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A BOUNDARY VALUE PROBLEM FOR ELLIPTIC-DIFFERENTIAL OPERATOR EQUATIONS WITH A SPECTRAL PARAMETER IN THE EQUATION AND IN THE BOUNDARY CONDITIONS

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Abstract. In this paper, in a separable Hilbert space H on a finite interval we study the solvability of boundary value problems for elliptic differential-operator equations of the second order in case, when the same spectral parameter enters quadratically in the equation and enters linearly in the boundary conditions. The obtained abstract result is applied to boundary value problems for elliptic partial differential equations.

1. Introduction

Boundary value problems for second order elliptic differential-operator equations in the case, when the same spectral parameter enters both in the equation and the boundary conditions, in various aspects, were studied in the works of V.I.Gorbachuk and M.A.Rybak [12], M.A.Rybak [13], L.A. Oleinik [11], M.Denche [7], B.A.Aliev [1,2,3], B.A.Aliev and Y.Yakubov [4], M.Bairamoglu and N.M. Aslanova [6] and others.

We note that, in all these papers [1-4,6,7,11-13], the order of the spectral parameter entering in the equation and the boundary conditions is the same, more exactly, the spectral parameters linearly enters both in the equation and in the boundary conditions. In contrast to [1-4,6,7,11-13], in this paper we study boundary value problems for second order elliptic differential – operator equations in the case when the same spectral parameter enters quadratically into the equation and linearly into the boundary conditions. So, in the paper, in a separable Hilbert spaces H, we consider the following boundary value problem on [0,1] for an elliptic differential-operator equation of the second order

$$L(\lambda, D)u := \lambda^2 u(x) - u''(x) + Au(x) = f(x), x \in (0,1), \tag{1.1}$$

$$L_1(\lambda) u := \alpha u'(1) + \lambda u(1) = f_1, L_2(\lambda) u := \beta u'(0) - \lambda u(0) = f_2, \tag{1.2}$$

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where λ is a spectral parameter ; α, β are some complex numbers from the right-hand side of the complex space; A is a linear, unbounded self-adjoint positive-definite operator in H; $D := \frac{d}{dx}$.

Simple sufficient conditions for the solvability of problem (1.1), (1.2) (in fact, a theorem on isomorphism is proved) and some estimates (with respect to u and λ) for the solution of the problem (1.1), (1.2) in the space $L_p\left((0,1);H\right), 1 < p < \infty$, are established.

One simple application of abstract results to boundary value problems for second order elliptic partial differential equations in the square is given. Note that when in equation (1.1) λ stands instead of λ^2 and boundary conditions (1.2) contain perturbed terms (with abstract operators), then the obtained problem was considered in paper [4]. Solvability, Fredholm property and also discreteness of the spectrum and completeness of system of roof vectors were studied in [4].

In the case when in equation (1.1) $Q(x)$ is taken instead of A, where $Q(x) \in W_p^1(0,1)$, $1 < p < \infty$, $Q(1) \neq 0$, and in boundary conditions (1.2) $\alpha = 1, L_2 u := u(0) = f_2$, then the obtained boundary value problem for second order ordinary differential equations (the Redgge problem) was studied in the monograph by S.Yakubov and Y.Yakubov [15, section 3.5.4] (see also the paper of S.Yakubov [16]). Solvability and discreteness of the spectrum and two fold completeness of a system of roof functions of the corresponding homogeneous boundary value problem were studied in [15] and [16].

We also mention the paper of N.B.Kerimov and Kh.R.Mamedov [8], where the asymptotic behavior of eigenvalues and eigenfunctions of boundary value problems for second order ordinary differential equations with a parameter is studied. The spectral parameter participates in the equation as λ^2, while in the boundary conditions both as λ and λ^2.

Introduce now some necessary definitions and notions used in the paper.

Let E_1 and E_2 be Banach spaces. The set $E_1 \dot{+} E_2$ of all vectors of the form (u,v), where $u \in E_1$, $v \in E_2$, with standard coordinatewise linear operations and with the norm

$$\|(u,v)\|_{E_1 \dot{+} E_2} := \|u\|_{E_1} + \|v\|_{E_2}$$

is a Banach space and is said to be a direct sum of Banach spaces E_1 and E_2.

Let E_1 and E be two Banach spaces. Denote by $B(E_1, E)$ a Banach space of all linear bounded operators acting from E_1 into E with the standard operator norm. If $E_1 = E$ then $B(E) := B(E,E)$.

Definition 1.1. A linear closed operator A, densely defined in a Hilbert space H, is said to be strongly positive if, for some $\varphi \in [0,\pi)$ and for all complex numbers μ such that $|\arg \mu| \leq \varphi$ (including $\mu = 0$), the operators $A + \mu I$ are (boundedly) invertible and the estimate

$$\left\|(A+\mu I)^{-1}\right\|_{B(H)} \leq C\left(1+|\mu|\right)^{-1}$$

holds, where I is the unit operator in H, $C = const > 0$. For $\varphi = 0$, the operator A is called positive.

A simple example of strongly positive operators are selfadjoint positive-definite operators acting in a Hilbert space. Note that from strong positivity of an operator A it follows strong positivity of the operator $A^\alpha, \alpha \in (0,1)$. Let A be a strongly positive operator in H. Since A^{-1} is bounded in H, then

$$H\left(A^n\right) := \left\{u : u \in D\left(A^n\right),\ \|u\|_{H(A^n)} = \|A^n u\|_H\right\},\ \ n \in \mathbb{N},$$

is a Hilbert space whose norm is equivalent to the norm of the graph of the operator A^n. If A is strongly positive in H, it is known that the operator $-A$ is a generating operator of the analytic, for $t > 0$, semigroup e^{-tA} and this semigroup exponentially decreases, i.e., there exist two numbers $C > 0$, $\sigma_0 > 0$ such that $\left\|e^{-tA}\right\| \leq Ce^{-\sigma_0 t}$, $0 \leq t < +\infty$. By virtue of [9, theorem 1.5.5], the operator $-A^{1/2}$ generates an analytic semigroup, for $t > 0$, decreasing at infinity.

