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FINITENESS OF THE NEGATIVE SPECTRUM OF THE ONE-DIMENSIONAL MAGNETIC SCHRÖDINGER OPERATOR

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Abstract. In the paper we find relation between the solutions of the basic homogeneous equation of perturbation theory and eigen functions of a self-adjoint magnetic Schrödinger operator responding to negative eigenvalues. Under certain conditions on magnetic and electric potentials, we prove finiteness of the negative spectrum of the one-dimensional magnetic Schrödinger operator.

1. Introduction

Depending on behavior of magnetic and electric potentials, the spectrum of the energy of the charged particle in the magnetic field contains, generally speaking, both a continuous and discrete component. It is known well that the discrete spectrum of the Schrödinger operator in quantum mechanics corresponds to the bound state described by the Schrodinger equation. Furthermore, it is known that the Schrödinger operator is an important device by integrating the Kortewag-de Fries equation by the inverse method of scattering theory. As each negative eigen value of the Schrödinger operator generates a solitone solution of the Kortewag-de-Friez equation, then knowing the number of negative eigenvalues is of interest. Furthermore, estimation of the number of negative eigen value plays an important role both in quantum mechanics and in spectral theory of differential operators.

A lot of papers have been devoted to investigation of the negative part of the spectrum of the Schrödinger operator. In the first turn point out the books [8,10,13,15] and the references therein, and also the papers [5,7,18,19].

Recently, the researchers directed their interests to the operators connected not only with electric and also with magnetic fields such as magnetic Schrödinger and Paouli operators (see e.i. [1, 4, 11, 12, 16]).

In the present paper, in the space $L_2(R_1)$ $(R_1 = (-\infty, +\infty))$ we study one-dimensional Schrödinger operator generated by the differential expression

$$\Delta_{a,V} = \left(\frac{1}{i}\frac{d}{dx} + a(x)\right)^2 + V(x),$$

where $a(x)$ and $V(x)$ are magnetic and electric potentials, respectively, and these potential are real functions satisfying the following conditions:

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a) $\Phi(x) \equiv a^2(x) + V(x) + ia'(x) \in L_1(R_1)$;
b) $a(x) \in L_1(R_1)$.

In spite of the fact that in one-dimensional case the magnetic potential is gauge equivalent to zero, in this paper we want to clear up how the interaction of electric and magnetic fields influences on spectral properties of the Schrödinger operator; how one can obtain from the found results of the general case the known results at no magnetic potential case; how one can approach to the solution of arising problems by applying in many-dimensional case the method developed in the papers [11,12].

Let us consider in $L_2(R_1)$ the self-adjoint operators $H_0 := -\frac{d^2}{dx^2}$ and $H = H_0 + W$,responding to quadratic forms $h_0(\varphi) = \int_{-\infty}^{+\infty} |\varphi'|^2 dx$ and $h_{a,V}(\varphi) = h_0(\varphi) + (W\varphi, \varphi)$ respectively, where W is an operator acting by the formula $W = -2i\frac{d}{dx}a(x) + \Phi(x)$. Note that the self-adjointness of the one-dimensional magnetic Schrödinger operator $H = H_0 + W$ was proved in the paper [6].

Denote by $C(R_1)$ a Banach space of continuous and bounded in R_1 functions with the norm $\sup\limits_{-\infty<x<+\infty} |f(x)| = \|f\|_{C(R_1)} < +\infty$.

