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ON BEHAVIOR AT INFINITY OF THE SOLUTIONS OF A SEMILINEAR ELLIPTIC EQUATION OF SECOND ORDER

SHIRMAIL G. BAGIROV

Abstract. Asymptotic behavior of the solution of a semilinear elliptic equation of second order in a cylindrical domain, satisfying the Neumann boundary condition on the lateral surface, is studied.

1. Introduction and the main result

Denote: $\Pi_{a,b} = G \times (a,b)$, $\Pi_{a,\infty} = \Pi_a$, $\Gamma_{a,b} = \partial G \times (a,b)$, $\Gamma_{a,\infty} = \Gamma_a$, where G is a bounded domain in R^n with the Lipschits boundary.

Let L be an operator of the form

$$L = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i},$$

where $x = (x_1, \ldots, x_n) \in R^n$, the coefficients $a_{ij}(x), a_i(x)$ are bounded, measurable functions, $a_{ij} = a_{ji}$ and the following ellipticity condition be fulfilled

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j > V_1 \left|\xi\right|^2, \ \ x \in G, \ \ V_1 = const > 0, \ \ \left|\xi\right|^2 = \sum_{i=1}^{n} \xi_i^2, \ \ \xi \in R^n.$$

We study the behavior at infinity of the solutions of the equation:

$$u_{tt} + Lu - (t+1)^{\mu} \left|u\right|^{\sigma} = 0 \ \text{ in } \Pi_0, \tag{1.1}$$

satisfying the condition:

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial u}{\partial x_j}\cos(x_i, n) \ \text{ on } \Gamma_0, \tag{1.2}$$

where $\sigma > 1$, $\mu > -2$, n is a unit vector of the external normal to ∂G.

The case $\mu = 0$ was considered in the paper [1].

Similar problems with nonlinearity of the form $\left|u\right|^{\sigma-1} u$ were investigated in the papers [2]–[6].

As the solution of problem (1.1), (1.2) we understand a generalized solution. The function $u(x,t)$ is said to be the generalized solution of equation (1.1) satisfying condition (1.2) if $u(x,t) \in W_2^1(\Pi_{a,b}) \cap L_\infty(\Pi_{a,b})$ for any $0 < a, b < \infty$ and it holds the equality:

$$\int\limits_{\Pi_{a,b}} u_t\varphi_t dxdt + \sum_{i,j=1}^{n} \int\limits_{\Pi_{a,b}} a_{ij}(x)u_{x_j}\varphi_{x_i}dxdt -$$

$$\sum_{i=1}^{n} \int\limits_{\Pi_{a,b}} a_i(x) u_{x_i} \varphi dxdt + \int\limits_{\Pi_{a,b}} (t+1)^{\mu} |u|^{\sigma} \varphi dxdt = 0$$

for any function $\varphi(x,t) \in W_2^1(\Pi_{a,b})$ such that $\varphi(x,a) = \varphi(x,b) = 0$.

From the classic results on smoothness of generalized solutions of linear elliptic equations it follows that $u(x,t)$ in any closed domain $\bar{\Pi}_{a,b}$ satisfies the Holder condition [7].

The following theorem is the main result

Theorem 1.1. *a) If $u(x,t) > 0$ is the solution of equation (1.1), satisfying condition (1.2), then $u(x,t) = O\left(t^{-\frac{\mu+2}{\sigma-1}}\right)$,*
b) If $u(x,t)$ is the solution of equation (1.1), satisfying condition (1.2), that changes the sign at any domain Π_a, $a > 0$, then

$$u(x,t) = O(e^{-ht}),$$

where h is independent of $u(x,t)$.

2. Absence of negative solutions and auxiliary lemma.

Let us consider the following problem:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x} \right) - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (a_i(x) u) = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial \nu^*} = \sum_{i,j=1}^{n} a_{ij} \frac{\partial u}{\partial x_j} \cos(n, x_i) - \sum_{i=1}^{n} a_i u \cos(n, x_i) = 0. \tag{2.2}$$

It is known that problem (2.1), (2.2) has a solution $k(x)$ such that $0 < m_1 \leqslant k(x) \leqslant m_2$, $m_1, m_2 = const > 0$ (see. [8]).