Definition 1.2. [14, theorem 1.14.5]. Interpolation spaces $\left(H\left(A^n\right), H\right)_{\theta,p}$ of Hilbert spaces $H\left(A^n\right)$ and H, where A is a strongly positive operator in H, are defined by the equality

$$\left(H\left(A^n\right), H\right)_{\theta,p} := \{u : u \in H,\ \ \|u\|_{(H(A^n),H)_{\theta,p}} :=$$

$$= \int_0^{+\infty} t^{-1+n\theta p} \left\|A^n e^{-tA} u\right\|_H^p dt < \infty\right\},\ \ \theta \in (0,1),\ \ p > 1,\ \ n \in \mathbb{N}.$$

We denote $\left(H\left(A^n\right), H\right)_{0,p} := H\left(A^n\right)$, $\left(H\left(A^n\right), H\right)_{1,p} := H$.

Denote by $L_p\left((0,1); H\right)$ $(1 < p < \infty)$ a Banach space (for $p = 2$ a Hilbert space) of vector-valued functions $x \to u(x) : [0,1] \to H$ strongly measurable and summable with order p and with the norm

$$\|u\|_{L_p((0,1);H)} := \left(\int_0^1 \|u(x)\|_H^p\, dx\right)^{1/p} < \infty.$$

Denote by $W_p^{2n}\left((0,1); H\left(A^n\right), H\right) := \left\{u : A^n u, u^{(2n)} \in L_p\left((0,1); H\right)\right\}$ a Banach space of vector-valued functions $x \to u(x) : [0,1] \to H$ strongly measurable and summable with order p and with the norm

$$\|u\|_{W_p^{2n}((0,1);H(A^n),H)} := \|A^n u\|_{L_p((0,1);H)} + \left\|u^{(2n)}\right\|_{L_p((0,1);H)} < \infty.$$

It is known [14, theorem 1.8.2] (see also [15, theorem 1.7.7/1]) that if $u \in W_p^{2n}((0,1);\ H\left(A^n\right), H)$ then, $\forall x_0 \in [0,1]$,

$$u^{(j)}\left(x_0\right) \in \left(H\left(A^n\right), H\right)_{\frac{j+\frac{1}{p}}{2n},p},\ \ j = 0, \ldots, 2n-1.$$

2. Homogeneous equations

First, consider the following boundary value problem, in a Hilbert space H,

$$L\left(\lambda, D\right) u := \lambda^2 u(x) - u''(x) + Au(x) = 0,\ \ x \in (0,1), \tag{2.1}$$

$$\begin{aligned} L_1\left(\lambda\right) u &:= \alpha u'(1) + \lambda u(1) = f_1, \\ L_2(\lambda)u &:= \beta u'(0) - \lambda u(0) = f_2. \end{aligned} \tag{2.2}$$

Theorem 2.1. *Let the following conditions be fulfilled:*

(1) *A is a selfadjoint, positive-definite operator $\left(A = A^* \geq \gamma^2 I\right)$ in a separable Hilbert space H;*

(2) *$\alpha \neq 0$, $\beta \neq 0$ are some complex numbers with $|\arg\alpha| \leq \frac{\pi}{2}$, $|\arg\beta| \leq \frac{\pi}{2}$.*

Then the problem (2.1), (2.2), for $f_k \in (H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}, p \in (1,\infty)$, *and for* $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, $|\lambda|$ *is sufficiently large, has a unique solution* $u(x)$ *which belongs to* $W_p^2((0,1); H(A), H)$ *and, for these* λ, *the following estimate holds for the solution of the problem (2.1), (2.2)*

$$|\lambda|^2 \|u\|_{L_p((0,1);H)} + \|u''\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)} \le$$
$$\le C\sum_{k=1}^{2}\left(\|f_k\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{p}}\|f_k\|_H\right). \quad (2.3)$$

Proof. Since $A = A^* \ge \gamma^2 I$ in H, by the spectral theorem (see, e.g., [10, chapter V, sections 5 and 6, chapter VI section 5]) there exists an operator-valued function $f(A) = \int_{\gamma^2}^{+\infty} f(\mu)\, dE_\mu$ for any measurable, bounded complex-valued function $f(\mu)$. Furthermore, $f(A)$ is a bounded operator in H and $\|f(A)\|_{B(H)} \le \underset{\gamma^2\le\mu<\infty}{ess\sup} |f(\mu)|$. Then, from condition 1, it follows that for any ψ, $0 \le \psi < \pi$ there exists $C_\psi > 0$ such that

$$\|R(\lambda, A)\| \le C_\psi (1+|\lambda|)^{-1}, \quad |\arg\lambda| \ge \pi - \psi,$$

where $R(\lambda, A) := (\lambda I - A)^{-1}$ is the resolvent of the operator A. Hence, by virtue of [15, lemma 5.4.2/6], for $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, there exists an analytic, for $x > 0$, and strongly continuous, for $x \ge 0$, semigroup $e^{-x\left(A+\lambda^2 I\right)^{1/2}}$. By virtue of [15, lemma 5.3.2/1], for a function $u(x)$ being a solution of the equation (2.1), for $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, belonging to $W_p^2((0,1); H(A), H)$, $1 < p < \infty$, it is necessary and sufficient that

$$u(x) = e^{-x\left(A+\lambda^2 I\right)^{1/2}} g_1 + e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}} g_2, \quad (2.4)$$

where $g_k \in (H(A), H)_{\frac{1}{2p},p}$, $k = 1, 2$.