Let $h(x) \in C(R_1)$ and $z = \lambda^2$, $\mathrm{Im}\lambda > 0$. Assume

$$u_0(\lambda) \equiv u_0(x,\lambda) = R_0(\lambda^2)\, h(x), \quad u(\lambda) \equiv u(x,\lambda) = R(\lambda^2)\, h(x),$$

where $R_0(\lambda^2) = (H_0 - \lambda^2)^{-1}$ and $R(\lambda^2) = (H - \lambda^2)^{-1}$ are the resolvents of the operators H_0 and H, respectively. Taking into account that the operators $-i\frac{d}{dx}$ and $R_0(\lambda^2)$ are permutable, $R_0(\lambda^2)$ is an integral operator with the kernel

$$G_0(x,y,\lambda) = -\frac{e^{i\lambda|x-y|}}{2i\lambda},$$

for $u(\lambda)$ we get the inhomogeneous equation

$$u(\lambda) + K(\lambda)\, u(\lambda) = u_0(\lambda),$$

where $K(\lambda)$ is an integral operator with the kernel

$$K(x,y,\lambda) = -\frac{e^{i\lambda|x-y|}}{2i\lambda}\left[\Phi(y) + 2\lambda sgn(x-y)\, a(y)\right].$$

Denote by $\mathbf{E}_+$ the set of those points from the half-plane $C_+ = \{\lambda \in C : \mathrm{Im}\lambda > 0\}$ for which the homogeneous equation

$$f + K(\lambda) f = 0 \tag{1.1}$$

has a nontrivial solution in $C(R_1)$.

In the paper [3, Theorem 1] it is proved that the operator $K(\lambda)$ is analytic with respect to λ in the upper part of the complex plane $C_+ = \{\lambda \in C : \mathrm{Im}\lambda > 0\}$ in uniform operator topology and for all λ from $\overline{C_+} \setminus \{0\} = \{\lambda \in C : \mathrm{Im}\lambda \geq 0, \lambda \neq 0\}$ is compact in $C(R_1)$ and is continuous in uniform operator topology. Using these results, in the paper [6] it is established that if $\sigma + i\tau = \lambda \in \mathbf{E}_+$ and $f(x)$ is a nontrivial solution of the homogeneous equation (1) from $C(R_1)$, and conditions a), b) are fulfilled, then

$$\sup_{-\infty<x<+\infty} e^{\tau} |f(x)| < +\infty \tag{1.2}$$

and $f(x) \in W_2^1(R_1)$.

In the paper [2] it is proved that under the conditions a) and b) for the operator H a positive semi-axis is a double continuous spectrum and on the interval $(0,+\infty)$ it has no eigenvalues.

The goal of the present paper is to study the negative spectrum of the one-dimensional magnetic Schrödinger operator H.

2. Basic results

Theorem 2.1. *Let $\sigma + i\tau = \lambda \in \mathbf{E}_+$. Then $\lambda = i\tau, \tau > 0$. Therewith $\lambda^2 = -\tau^2$ is the eigenvalue of the operator H of finite multiplicity.*

Proof. Let $\sigma + i\tau = \lambda \in \mathbf{E}_+$. Show that the number λ^2 is the eigen value of the operator H. It follows from estimation (1.2) that equation (1.1) has a nontrivial, exponentially decreasing solution from the space $W_2^1(R_1)$. Show that

$$h_{a,V}(f) = h_0(f) + (Wf, f) = \lambda^2 (f, f)$$

is valid.

From the everywhere density of the space of basic functions $C_0^\infty(R_1)$ in $W_2^1(R_1)$ it follows that there exists the sequence $\{f_n(x)\}_{n=1}^\infty \subset C_0^\infty(R_1)$ such that