Lemma 2.1. *For any $\sigma > 1$, $\mu > -2$ problem (1.1), (1.2) has no negative solutions.*

Proof. In defining the solution, as a test function we take $\varphi(x,t) = t\psi(t)k(x)$, where $\psi(t) \in C_0^{\infty}(R)$, $\psi(t) = 1$ for $0 \leqslant t \leqslant R$, $\psi(t) = 0$ for $t \geqslant 2R$ and $k(x)$ is the solution of problem (2.1), (2.2).

Then we get:

$$\int\limits_{\Pi_{0,2R}} |u|^{\sigma} (t+1)^{\mu} t\psi k dxdt = -\int u_t (t\psi' + \psi) k dxdt -$$

$$-\sum_{i,j=1}^{n} \int\limits_{\Pi_{0,2R}} a_{ij}(x) u_{x_j} k_{x_i} t\psi dxdt + \sum_{i=1}^{n} \int\limits_{\Pi_{0,2R}} a_i(x) u_{x_i} k t\psi dxdt =$$

$$= -\int\limits_{\Pi_{0,2R}} u_t (t\psi' + \psi) k \, dxdt = \int\limits_{\Pi_{0,2R}} uk(t\psi'' + 2\psi') dxdt + \int\limits_{G} ku(x,0) dx \leqslant$$

$$\leqslant m_2 \left(\int\limits_{\Pi_{0,2R}} |u|^\sigma (t+1)^\mu t\psi dxdt \right)^{1/\sigma} \left(\int\limits_{\Pi_{0,2R}} \frac{|t\psi'' + 2\psi'|^q}{t^{q-1}(t+1)^{\mu(q-1)}\psi^{q-1}} dxdt \right)^{1/q} \leqslant$$

$$\leqslant \frac{\varepsilon m_2}{\sigma} \int\limits_{\Pi_{0,2R}} |u|^\sigma (t+1)^\mu t\psi dxdt +$$

$$+ \frac{m_2}{\varepsilon^{q-1} q} \int\limits_{\Pi_{0,2R}} \frac{|t\psi'' + 2\psi'|^q}{t^{q-1}(t+1)^{\mu(q-1)}\psi^{q-1}} dxdt + m_2 \int\limits_G u(x,0)dx,$$

where $\frac{1}{\sigma} + \frac{1}{q} = 1$.

Then as a result we have:

$$\left(\frac{m_1}{m_2} - \frac{\varepsilon}{\sigma} \right) \cdot \int\limits_{\Pi_{0,2R}} |u|^\sigma (t+1)^\mu t\psi dxdt \leqslant$$

$$\leqslant \frac{1}{\varepsilon^{q-1} q} \cdot \int\limits_{\Pi_{0,2R}} \frac{|t\psi'' + 2\psi'|^q}{(t+1)^{\mu(q-1)} t^{q-1}\psi^{q-1}} dxdt + \int\limits_G u(x,0)dx. \tag{2.3}$$

we take $\psi(t)$ in the form $\psi(t) = \psi(\tau R) = (\varphi_0(\tau))^\lambda = \theta(\tau)$, where $\tau = t/R$, $\varphi_0(\tau) = 0$ for $\tau \leq 1$, $\tau \geq 2$, $\varphi_0(t) \in C_0^\infty(R)$, λ is a rather large positive number. Estimate the first term in the right hand side of (2.3):

$$\int\limits_{\Pi_{0,2R}} \frac{|t\psi'' + 2\psi'|^q}{(t+1)^{\mu(q-1)} t^{q-1}\psi^{q-1}} dxdt \leqslant$$