A function $u(x)$ of the form (2.4) satisfies the boundary condition (2.2) if

$$\begin{cases} \left[-\alpha\left(A+\lambda^2 I\right)^{1/2}+\lambda I\right] e^{-\left(A+\lambda^2 I\right)^{1/2}} g_1 + \left[\alpha\left(A+\lambda^2 I\right)^{1/2}+\lambda I\right] g_2 = f_1, \\ -\left[\beta(A+\lambda^2 I)^{1/2}+\lambda I\right] g_1 + \left[\beta(A+\lambda^2 I)^{1/2} - \lambda I\right] e^{-(A+\lambda^2 I)^{1/2}} g_2 = f_2. \end{cases} \quad (2.5)$$

We rewrite the system (2.5), in the space $\mathbb{H} := (H(A), H)_{\frac{1}{2}+\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2p},p}$, in the operator form

$$(A(\lambda) + R(\lambda))\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad (2.6)$$

where $A(\lambda)$ and $R(\lambda)$ are operator-matrices of dimension 2×2:

$$A(\lambda) := \begin{pmatrix} 0 & \alpha\left(A+\lambda^2 I\right)^{1/2} + \lambda I \\ -\left[\beta(A+\lambda^2 I)^{1/2} + \lambda I\right], & 0 \end{pmatrix},$$
$$D(A(\lambda)) := (H(A), H)_{\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2p},p}$$

and

$$R(\lambda) := \begin{pmatrix} \left[-\alpha\left(A+\lambda^2 I\right)^{1/2} + \lambda I\right] e^{-\left(A+\lambda^2 I\right)^{1/2}} & 0 \\ 0 & \left[\beta(A+\lambda^2 I)^{1/2} - \lambda I\right] e^{-\left(A+\lambda^2 I\right)^{1/2}} \end{pmatrix},$$

$$D\left(R\left(\lambda\right)\right):=\mathbb{H}.$$

Show that the operator $A\left(\lambda\right)$, in the space $\mathbb{H}$, for λ from the sector $\left|\arg\lambda\right|\leq\varphi<\frac{\pi}{2}$, has a bounded inverse $A\left(\lambda\right)^{-1}$ acting from $\mathbb{H}$ into $\left(H\left(A\right),H\right)_{\frac{1}{2p},p}\dot{+}\left(H\left(A\right),H\right)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\|A\left(\lambda\right)^{-1}\right\|_{B\left(\mathbb{H},\left(H\left(A\right),H\right)_{\frac{1}{2p},p}\dot{+}\left(H\left(A\right),H\right)_{\frac{1}{2p},p}\right)}\leq C, \tag{2.7}$$

where $C>0$ is a constant independent on λ. Since $A\left(\lambda\right)^{-1}$ formally has the form

$$A\left(\lambda\right)^{-1}=\begin{pmatrix} 0 & -\left[\beta(A+\lambda^2 I)^{1/2}+\lambda I\right]^{-1} \\ \left[\alpha\left(A+\lambda^2 I\right)^{1/2}+\lambda I\right]^{-1} & 0\end{pmatrix}$$

then it is sufficient to show that the operator $\left[\alpha\left(A+\lambda^2 I\right)^{1/2}+\lambda I\right]^{-1}$, for $\left|\arg\lambda\right|\leq\varphi<\frac{\pi}{2}$, is bounded from $\left(H\left(A\right),H\right)_{\frac{1}{2}+\frac{1}{2p},p}$ into $\left(H\left(A\right),H\right)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\|\left[\alpha\left(A+\lambda^2 I\right)^{1/2}+\lambda I\right]^{-1}\right\|_{B\left(\left(H\left(A\right),H\right)_{\frac{1}{2}+\frac{1}{2p},p},\left(H\left(A\right),H\right)_{\frac{1}{2p},p}\right)}\leq C, \tag{2.8}$$

where $C>0$ is a constant independent on λ.

Consider the function $f\left(\mu\right)=\left(1+\alpha^{-1}\lambda\left(\mu+\lambda^2\right)^{-1/2}\right)^{-1}$, for a fixed $\alpha\neq 0$ with $\left|\arg\alpha\right|\leq\frac{\pi}{2}$. Note that $z^{\frac{1}{2}}=\left|z\right|^{\frac{1}{2}}e^{i\frac{\arg z}{2}}$, where $-\pi<\arg z\leq\pi$. Show now that, for $\left|\arg\lambda\right|\leq\varphi<\frac{\pi}{2}$,

$$\inf_{\gamma^2\leq\mu<\infty}\left|\left(f(\mu)\right)^{-1}\right|=\inf_{\gamma^2\leq\mu<\infty}\left|\left(1+\alpha^{-1}\lambda\left(\mu+\lambda^2\right)^{-1/2}\right)\right|\geq C,\ \ \exists C>0. \tag{2.9}$$

Since $\left|\arg\alpha\right|\leq\frac{\pi}{2}$ then $\left|\arg\alpha^{-1}\right|\leq\frac{\pi}{2}$. If $0\leq\arg\lambda\leq\varphi<\frac{\pi}{2}$ and $\gamma^2\leq\mu<\infty$, then $0\leq\arg\left(\mu+\lambda^2\right)\leq 2\varphi$ and $-2\varphi\leq\arg\left(\mu+\lambda^2\right)^{-1}\leq 0$. Therefore $-\varphi\leq\arg\left(\mu+\lambda^2\right)^{-1/2}\leq 0$. Consequently, $-\varphi\leq\arg\left(\lambda\left(\mu+\lambda^2\right)^{-1/2}\right)\leq\varphi$. If $-\varphi\leq\arg\lambda\leq 0$ and $\gamma^2\leq\mu<\infty$, we have $-2\varphi\leq\arg\left(\mu+\lambda^2\right)\leq 0$. Hence, $0\leq\arg\left(\mu+\lambda^2\right)^{-1/2}\leq\varphi$. Consequently, $-\varphi\leq\arg\left(\lambda\left(\mu+\lambda^2\right)^{-1/2}\right)\leq\varphi$. So, for $\left|\arg\lambda\right|\leq\varphi<\frac{\pi}{2}$ and $\gamma^2\leq\mu<\infty$, we have $\left|\arg\left(\lambda\left(\mu+\lambda^2\right)^{-1/2}\right)\right|\leq\varphi<\frac{\pi}{2}$. Therefore,

$$\left|\arg\left(\alpha^{-1}\lambda\left(\mu+\lambda^2\right)^{-1/2}\right)\right|\leq\frac{\pi}{2}+\varphi<\pi. \tag{2.10}$$