$$\lim_{n\to\infty} \|f_n(x) - f(x)\|_{W_2^1(R_1)=0} .$$

As $\lambda \in \mathbf{E}_+$, the operator $-\frac{d^2}{dx^2} - \lambda^2$, is one-to-one maps the space of distributions of slow growth S' onto itself (S is the Schwarts space [17, p. 87]). It is known that $C_0^\infty(R_1) \subset S'$ and $W_2^1(R_1) \subset S'$. Hence it follows that the images of the elements of the spaces $C_0^\infty(R_1)$ and $W_2^1(R_1)$ by mapping $-\frac{d^2}{dx^2} - \lambda^2$ become the elements of the space distributions of slow growth S'. In particular, the linear manifold $\left(-\frac{d^2}{dx^2} - \lambda^2\right) C_0^\infty(R_1)$ is everywhere dense both in $L_2(R_1)$ and in $W_2^1(R_1)$. The similar results hold for the operator $-\frac{d^2}{dx^2} - \overline{\lambda}^2$ as well. Let now $\psi \in \left(-\frac{d^2}{dx^2} - \overline{\lambda}^2\right) C_0^\infty(R_1)$. Then there exists a unique element $\varphi \in C_0^\infty(R_1)$ such that $\psi = \left(-\frac{d^2}{dx^2} - \overline{\lambda}^2\right)\varphi$. Taking into account (1.1) and the equality $K(\lambda) = R_0(\lambda^2) W$, we have:

$$0 = (f + K(\lambda) f, \psi) = \lim_{n\to\infty} (f_n + K(\lambda) f_n, \psi) =$$

$$\lim_{n\to\infty} \left(f_n + K(\lambda) f_n, \left(-\frac{d^2}{dx^2} - \overline{\lambda}^2\right)\varphi\right) =$$

$$\lim_{n\to\infty} \left(\left(-\frac{d^2}{dx^2} - \lambda^2\right)(f_n + K(\lambda) f_n), \varphi\right) =$$

$$\lim_{n\to\infty} \left(\left(-\frac{d^2}{dx^2} - \lambda^2\right)\left(f_n + R_0(\lambda^2) W f_n\right), \varphi\right) =$$

$$\lim_{n\to\infty} \left(\left(-\frac{d^2}{dx^2} - \lambda^2\right) f_n + W f_n, \varphi\right) =$$

$$\left(\left(-\frac{d^2}{dx^2} - \lambda^2\right) f + W f, \varphi\right) = \left(Hf - \lambda^2 f, \varphi\right).$$

By arbitrariness of ψ (together with it the arbitrariness of φ) we get

$$Hf = -\frac{d^2}{dx^2} f + Wf = \lambda^2 f. \tag{2.1}$$

For proving the equality $\lambda = i\tau, \tau > 0$ it suffices to notice that $\lambda \in \mathbf{E}_+$ and the eigenvalue λ^2 of the self-adjoint operator H should be real. Note that the finiteness of multiplicity of the eigenvalue $\lambda^2 = -\tau^2$ of the operator H follows from Fredholm's analytic theorem (see e.i. [14, p. 87]). However, the last part of the statement of the theorem is the consequence of general theory of ordinary differential equations with regular generalized coefficients. □

Remark 2.1. In equality (3) the sum $-\frac{d^2}{dx^2} f + Wf$ should be understood in the sense of quadratic forms but not the sum of operators. The matter is that though both functions $f(x)$ and $(Hf)(x)$ belong to the space $L_2(R_1)$, but there may happen so that none of the functions $-\frac{d^2}{dx^2} f$ and Wf belong to the space $L_2(R_1)$.

Theorem 2.2. *Let conditions a), b) be fulfilled, and furthermore,*
c) $\int_{-\infty}^{+\infty} |x\Phi(x)|\, dx < +\infty$.
Then the set $\mathbf{E}_+$ *is finite.*

Proof. Using the Fredholm analytic theorem, the lower boundedness of the self-adjoint operator H, and Theorem 1, we get that $\mathbf{E}_+$ is a bounded set without limit points, except may be $\lambda = 0$. Show that $\lambda = 0$ also may not be a limit point of $\mathbf{E}_+$. Let $\lambda = 0$ be a limit point of the set $\mathbf{E}_+$. Then according to theorem 1, there exists a sequence of numbers λ_n and sequence of functions $f_n(x) \in W_2^1(R_1)$ such that for any n