$$\leqslant \int\limits_G \int\limits_{1 \leqslant \tau \leqslant 2} \frac{|\tau\theta'' + 2\theta'|^q}{R^{(\mu+2)(q-1)} \left(\tau + \frac{1}{R}\right)^{\mu(q-1)} \tau^{q-1}\theta^{q-1}} dxdt \leqslant R^{(\mu+2)(1+q)} \cdot A(\varphi_0),$$

where

$$A(\varphi_0) = \text{mes}G \int\limits_{1 \leqslant \tau \leqslant 2} \frac{\left| \tau\lambda\varphi_0^{\lambda-1}\varphi_0'' + \tau\lambda(\lambda-1)\varphi_0^{\lambda-2} + 2\lambda\varphi_0^{\lambda-1}\varphi_0' \right|^q}{\left(\tau + \frac{1}{R}\right)^{\mu(q-1)} \tau^{q-1}\varphi_0^{\lambda(q-1)}} d\tau.$$

Obviously, by chosing λ, φ_0 we can choose so that there would be $A(\varphi_0) < \infty$. Then from (2.3) we obtain:

$$\left(\frac{m_1}{m_2} - \frac{\varepsilon}{\sigma} \right) \cdot \int\limits_{\Pi_{0,R}} |u|^\sigma (t+1)^\mu t dxdt \leqslant \left(\frac{m_1}{m_2} - \frac{\varepsilon}{\sigma} \right) \cdot \int\limits_{\Pi_{0,2R}} |u|^\sigma (t+1)^\mu t\psi dxdt \leqslant$$

$$\leqslant \frac{1}{q\varepsilon^{q-1}} \cdot R^{(\mu+2)(1-q)} A(\varphi_0) + \int\limits_G u(x,0)dx. \tag{2.4}$$

Since $q > 1$, then for $\int\limits_G u(x,0)dx \leqslant 0$ and as $R \to \infty$ from (2.4) we get $u \equiv 0$ in Π_0. This proves Lemma 2.1. □

If in the proof of Lemma 2.1 as a test function we take $\varphi(x,t) = (t-t_0)\psi(t)k(x)$ for $t \geqslant t_0$, $\varphi(x,t) = 0$ for $t < t_0$, then we get that for any nontrivial solution $u(x,t)$ of problem (1.1), (1.2)

$$\int_G u(x,t_0)dx > 0. \tag{2.5}$$

Lemma 2.2. *If $u(x,t)$ is the non-trivial solution of problem (1.1), (1.2), then*

$$u(x,t) = O\left(t^{-\frac{\mu+2}{\sigma-1}}\right).$$

Since

$$u_{tt} + Lu - (t+1)^\mu |u|^{\sigma-1} u \geqslant u_{tt} + Lu - (t+1)^\mu |u|^\sigma,$$

then any solution of equation (1.1) is the subsolution of the equation

$$u_{tt} + Lu - (t+1)^\mu |u|^{\sigma-1} u = 0. \tag{2.6}$$

Equation (2.6) has a strong positive solution $\omega(t)$, satisfying the relations $\omega(t_0) = 1$, $\omega'(t_0) = 0$ at the points $t_0 \pm T$ (where T is independent of t_0) (see. [4]). Then for rather large t from the maximum principle the subsolution is less than the solution i.e. $u(x,t) \leqslant \omega(t)$ in Π_{t_0-T,t_0+T}.

Thus, $u(x,t)$ is upper bounded, as for large t is less than the value at the vertex of the parabola.

The function $v(x,t) = u(x,t) - c_0 t^{-\frac{\mu+2}{\sigma-1}}$, where $c_0 = \left[\frac{(\mu+\sigma+1)(\mu+2)}{(\sigma-1)^2}\right]^{\frac{1}{\sigma-1}}$, is also an upper bounded subsolution of equation (2.6). Then:

$$v_{tt} + Lv - a(x,t)v \geqslant 0, \tag{2.7}$$

where $a(x,t) \geqslant 0$.