If (2.9) is not true then there necessarily exist sequences μ_n and λ_n such that $\gamma^2\leq\mu_n<\infty$, $\left|\arg\lambda_n\right|\leq\varphi$ and $\alpha^{-1}\lambda_n\left(\mu_n+\lambda_n^2\right)^{-1/2}+1\to 0$ or $\alpha^{-1}\lambda_n\left(\mu_n+\lambda_n^2\right)^{-1/2}\to -1$ and this contradicts with (2.10). Consequently, (2.9) holds. It means that $f\left(\mu\right)$ is a bounded function on $[\gamma^2,\infty)$, uniformly on λ, $\left|\arg\lambda\right|\leq\varphi<\frac{\pi}{2}$. Then, according to the remark at the beginning of the proof,

$\exists C>0$ such that

$$\left\|\left[I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right]^{-1}\right\|_{B(H)}\le$$

$$\le \sup_{\gamma^2\le\mu<\infty}\left|1+\alpha^{-1}\lambda\left(\mu+\lambda^2\right)^{-1/2}\right|^{-1}\le C, \tag{2.11}$$

uniformly on λ, $|\arg\lambda|\le\varphi<\dfrac{\pi}{2}$. Note that "*ess* sup" = "sup" since $f(\mu)$ is a continuous function.

On the other hand, by the same remark at the beginning of the proof, for λ from the sector $|\arg\lambda|\le\varphi<\dfrac{\pi}{2}$, we have

$$\left\|\left(A+\lambda^2 I\right)^{-1}\right\|_{B(H)}\le\frac{C\varphi}{1+|\lambda|^2}. \tag{2.12}$$

Similarly,

$$\left\|\left(A+\lambda^2 I\right)^{-1/2}\right\|_{B(H)}\le\frac{C\varphi}{1+|\lambda|}. \tag{2.13}$$

Then, from the representation

$$\left[\alpha\left(A+\lambda^2 I\right)^{1/2}+\lambda I\right]^{-1}=\frac{1}{\alpha}\left[I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right]^{-1}\left(A+\lambda^2 I\right)^{-1/2}, \tag{2.14}$$

by virtue of (2.11) and (2.13), for λ from the sector $|\arg\lambda|\le\varphi<\dfrac{\pi}{2}$, we have

$$\left\|\left[\alpha\left(A+\lambda^2 I\right)^{1/2}+\lambda I\right]^{-1}\right\|_{B(H)}\le\frac{C}{1+|\lambda|},\quad \exists C>0. \tag{2.15}$$

Now prove the estimate (2.8). According to (2.14), it is sufficient to show that

a) the operator $\left(A+\lambda^2 I\right)^{-1/2}$ for λ from the sector $|\arg\lambda|\le\varphi<\dfrac{\pi}{2}$ is bounded from $(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}$ into $(H(A),H)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\|\left(A+\lambda^2 I\right)^{-1/2}\right\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p},(H(A),H)_{\frac{1}{2p},p}\right)}\le C, \tag{2.16}$$

where $C>0$ is some constant independent on λ;

b) the operator $\left(I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right)^{-1}$, for λ from the sector $|\arg\lambda|\le\varphi<\dfrac{\pi}{2}$, is bounded from $(H(A),H)_{\frac{1}{2p},p}$ into $(H(A),H)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\|\left(I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right)^{-1}\right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)}\le C, \tag{2.17}$$

where $C>0$ is some constant independent on λ.

Note that a) was proved in [5]. Prove b). From the estimate (2.11) it follows that, for $|\arg\lambda|\le\varphi$, it also holds the estimate

$$\left\|\left(I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right)^{-1}\right\|_{B(H(A))}\le C,\quad \exists C>0. \tag{2.18}$$

Then, according to the interpolation theorem [14, theorem 1.3.3/(a)] (see also [15, section 1.7.9]), from the estimates (2.11) and (2.18) it follows that, for λ

from the sector $|\arg\lambda| \leq \varphi < \frac{\pi}{2}$, the operator $\left[I + \alpha^{-1}\lambda\left(A + \lambda^2 I\right)^{-1/2}\right]^{-1}$ is bounded from $(H(A), H)_{\theta,p}$ into $(H(A), H)_{\theta,p}$, for any $\theta \in (0,1)$, and it holds the estimate

$$\left\|\left(I + \alpha^{-1}\lambda\left(A + \lambda^2 I\right)^{-1/2}\right)^{-1}\right\|_{B\left((H(A),H)_{\theta,p}\right)} \leq$$

$$\leq \left\|\left(I + \alpha^{-1}\lambda\left(A + \lambda^2 I\right)^{-1/2}\right)^{-1}\right\|_{B(H(A))}^{1-\theta} \times$$

$$\times \left\|\left(I + \alpha^{-1}\lambda\left(A + \lambda^2 I\right)^{-1/2}\right)^{-1}\right\|_{B(H)}^{\theta} \leq C. \tag{2.19}$$

Take $\theta = \dfrac{1}{2p}$ in (2.19). Then we get (2.17). So, from the representation (2.14), by virtue of the estimates (2.16) and (2.17), it follows that for λ from the sector $|\arg\lambda| \leq \varphi < \dfrac{\pi}{2}$ the estimate (2.8) holds. Consequently, for λ from the sector $|\arg\lambda| \leq \varphi$, the operator $A(\lambda)^{-1}$ is bounded from $\mathbb{H}$ into $(H(A), H)_{\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2p},p}$ and the estimate (2.7) holds. Then, from the equation (2.6), we have

$$\left(I + A(\lambda)^{-1} R(\lambda)\right)\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = A(\lambda)^{-1}\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}. \tag{2.20}$$

We can now show that all the operators in the operator–matrix $A(\lambda)^{-1} R(\lambda)$, for sufficiently large $|\lambda|$ from the sector $|\arg\lambda| \leq \varphi$, are bounded from $(H(A), H)_{\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p}$. It is sufficient to show this for the operator

$$\left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I\right]^{-1}\left[-\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I\right] e^{-\left(A+\lambda^2 I\right)^{1/2}}.$$