$$f_n(x) + K(\lambda_n) f_n(x) = 0, \tag{2.2}$$

where $\lambda_n = i\tau_n$, $\tau_n > 0$, $\lim\limits_{n\to\infty} \tau_n = 0$. In view of inequality (1.2), without loss of generality, we can choose the normalization $f_n(x)$ by the equality

$$f_n(x) = e^{-\tau_n |x|} g_n(x), \quad \|g_n\|_{C(R_1)} = 1, \tag{2.3}$$

where $\lim\limits_{n\to\infty} \|g_n(x) - g_0(x)\|_{C(R_1)} = 0$, and the function $f_0(x) \equiv g_0(x) \in C(R_1)$ is the solution of the integral equation

$$f_0(x) + K(0) f_0(x) = 0 \tag{2.4}$$

with the kernel

$$K(x, y) = \frac{1}{2} |x - y| \Phi(x) + i sgn(x - y)\, a(y).$$

Using

$$\int_{-\infty}^{+\infty} \left[-\frac{1}{2}|x|\right] e^{-ixp} dx = \frac{1}{p^2}, \quad \int_{-\infty}^{+\infty} \left[-\frac{e^{i\lambda|x|}}{2i\lambda}\right] e^{-ixp} dx = \frac{1}{p^2 - \lambda^2}$$

and passing to Fourier transformations

$$\hat{u}(p) = \int_{-\infty}^{+\infty} e^{-ixp} u(x)\, dx$$

in equations (2.2) and (2.4) we get their p-representations

$$\hat{f}_n(p) = -\frac{1}{p^2+\tau_n^2}\left[\left(\Phi \hat{f}_n\right)(p) + 2p\left(a\hat{f}_n\right)(p)\right],$$

$$\hat{f}_0(p) = -\frac{1}{p^2}\left[\left(\Phi \hat{f}_0\right)(p) + 2p\left(a\hat{f}_0\right)(p)\right]. \tag{2.5}$$

By Theorem 2.1, the numbers $\lambda_n^2 = -\tau_n^2$ are the eigenvalues of the self-adjoint operator H, therefore for $n \neq m$ the eigen functions $f_n(x)$ and $f_m(x)$ of the operator H will be orthogonal in the space $L_2(R_1)$. Then from the Parseval equality we have $(f_n, f_m) = \left(\hat{f}_n, \hat{f}_m\right) = 0$ for $n \neq m$. Hence, by (2.5) we have

$$\int_{-\infty}^{+\infty} \frac{1}{p^2+\tau_n^2}\frac{1}{p^2+\tau_m^2}\varphi_n(p)\overline{\varphi_m(p)}dp = 0, \ \ n \neq m, n, m = 1, 2, ..., \tag{2.6}$$

where $\varphi_n(p) = \left(\Phi \hat{f}_n\right)(p) + 2p\left(a\hat{f}_n\right)(p)$, $n = 0, 1, 2, ...$ From normalization (2.3) it follows that

$$|\varphi_n(p)|^2 \leq 2\left|\left(\Phi \hat{f}_n\right)(p)\right|^2 + 4p^2\left|\left(a\hat{f}_n\right)(p)\right|^2 \leq$$

$$2\left(\int_{-\infty}^{+\infty}|\Phi(x)|\,dx\right)^2 + 4p^2\left(\int_{-\infty}^{+\infty}|a(x)|\,dx\right)^2.$$

Hence, for the arbitrary positive number δ we obtain:

$$\left|\int_{|p|\geq\delta}\frac{1}{p^2+\tau_n^2}\frac{1}{p^2+\tau_m^2}\varphi_n(p)\overline{\varphi_m(p)}dp\right| \leq \int_{|p|\geq\delta}\frac{1}{p^4}|\varphi_n(p)|\left|\overline{\varphi_m(p)}\right|dp \leq$$

$$\frac{1}{2}\int_{|p|\geq\delta}\frac{1}{p^4}|\varphi_n(p)|^2\,dp + \frac{1}{2}\int_{|p|\geq\delta}\frac{1}{p^4}|\varphi_m(p)|^2\,dp \leq$$