Let us consider the function $v - \varepsilon t$. This function also satisfies inequality (2.7) and is negative for $t = 0$. There exists $T_0(\varepsilon)$ such that for $T \geqslant T_0(\varepsilon)$, $v - \varepsilon T \leqslant 0$. Then from the maximum principle it follows that $v - \varepsilon T \leqslant 0$ for $t \geqslant 0$. Having tending ε to zero, we get $v \leqslant 0$.

So:

$$u^+(x,t) \leqslant c_0 t^{-\frac{\mu+2}{\sigma-1}}. \tag{2.8}$$

As

$$|u| = u^+ - u^-, \quad u = u^+ + u^-,$$

then by (2.5) we get:

$$\int_G |u|\, dx = \int_G (u^+ - u^-)dx = \int_G (2u^+ - u)dx \leqslant 2\int_G u^+ dx \leqslant c_1 \cdot t^{-\frac{\mu+2}{\sigma-1}}.$$

Hence for large T

$$\int_{\Pi_{T-2,T+2}} |u|\, dxdt \leqslant c_1 \cdot \int_{T-2}^{T+2} t^{-\frac{\mu+2}{\sigma-1}} dt \leqslant 4c_1(T-2)^{-\frac{\mu+2}{\sigma-1}} =$$

$$= 4c_1(T+1)^{-\frac{\mu+2}{\sigma-1}} \left(1 - \frac{3}{T+1}\right)^{-\frac{\mu+2}{\sigma-1}} \leqslant c_2(T+1)^{-\frac{\mu+2}{\sigma-1}}. \tag{2.9}$$

From the theory of linear differential equations we know that (see. [7])

$$\max_{\Pi_{T-1,T+1}} |u| \leqslant c \int\limits_{\Pi_{T-1,T+1}} |u|\, dxdt.$$

Using (2.9), from this inequality we get that for large T

$$\max_{\Pi_{T-1,T+1}} |u| \leqslant c_2 (T+1)^{-\frac{\mu+2}{\sigma-1}}.$$

Hence for $t \in [T-1, T+1]$

$$|u(x,t)| \leqslant c_2 t^{-\frac{\mu+2}{\sigma-1}}.$$

So

$$u(x,t) = O\left(t^{-\frac{\mu+2}{\sigma-1}}\right).$$

This proves lemma 2.2.

3. The proof of the main result.

Point a) was already proved, prove b). Write equation (1.1) in the form

$$u_{tt} + Lu - q_0(x,t)u = 0,$$

where $q_0(x,t) = (t+1)^{\mu} |u|^{\sigma-1} \operatorname{sign} u$.

As $q_0(x,t) = O(t^{-2})$, then there exists t_0 such that for $t \geqslant t_0$

$$|q_0(x,t)| < \varepsilon.$$

Take $\theta(t) \in C_0^{\infty}$ such that $\theta(t) = 1$ for $t > t_0 + 1$, $\theta(t) = 0$ for $t \leqslant t_0$ and $0 \leqslant \theta(t) \leqslant 1$.

Assume

$$v(x,t) = \theta(t)u(x,t).$$

The function $v(x,t)$ satisfies the equation

$$v_{tt} + Lv - q(x,t)v = F(x,t), \tag{3.1}$$

and the boundary condition

$$\frac{\partial v}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0, \tag{3.2}$$

where

$$q(x,t) = \begin{cases} q_0(x,t) & \text{for} \quad t \geqslant t_0 + 1 \\ 0 & \text{for} \quad t \leqslant t_0, \end{cases}$$

$$F(x,t) = (\theta_t \cdot u)_t + \theta_t \cdot u_t.$$

Show that $|v(x,t)| \leqslant c \cdot \exp\{-ht\}$, $c = const$. As the function $F(x,t)$ has a compact support, then from theory of linear equations (see. [9], [10]) it follows that problem (3.1), (3.2) has the solution $v_1(x,t)$ such that

$$v_1(x,t) = \begin{cases} O(e^{-ht}) & \text{as} \quad t \to +\infty \\ at + b + O(e^{ht}) & \text{as} \quad t \to -\infty. \end{cases} \tag{3.3}$$