From (2.15) it follows that for λ from the sector $|\arg\lambda| \leq \varphi < \dfrac{\pi}{2}$ it holds the estimate

$$\left\|\left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I\right]^{-1}\right\|_{B(H(A))} \leq \frac{C}{1 + |\lambda|}. \tag{2.21}$$

Then, by virtue of the interpolation theorem [14, theorem 1.3.3/(a)] (see also [15, section 1.7.9]), it follows from (2.15) and (2.21) that for λ from the sector $|\arg\lambda| \leq \varphi < \dfrac{\pi}{2}$ it holds the estimate

$$\left\|\left[\alpha\left(A + \lambda^2 I\right)^{1/2} + \lambda I\right]^{-1}\right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)} \leq \frac{C}{1 + |\lambda|}. \tag{2.22}$$

By virtue of [15, lemma 5.4.2/6], from the interpolation theorem [14, theorem 1.3.3/(a)] it also follows that, for $|\arg\lambda| \leq \varphi < \dfrac{\pi}{2}$, the estimates

$$\left\|\left(A + \lambda^2 I\right)^{1/2} e^{-\left(A+\lambda^2 I\right)^{1/2}}\right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)} \leq Ce^{-\omega|\lambda|}, \ \exists C, \omega > 0, \tag{2.23}$$

and

$$\left\| e^{-\left(A+\lambda^2 I\right)^{1/2}} \right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)} \leq Ce^{-\omega|\lambda|}, \ \exists C, \omega > 0, \tag{2.24}$$

hold. Then, by virtue of the estimates (2.22)–(2.24), for λ from the sector $|\arg \lambda| \leq \varphi < \frac{\pi}{2}$, we have

$$\left\| \left[\alpha \left(A+\lambda^2 I\right)^{1/2} + \lambda I\right]^{-1} \left[-\alpha \left(A+\lambda^2 I\right)^{1/2} + \lambda I\right] e^{-\left(A+\lambda^2 I\right)^{1/2}} \right\|_{B\left((H(A),H)_{\frac{1}{2p},p}\right)} \leq$$

$$\leq C\left((1+|\lambda|)^{-1} e^{-\omega|\lambda|} + e^{-\omega|\lambda|}\right) \leq Ce^{-\omega|\lambda|}.$$

Consequently, for sufficiently large $|\lambda|$ from the sector $|\arg \lambda| \leq \varphi < \dfrac{\pi}{2}$, the operator $A(\lambda)^{-1} R(\lambda)$ is bounded from $(H(A), H)_{\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2p},p}$ and it holds the estimate

$$\left\| A(\lambda)^{-1} R(\lambda) \right\|_{B\left((H(A),H)_{\frac{1}{2p},p} \dot{+} (H(A),H)_{\frac{1}{2p},p}\right)} \leq Ce^{-\omega|\lambda|} < 1. \tag{2.25}$$

Hence, according to the Neumann identity, for $|\arg \lambda| \leq \varphi < \dfrac{\pi}{2}$ and sufficiently large $|\lambda|$,

$$\left(I + A(\lambda)^{-1} R(\lambda)\right)^{-1} = I + \sum_{k=1}^{\infty} \left(-A(\lambda)^{-1} R(\lambda)\right)^k, \tag{2.26}$$

where the series converges in the norm of the space of bounded operators in $(H(A), H)_{\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2p},p}$. Then, from (2.20), for sufficiently large $|\lambda|$ from the sector $|\arg \lambda| \leq \varphi < \dfrac{\pi}{2}$, we have

$$\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \left(I + A(\lambda)^{-1} R(\lambda)\right)^{-1} A(\lambda)^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Consequently, using the formulas of $A(\lambda)^{-1}$ and $R(\lambda)$ and (2.26), for sufficiently large $|\lambda|$ from the sector $|\arg \lambda| \leq \varphi$, the elements g_1 and g_2 can be represented in the form

$$g_k = (C_{k1}(\lambda) + R_{k1}(\lambda)) f_1 + (C_{k2}(\lambda) + R_{k2}(\lambda)) f_2, \ \ k = 1, 2, \tag{2.27}$$

where $C_{11}(\lambda) = 0, \ C_{12}(\lambda) = \left[\beta(A+\lambda^2 I)^{1/2} + \lambda I\right]^{-1}$,
$C_{21}(\lambda) = \left[\alpha\left(A+\lambda^2 I\right)^{1/2} + \lambda I\right]^{-1}$, $C_{22}(\lambda) = 0$, and $R_{kj}(\lambda)$ are some bounded operators acting from $(H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}$ into $(H(A), H)_{\frac{1}{2p},p}$. Furthermore, from the estimaties (2.7) and (2.25) it follows that, for $|\arg \lambda| \leq \varphi$ and $|\lambda| \to \infty$,

$$\|R_{kj}(\lambda)\|_{B\left((H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}, (H(A),H)_{\frac{1}{2p},p}\right)} \leq Ce^{-\omega|\lambda|}, \ \exists C, \omega > 0. \tag{2.28}$$

From the representations of $A(\lambda)^{-1}$ and $R(\lambda)$ it also follows that, for sufficiently large $|\lambda|$ from the sector $|\arg \lambda| \leq \varphi < \dfrac{\pi}{2}$, for the operators $R_{kj}(\lambda)$ we have

$$\|R_{kj}(\lambda)\|_{B(H)} \leq Ce^{-\omega|\lambda|}, \ \exists C, \omega > 0. \tag{2.29}$$

Substituting (2.27) into (2.4), we get

$$u(x)=\sum_{k=1}^{2}\left\{e^{-x\left(A+\lambda^2 I\right)^{1/2}}\left(C_{1k}(\lambda)+R_{1k}(\lambda)\right)+\right.$$

$$\left.+e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}}\left(C_{2k}(\lambda)+R_{2k}(\lambda)\right)\right\}f_k. \tag{2.30}$$

