in the study of Schrödinger equation with perturbed potentials W.

Corollary 3.2. *Let* $\alpha,\Omega,s,p_j,\lambda_j,p$ *and* λ *are as in Theorem 3.2 and let* $W\in L^{(n-\lambda)/\alpha,\lambda}$. *If* $p>s'$ *and* $1/p-1/q=\alpha/(n-\lambda)$, *then there exists a positive constant* C *such that*

$$\|W\cdot I_{\Omega,\alpha,m}\boldsymbol{f}\|_{L^{p,\lambda}(\mathbb{R}^n)}\leq C\|W\|_{L^{(n-\lambda)/\alpha,\lambda}(\mathbb{R}^n)}\prod_{j=1}^{m}\|f_j\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Moreover, similar estimates hold for $M_{\Omega,\alpha,m}$.

Proof of Theorem 3.3. By the Hölder inequality, we have

$$
\begin{aligned}
M_{\Omega,\alpha}\mathbf{f}(x) &= \sup_{r>0} \frac{1}{r^{n-\alpha}} \int_{B(0,r)} |\Omega(y)| \prod_{j=1}^{m} |f_j(x-\theta_j y)| dy \\
&\le C \sup_{r>0} \frac{1}{r^{n-\alpha}} \Big(\int_{B(0,r)} |\Omega(y)|^{p'} dy \Big)^{1p'} \Big(\int_{B(0,r)} \prod_{j=1}^{m} |f_j(x-\theta_j y)|^{p} dy \Big)^{1/p} \\
&\le C \sup_{r>0} \frac{1}{r^{n-\alpha}} \Big(\int_{B(0,r)} |\Omega(y)|^{p'} dy \Big)^{1/p'} \prod_{j=1}^{m} \Big(\int_{B(0,r)} |f_j(x-\theta_j y)|^{p_j} dy \Big)^{1/p_j} \\
&\le C \sup_{r>0} r^{\alpha-n/p} \Big(\frac{1}{r^n} \int_{B(0,r)} |\Omega(y)|^{s} dy \Big)^{1/s} \prod_{j=1}^{m} \Big(\int_{B(0,r)} |f_j(x-\theta_j y)|^{p_j} dy \Big)^{1/p_j} \\
&\le C \sup_{r>0} r^{\alpha-n/p} \prod_{j=1}^{m} \Big(\int_{B(0,r)} |f_j(x-\theta_j y)|^{p_j} dy \Big)^{1/p_j}
\end{aligned}
$$

If $p=(n-\lambda)/\alpha \ge s'$, from the condition (2.1), we have

$$
\begin{aligned}
M_{\Omega,\alpha}\mathbf{f}(x) &\le C \sup_{r>0} r^{\alpha-(n-\lambda)/p} \prod_{j=1}^{m} \Big(\frac{1}{r^{\lambda_j}} \int_{B(0,r)} |f_j(x-\theta_j y)|^{p_j} dy \Big)^{1/p_j} \\
&= C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.
\end{aligned}
$$

Therefore, we complete the proof of Theorem 3.3.

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