The function $\omega(x,t) = v_1(x,t) - v(x,t)$ satisfies the equation:

$$\omega_{tt} + L\omega - q(x,t)\omega = 0 \tag{3.4}$$

and the boundary condition :

$$\frac{\partial\omega}{\partial\nu} = 0 \quad \text{on} \quad \Gamma_0,$$

$\omega(x,t) \to 0$ as $t \to +\infty$ and $\omega = at + b + O(e^{ht})$ as $t \to -\infty$. If we prove that $\omega \equiv 0$, then the problem statement follows. Show that $a = 0$, $b = 0$. Suppose that $a > 0$. So, $\omega(x,t) < 0$ for $t < -T_1$, where T_1 is a rather large positive number. Prove $\omega < 0$ for $t > -T_1$. As $q(x,t) = q_0(x,t)$ for $t \geqslant t_0 + 1$, then $q(x,t) = O(t^{-2})$ as $t \to +\infty$.

Denote $l = \max\limits_{t=T} \omega(x,t)$, and $W(x,t) = (\omega - l)^+$, where T is a rather large positive number. Obviously, $W(x,t) = 0$ for $t = -T_1$, $t = T$ and

$$W(x,t) \in \overset{\circ}{W}{}_2^1(Q_{T_1,T}).$$

In defining the solution, as a test function we take $\varphi(x,t) = W(x,t) \cdot k(x)$, where $k(x)$ is the positive solution of problem (2.1),(2.2) such that $0 < m_1 \leqslant k(x) \leqslant m_2$.

Then from the definition of the solution we have:

$$\int\limits_{A_l^+} |\omega_t|^2 \, k dxdt + \nu_1 \int\limits_{A_l^+} |\nabla\omega|^2 \, k dxdt \leqslant - \int\limits_{A_l^+} q(x,t) \cdot k \cdot \omega \cdot (\omega - l)^+ dxdt, \qquad (3.5)$$

where $A_l^+ = \{(x,t),\ W > 0\}$. Here we used that

$$-\sum_{i,j=1}^{n} \int\limits_{\Pi_{-T_1,T}} a_{ij} \frac{\partial\omega}{\partial x_j}\frac{\partial k}{\partial x_i} W dxdt + \sum_{i=1}^{n} \int\limits_{\Pi_{-T_1,T}} a_i \frac{\partial\omega}{\partial x_i} \cdot W \cdot k dxdt =$$

$$= -\frac{1}{2}\sum_{i,j=1}^{n} \int\limits_{\Pi_{-T_1,T}} a_{i,j} \frac{\partial\omega^2}{\partial x_j}\frac{\partial k}{\partial x_i} dxdt + \frac{1}{2}\sum_{i=1}^{n} \int\limits_{\Pi_{-T_1,T}} a_i \frac{\partial\omega^2}{\partial x_i} k dxdt = 0.$$

Estimate the right hand side of (3.5) using the inequality [see. 7]

$$\|u\|_{\frac{2n}{n-2}} \leqslant C \, \|\nabla u\|_{2,\Omega} \, ,$$

where C is a constant dependent on the dimension of n.

Then:

$$- \int\limits_{A_l^+} q(x,t)\omega(\omega - l)^+ k dxdt \leqslant \int\limits_{A_l^+} |q(x,t)| \, (\omega - l + l)(\omega - l) k dxdt =$$

$$= \int\limits_{A_l^+} |q(x,t)| \cdot |\omega - l|^2 \, k dxdt + l \cdot \int\limits_{A_l^+} |q(x,t)| \cdot |\omega - l| \, k dxdt \leqslant$$

$$\leqslant m_2 \int\limits_{\substack{A_l^+ \\ t>t_0}} |q(x,t)| \cdot |\omega - l|^2 \, dxdt + l \cdot m_2 \int\limits_{\substack{A_l^+ \\ t>t_0}} |q(x,t)| \cdot |\omega - l| \, dxdt. \qquad (3.6)$$