In order to show the estimate (2.3), it is necessary to estimate, for sufficiently large $|\lambda|$ from the sector $|\arg\lambda|\leq\varphi<\frac{\pi}{2}$, some finite number of integrals in the space $L_p((0,1);H)$. The integrand expressions are obtained from the functions $u(x)$, $u''(x)$, $Au(x)$, where $u(x)$ is determined by the equality (2.30). Here, [15, theorem 5.4.2/1 and lemma 5.4.2/6] and the estimates (2.11), (2.12), (2.22), (2.28), and (2.29) are essentially used. Estimate one of these integrals, for example, the integral

$$|\lambda|^2\left(\int_0^1\left\|e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}}C_{21}(\lambda)f_1\right\|_H^p dx\right)^{1/p}.$$

By virtue of [15, theorem 5.4.2/1 and lemma 5.4.2/6] and the estimates (2.11), (2.12), and (2.19), for sufficiently large $|\lambda|$ from the sector $|\arg\lambda|\leq\varphi<\frac{\pi}{2}$, we have

$$|\lambda|^2\left(\int_0^1\left\|e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}}C_{21}(\lambda)f_1\right\|_H^p dx\right)^{1/p}=$$

$$=|\lambda|^2\left(\int_0^1\left\|e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}}\left[\alpha\left(A+\lambda^2 I\right)^{1/2}+\lambda I\right]^{-1}f_1\right\|_H^p dx\right)^{1/p}=$$

$$=|\lambda|^2\left(\int_0^1\left\|e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}}\frac{1}{\alpha}\left(A+\lambda^2 I\right)^{-1/2}\left[I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right]^{-1}f_1\right\|_H^p dx\right)^{1/p}\leq$$

$$\leq\frac{1}{|\alpha|}|\lambda|^2\left\|\left(A+\lambda^2 I\right)^{-1}\right\|_{B(H)}\times$$

$$\times\left(\int_0^1\left\|\left(A+\lambda^2 I\right)^{1/2}e^{-(1-x)\left(A+\lambda^2 I\right)^{1/2}}\left[I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right]^{-1}f_1\right\|_H^p dx\right)^{1/p}\leq$$

$$\leq\frac{1}{|\alpha|}|\lambda|^2\frac{1}{1+|\lambda|^2}\left(\left\|\left[I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right]^{-1}f_1\right\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+\right.$$

$$\left.+|\lambda|^{2\left(\frac{1}{2}-\frac{1}{2p}\right)}\left\|\left[I+\alpha^{-1}\lambda\left(A+\lambda^2 I\right)^{-1/2}\right]^{-1}f_1\right\|_H\right)\leq$$

$$\leq C\left(\|f_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}}+|\lambda|^{1-\frac{1}{p}}\|f_1\|_H\right).$$

□

3. Nonhomogeneous equations

Let us now consider our full boundary value problem in a separable Hilbert space H, i.e.,

$$L(\lambda, D)\, u := \lambda^2 u(x) - u''(x) + Au(x) = f(x), \;\; x \in (0,1), \tag{3.1}$$

$$\begin{aligned} L_1(\lambda)\, u &:= \alpha u'(1) + \lambda u(1) = f_1, \\ L_2(\lambda) u &:= \beta u'(0) - \lambda u(0) = f_2. \end{aligned} \tag{3.2}$$

Theorem 3.1. *Let conditions of theorem 1 be satisfied.*

Then the operator $\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda) u := (L(\lambda, D) u, L_1(\lambda) u,\; L_2(\lambda) u)$, *for sufficiently large* $|\lambda|$ *from the sector* $|\arg \lambda| \le \varphi < \frac{\pi}{2}$, *is an isomorphism from* $W_p^2((0,1); H(A), H)$ *onto* $L_p((0,1); H) \dot{+} (H(A), H)_{\frac{1}{2}+\frac{1}{2p},p} \dot{+} (H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}$, $p \in (1,\infty)$, *and, for these* λ, *the following estimate is valid for the solution of the problem (3.1), (3.2)*

$$|\lambda|^2 \|u\|_{L_p((0,1);H)} + \|u''\|_{L_p((0,1);H)} + \|Au\|_{L_p((0,1);H)} \le$$
$$\le C\left[|\lambda| \|f\|_{L_p((0,1);H)} + \sum_{k=1}^{2}\left(\|f_k\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{p}} \|f_k\|_H\right)\right]. \tag{3.3}$$

Proof. The injectivity of the mapping $\mathbb{L}(\lambda)$ follows from theorem 2.1, since the homogeneous boundary value problem corresponding to the boundary value problem (3.1), (3.2), for sufficiently large $|\lambda|$ from the sector $|\arg \lambda| \le \varphi < \frac{\pi}{2}$, has only a trivial solution. Thus, it is sufficient to show that $\mathbb{L}(\lambda)$ is surjective, i.e., for any $f \in L_p((0,1); H)$ and any $f_1 \in (H(A); H)_{\theta_1}$, $f_2 \in L_p(H(A); H)_{\theta_2}$, there exists a solution of problem (3.1), (3.2) belonging to $W_p^2((0,1); H(A), H)$. Define $\tilde{f}(x) := f(x)$ if $x \in (0,1)$ and $\tilde{f}(x) = 0$ if $x \notin (0,1)$.

A solution of the problem (3.1), (3.2) can be represented in the form of the sum $u(x) = u_1(x) + u_2(x)$, where $u_1(x)$ is the restriction on $(0,1)$ of the solution $\tilde{u}_1(x)$ of the equation

$$L(\lambda, D)\, \tilde{u}_1(x) = \tilde{f}(x), \;\; x \in \mathbb{R} = (-\infty, +\infty), \tag{3.4}$$

and $u_2(x)$ is a solution of the problem

$$L(\lambda, D)\, u_2(x) = 0, x \in (0,1), L_1(\lambda)\, u_2 = f_1 - L_1(\lambda)\, u_1,$$
$$L_2(\lambda)\, u_2 = f_2 - L_2(\lambda)\, u_1. \tag{3.5}$$