At first estimate the first term:

$$F_1 = \int\limits_{\substack{A_l^+ \\ t>t_0}} |q(x,t)| \cdot |\omega - l|^2 \, dxdt \leqslant$$

$$\leqslant \left(\int\limits_{\substack{A_l^+ \\ t>t_0}} |\omega - l|^{\frac{2(n+1)}{n-1}} \, dxdt \right)^{\frac{n-1}{n+1}} \left(\int\limits_{\substack{A_l^+ \\ t>t_0}} |q(x,t|^{\frac{n+1}{2}} \, dxdt \right)^{\frac{2}{n+1}} \leqslant$$

$$\leqslant \left(\int\limits_{A_l^+ \cap Q_{T_1,T_2}} |\omega - l|^{\frac{2(n+1)}{n-1}} \, dxdt \right)^{\frac{n-1}{n+1}} \left(\int\limits_{A_l^+ \cap \{t>t_0\}} |q(x,t)|^{\frac{n+1}{2}} \, dxdt \right)^{\frac{2}{n+1}} \leqslant$$

$$\leqslant \left[\left(\int\limits_{A_l^+ \cap Q_{T_1,T_2}} |\omega - k|^{\frac{2(n+1)}{n-1}} \, dxdt \right)^{\frac{n-2}{2(n+1)}} \right]^2 \left(\int\limits_{A_k^+ \cap \{t>t_0\}} |q(x,t)|^{\frac{n+1}{2}} \, dxdt \right)^{\frac{2}{n+1}} \leqslant$$

$$\leqslant C \cdot \left(\int\limits_{A_l^+ \cap Q_{T_1,T_2}} |\nabla(\omega - l)|^2 \, dxdt \right) \cdot I_2, \tag{3.7}$$

where

$$I_2 = \left(\int\limits_{A_l^+ \cap \{t>t_0\}} |q(x,t)|^{\frac{n+1}{2}} \, dxdt \right)^{\frac{2}{n+1}}.$$

Then estimate I_2.

$$I_2 = \left(\int\limits_{A_l^+ \cap \{t>t_0\}} |q(x,t)|^{\frac{n+1}{2}} \, dxdt \right)^{\frac{2}{n+1}} \leqslant C_1 \cdot \left(\int\limits_{A_l^+ \cap \{t>t_0\}} t^{-(n+1)} dxdt \right)^{\frac{2}{n+1}} \leqslant$$

$$\leqslant C_1 \cdot \left(\int\limits_{t_0}^{T} t^{-(n+1)} dxdt \right)^{\frac{2}{n+1}} \leqslant C_2 \cdot \left(\frac{t^{-n}}{-n} \Big|_{t_0}^{T} \right)^{\frac{2}{n+1}} =$$

$$= C_2 \cdot \left(\frac{T^{-n}}{-n} + \frac{t_0^{-n}}{n} \right)^{\frac{2}{n+1}} = C_3 \cdot \left(t_0^{-n} - T^{-n} \right)^{\frac{2}{n+1}}.$$

take t_0 so that $|u(x,t)| < \varepsilon$ and $C_3 \cdot t_0^{-\frac{2n}{n+1}} < \frac{m_1 \nu_1}{4C m_2 \cdot}$. Then we get

$$I_2 \leqslant \frac{m_1 \nu_1}{4C \cdot m_2}.$$

From (3.7) it follows

$$F_1 \leqslant \frac{\nu_1 m_1}{4m_2} \cdot \int\limits_{A_l^+ \cap Q_{T_1,T_2}} |\nabla(\omega - l)|^2 \, dxdt. \tag{3.8}$$

Estimate that second term in the right side of (3.6).