It is obvious that a solution of the equation (3.4) is given by the following formula

$$\tilde{u}_1(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\mu x} L(\lambda, i\mu)^{-1} F\tilde{f}(\mu)\, d\mu$$

where $F\tilde{f}$ is the Fourier transform of the function $\tilde{f}(x)$ and $L(\lambda, \sigma) = -\sigma^2 I + A + \lambda^2 I$. Further, it is proved in [15, theorem 5.4.4] that the solution $\tilde{u}_1$ belongs to $W_p^2(\mathbb{R}; H(A), H)$ and, for the solution, it holds the estimate

$$|\lambda|^2 \|\tilde{u}_1\|_{L_p(\mathbb{R};H)} + \|\tilde{u}_1\|_{W_p^2(\mathbb{R};H(A),H)} \le C\left\|\tilde{f}\right\|_{L_p(\mathbb{R};H)}, \;\; |\arg \lambda| \le \varphi. \tag{3.6}$$

Therefore, $u_1 \in W_p^2((0,1); H(A), H)$ and, from (3.6), for $|\arg \lambda| \le \varphi$, we have

$$|\lambda|^2 \|u_1\|_{L_p((0,1);H)} + \|u_1\|_{W_p^2((0,1);H(A),H)} \le C\|f\|_{L_p((0,1);H)}. \tag{3.7}$$

By virtue of [15, theorem 1.7.7/1] (see also [14, theorem 1.8.2]) and the inequality (3.7), we have

$$u_1^{(s)}(x_0) \in (H(A), H)_{\frac{s}{2}+\frac{1}{2p},p}, \quad \forall x_0 \in [0,1], \; s = 0,1.$$

Hence, $L_1(\lambda) u_1 \in (H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}$, $L_2(\lambda) u_1 \in (H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}$ since $(H(A), H)_{\frac{1}{2p},p} \subset (H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}$. Thus, by virtue of theorem 1, the problem (3.5) has a unique solution $u_2(x)$ that belongs to $W_p^2((0,1); H(A), H)$, for sufficiently large $|\lambda|$ from the sector $|\arg\lambda| \le \varphi$. Furthermore, for the solution of the problem (3.5), for $|\arg\lambda| \le \varphi$, $|\lambda| \to \infty$, we have

$$|\lambda|^2 \|u_2\|_{L_p((0,1);H)} + \|u_2''\|_{L_p((0,1);H)} + \|Au_2\|_{L_p((0,1);H)} \le$$
$$\le C\Big[\|f_1 - L_1(\lambda) u_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + \|f_2 - L_2(\lambda) u_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} +$$
$$+ |\lambda|^{1-\frac{1}{p}} \left(\|f_1 - L_1(\lambda) u_1\|_H + \|f_2 - L_2(\lambda) u_1\|_H\right)\Big] \le$$
$$\le C\Big[\|f_1\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + \left\|u_1'(1)\right\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} +$$
$$+ |\lambda| \|u_1(1)\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + \|f_2\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} +$$
$$+ \left\|u_1'(0)\right\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda| \|u_1(0)\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} +$$
$$+ |\lambda|^{1-\frac{1}{p}} \left(\|f_1\|_H + \left\|u_1'(1)\right\|_H + |\lambda| \|u_1(1)\|_H + \|f_2\|_H + \left\|u_1'(0)\right\|_H + |\lambda| \|u_1(0)\|\right)\Big]. \tag{3.8}$$

By virtue of [15, theorem 1.7.7/1] (see also [14, theorem 1.8.2]) and (3.7), for any $x_0 \in [0,1]$, we have

$$\left\|u_1^{(s)}(x_0)\right\|_{(H(A),H)_{\frac{s}{2}+\frac{1}{2p},p}} \le$$
$$\le C \|u_1\|_{W_p^2((0,1);H(A),H)} \le C \|f\|_{L_p((0,1);H)}, \quad s = 0,1. \tag{3.9}$$

By virtue of [15, theorem 1.7.7/2], for any complex number λ and any $u \in W_p^2((0,1); H)$, $s = 0,1$,

$$|\lambda|^{2-s} \left\|u^{(s)}(x_0)\right\|_H \le C\left(|\lambda|^{\frac{1}{p}} \|u\|_{W_p^2((0,1);H)} + |\lambda|^{2+\frac{1}{p}} \|u\|_{L_p((0,1);H)}\right). \tag{3.10}$$

Dividing (3.10) by $|\lambda|^{\frac{1}{p}}$, for $\lambda \in \mathbb{C}$, $u \in W_p^2((0,1); H)$, $s = 0,1$, we have

$$|\lambda|^{2-s-\frac{1}{p}} \left\|u^{(s)}(x_0)\right\|_H \le C\left(\|u\|_{W_p^2((0,1);H)} + |\lambda|^2 \|u\|_{L_p((0,1);H)}\right), \quad s = 0,1. \tag{3.11}$$

Then, from (3.7) and (3.11), for $|\arg\lambda| \le \varphi < \frac{\pi}{2}$, we have

$$|\lambda|^{2\left(1-\frac{s}{2}-\frac{1}{2p}\right)} \left\|u_1^{(s)}(x_0)\right\|_H \le C\left(\|u_1\|_{W_p^2((0,1);H)} + |\lambda|^2 \|u_1\|_{L_p((0,1);H)}\right) \le$$
$$\le C\left(\|u_1\|_{W_p^2((0,1);H(A),H)} + |\lambda|^2 \|u_1\|_{L_p((0,1);H)}\right) \le$$
$$\le C \|f\|_{L_p((0,1);H)}, \quad s = 0,1. \tag{3.12}$$

According to the estimates (3.9) and (3.12), from (3.8), for $|\arg \lambda| \leq \varphi < \frac{\pi}{2}$, $|\lambda| \to \infty$, we have

$$|\lambda|^2 \|u_2\|_{L_p((0,1);H)} + \left\|u_2^{''}\right\|_{L_p((0,1);H)} + \|Au_2\|_{L_p((0,1);H)} \leq$$

$$\leq C \left[|\lambda| \|f\|_{L_p((0,1);H)} + \sum_{k=1}^{2} \left(\|f_k\|_{(H(A),H)_{\frac{1}{2}+\frac{1}{2p},p}} + |\lambda|^{1-\frac{1}{p}} \|f_k\|_H \right) \right]. \quad (3.13)$$

Then, from (3.7) and (3.13) it follows (3.3) since $u = u_1 + u_2$. The theorem is proved. □

4. Application of abstract results to elliptic partial differential equations

Let us consider a boundary value problem with a parameter for an elliptic partial differential equation of the second order in the square $[0,1] \times [0,1]$

$$L(\lambda, D_x, D_y)\, u := \lambda^2 u(x,y) - D_x^2 u(x,y) - D_y(a(y)\, D_y u(x,y)) = f(x,y), \quad (4.1)$$

$$\begin{array}{l} L_1(\lambda)\, u := \alpha D_x u(1,y) + \lambda u(1,y) = f_1(y), \quad y \in [0,1], \\ L_2(\lambda) u := \beta D_x u(0,y) - \lambda u(0,y) = f_2(y), \quad y \in [0,1], \end{array} \quad (4.2)$$

$$u(x,0) = u(x,1) = 0, \quad x \in [0,1], \quad (4.3)$$

where $D_x := \dfrac{\partial}{\partial x}$, $D_y := \dfrac{\partial}{\partial y}$.

Denote the interpolation space of Sobolev spaces by

$$B^s_{q,p}(0,1) := \left(W_q^{s_0}(0,1), W_q^{s_1}(0,1)\right)_{\theta,p},$$

where $0 \leq s_0, s_1$ are integers, $0 < \theta < 1$, $1 < q < \infty$, $1 \leq p \leq \infty$ and $s = (1-\theta)\, s_0 + \theta s_1$. Set

$$W_p^s(0,1) := B^s_{p,p}(0,1) := \left(W_p^{s_0}(0,1), W_p^{s_1}(0,1)\right)_{\theta,p},$$

if $0 < s \neq$ integer.

Theorem 4.1. *Let the following conditions be fulfilled:*

1. $a(\cdot) \in C^1[0,1]$, $a(y) > 0$ for $y \in [0,1]$;

2. $\alpha \neq 0$, $\beta \neq 0$ are complex numbers with $|\arg \alpha| \leq \dfrac{\pi}{2}$, $|\arg \beta| \leq \dfrac{\pi}{2}$.

Then the operator $\mathbb{L}(\lambda) : u \to \mathbb{L}(\lambda)\, u := (L(\lambda, D_x, D_y)\, u, L_1(\lambda)\, u, L_2(\lambda) u)$, for $|\arg \lambda| \leq \varphi < \frac{\pi}{2}$ and sufficiently large $|\lambda|$, is an isomorphism from $W_p^2\left((0,1); W_2^2(0,1), L_2(0,1)\right)$, $1 < p < \infty$, onto

$$L_p((0,1); L_2(0,1)) \dot{+} B^{1-\frac{1}{p}}_{2,p,*}(0,1) \dot{+} B^{1-\frac{1}{p}}_{2,p,*}(0,1),$$

where

$$B^{1-\frac{1}{p}}_{2,p,*}(0,1) := \begin{cases} B^{1-\frac{1}{p}}_{2,p}(0,1), \quad 1 < p < 2, \\ W_2^{\frac{1}{2}}\left((0,1);\ \int_0^1 (\min\{x, 1-x\})^{-1} |u(x)|^2\, dx < \infty\right), p = 2, \\ B^{1-\frac{1}{p}}_{2,p}((0,1); u(0) = u(1) = 0), \ p > 2, \end{cases}$$

and, for these λ, for the solution of the problem (4.1)–(4.3) it holds the following estimate

$$|\lambda|^2 \|u(x,y)\|_{L_p((0,1),L_2(0,1))} + \left\|D_x^2 u(x,y)\right\|_{L_p((0,1);L_2(0,1))} +$$
$$+ \|D_y(a(y)D_y u(x,y))\|_{L_p((0,1);L_2(0,1))} \leq$$
$$\leq C\left[|\lambda|\,\|f(x,y)\|_{L_p((0,1);L_2(0,1))} + \|f_1(y)\|_{B_{2,p}^{1-\frac{1}{p}}(0,1)} + \right.$$
$$\left. + \|f_2(y)\|_{B_{2,p}^{1-\frac{1}{p}}(0,1)} + |\lambda|^{1-\frac{1}{p}}\left(\|f_1(y)\|_{L_2(0,1)} + \|f_2(y)\|_{L_2(0,1)}\right)\right]. \tag{4.4}$$

Proof. In the space $H = L_2(0,1)$, consider an operator A which is defined by the following equalities

$$D(A) := W_2^2((0,1), u(0) = u(1) = 0), \quad Au := \left(-a(y)u'(y)\right)'.$$

Then, we can rewrite the problem (4.1)–(4.3) in the operator form (3.1), (3.2) and apply theorem 3.1. From condition 1 it follows that the operator A is self-adjoint, positive-definite in $H = L_2(0,1)$, i.e., the only thing remains is to write down explicitly the interpolation space $(H(A), H)_{\frac{1}{2}+\frac{1}{2p},p}$.

By virtue of [14, theorem 4.3.3],

$$(H(A), H)_{\frac{1}{2}+\frac{1}{2p},p} = \left(W_2^2((0,1); u(0) = u(1) = 0),\ L_2(0,1)\right)_{\frac{1}{2}+\frac{1}{2p},p} =$$
$$= \begin{cases} B_{2,p}^{1-\frac{1}{p}}(0,1), & 1 < p < 2, \\ W_2^{\frac{1}{2}}\left((0,1);\ \int_0^1 (\min\{x, 1-x\})^{-1} |u(x)|^2 dx < \infty\right), & p = 2, \\ B_{2,p}^{1-\frac{1}{p}}((0,1); u(0) = u(1) = 0),\ & p > 2, \end{cases}$$

i.e., $(H(A), H)_{\frac{1}{2}+\frac{1}{2p},p} = B_{2,p,*}^{1-\frac{1}{p}}(0,1)$.

The theorem is proved. □

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