$$F_2 = l \cdot \int\limits_{A_l^+} |q(x,t)| \cdot |\omega - l| \, dxdt \leqslant$$

$$\leqslant l \cdot \left(\int\limits_{A_l^+} |q(x,t)|^{p_1} \, dxdt \right)^{\frac{1}{p_1}} \left(\int\limits_{A_l^+} |\omega - l|^{\frac{2(n+1)}{n-1}} \, dxdt \right)^{\frac{n-1}{2(n+1)}} \leqslant$$

$$\leqslant l \cdot C_1 \left(\int\limits_{\substack{t>t_0 \\ A_l^+}} t^{-2p_1} dt \right)^{\frac{1}{p_1}} \left(\int\limits_{A_l^+} |\nabla(\omega - l)|^2 \, dxdt \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant \frac{\nu_1 m_1}{4m_2} \int\limits_{A_l^+} |\nabla(\omega - l)|^2 \, dxdt + l^2 \cdot C_2 \left(\int\limits_{\substack{t>t_0 \\ A_l^+}} t^{-2p_1} dt \right)^{\frac{2}{p_1}}, \tag{3.9}$$

here $\frac{1}{p_1} + \frac{n-1}{2(n+1)} = 1$.

Hence $p_1 = 1 + \frac{n-1}{n+3}$.

Combining (3.8) and (3.9), we get

$$m_1 \int\limits_{A_l^+} |\omega_t|^2 \, dxdt + m_1 \nu_1 \int\limits_{A_l^+} |\nabla \omega|^2 \, dxdt \leqslant \frac{m_1}{2} \int\limits_{A_l^+} |\omega_t|^2 \, dxdt +$$

$$+ \frac{m_1 \nu_1}{2} \int\limits_{A_l^+} |\nabla \omega|^2 \, dxdt + l^2 \cdot C_2 \left(\int\limits_{t_0}^{T} t^{-2p_1} dt \right)^{2/p_1}.$$

As a result, for $n > 1$ we have:

$$\frac{m_1}{2} \cdot \int\limits_{A_l^+} |\omega_t|^2 \, dxdt + \frac{m_1 \nu_1}{2} \cdot \int\limits_{A_l^+} |\nabla \omega|^2 \, dxdt \leqslant l^2 \cdot C \left(\int\limits_{t_0}^{T} t^{-2p_1} dt \right)^{2/p_1}.$$

From the fact $l(T) \to 0$ as $T \to 0$ and from the convergence of the integral $\int\limits_{t_0}^{T} t^{-2p_1} dt$ we get $mes\, A_l^+ = 0$.

So $\omega - l \leqslant 0$. As l converges to zero, then $\omega < 0$.

In the similar way, we can prove that if $a < 0$, then $\omega(x,t) > 0$.

Show that $a = b = 0$. Assume $a > 0$. So, $\omega(x,t) < 0$ for $t > t_1$. The function $\omega_1 = -t^\beta$ will be a supersolution of equation (3.4) for large in modulus negative β.

Indeed:

$$\omega_{1tt} + L\omega_1 - q(x,t)\omega_1 = -\beta(\beta-1)t^{\beta-2} + q(x,t)t^\beta = -t^{\beta-2}(\beta(\beta-1) - qt^{-2}) < 0.$$

Let t_2 be rather large. Choose A so small positive number that $-At_2^\beta \geqslant \omega(x,t_2)$. Then from $W = \omega(x,t_2) + At_2^\beta \leqslant 0$, $\omega(x,t) + At^\beta \to 0$ as $t \to +\infty$ and

$$W_{tt} + LW - q\,(x,t)\,W \geqslant 0,$$

as above we can prove that

$$\omega(x,t) + A\,t^\beta \leqslant 0 \ \text{ as } \ t \geqslant t_2.$$

Let us consider the points set, where $v = u < 0$, for them we have

$$-A \cdot t^\beta \geqslant \omega(x,t) \geqslant v_1 \geqslant -C_1 e^{-h\,t}.$$

This contradiction proves that a can not be positive. We can similarly show that a can not be negative, and that $b = 0$. Since $\omega \to 0$ as $t \to \pm\infty$, consequently $\omega \equiv 0$.

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Shirmail G. Bagirov

Baku State University, 23 Z. Khalilov str., AZ 1148, Baku, Azerbaijan

Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9 B. Vahabzadeh str., AZ 1141, Baku, Azerbaijan
E-mail address: bagirov@yahoo.com

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