$$2\left(\int_{-\infty}^{+\infty}|\Phi(x)|\,dx\right)^2\int_{|p|\geq\delta}\frac{1}{p^4}dp +$$

$$4\left(\int_{-\infty}^{+\infty}|a(x)|\,dx\right)^2\int_{|p|\geq\delta}\frac{1}{p^2}dp \leq c,$$

where $c > 0$ is a constant dependent only on δ. From the weak compactness of the space $L_2(R_1)$ it follows that for the arbitrary positive number δ from the sequence $\varphi_n(p)$ one can choose a subsequence (for simplicity we again denote it by $\varphi_n(p)$), that

$$\lim_{m,n\to\infty}\int_{|p|\geq\delta}\frac{1}{p^2+\tau_n^2}\frac{1}{p^2+\tau_m^2}\varphi_n(p)\overline{\varphi_m(p)}dp =$$

$$\lim_{n\to\infty}\int_{|p|\geq\delta}\frac{1}{p^2+\tau_n^2}\frac{1}{p^2}\varphi_n(p)\overline{\varphi_0(p)}dp = \int_{|p|\geq\delta}\frac{1}{p^4}|\varphi_0(p)|^2\,dp. \tag{2.7}$$

Then from (2.6) and (2.7) it follows that for the arbitrary positive number δ there should exist the finite

$$\lim_{n\to\infty}\int_{|p|\geq\delta}\frac{1}{p^2+\tau_n^2}\frac{1}{p^2}\varphi_n(p)\overline{\varphi_0(p)}dp. \tag{2.8}$$

Now we should answer the following question. Which conditions should satisfy the function $f_0(x)$ for the existence of the limit (2.8) To this end we investigate the function $\varphi_n(p)$ in the vicinity of the point $p=0$. According to the expansion

$$e^{-ipx} = 1 - ixp + O\left(p^2x^2\right)$$

in δ-vicinity of the point $p=0$, we have

$$\varphi_n(p) = A_n - ipB_n + 2pC_n + \psi_n(p), \quad n = 1,2,...,$$

where

$$A_n = \int_{-\infty}^{+\infty} \Phi(x) f_n(x)\,dx, \ B_n = \int_{-\infty}^{+\infty} x\Phi(x) f_n(x)\,dx,$$

$$C_n = \int_{-\infty}^{+\infty} a(x) f_n(x)\,dx,$$

$$\psi_n(p) = O\left(p^2\right), \quad n = 1,2,....$$

From the equalities

$$\int_{|p|\le\delta} \frac{1}{p^2+\tau_n^2}\frac{1}{p^2+\tau_m^2}dp = \frac{2}{\tau_m^2-\tau_n^2}\left[\frac{1}{\tau_n}arctg\frac{\delta}{\tau_n} - \frac{1}{\tau_m}arctg\frac{\delta}{\tau_m}\right],$$

$$\int_{|p|\le\delta} \frac{p^2}{p^2+\tau_n^2}\frac{1}{p^2+\tau_m^2}dp = \frac{2}{\tau_m}arctg\frac{\delta}{\tau_m} - \\ -\frac{2\tau_n^2}{\tau_m^2-\tau_n^2}\left[\frac{1}{\tau_n}arctg\frac{\delta}{\tau_n} - \frac{1}{\tau_m}arctg\frac{\delta}{\tau_m}\right]$$

and

$$\int_{|p|\le\delta} \frac{p}{p^2+\tau_n^2}\frac{1}{p^2+\tau_m^2}dp = 0$$

it follows that for the existence of the limit

$$\lim_{m,n\to\infty}\int_{|p|<\delta} \frac{1}{p^2+\tau_n^2}\frac{1}{p^2+\tau_m^2}\varphi_n(p)\overline{\varphi_m(p)}dp$$

the solution $f_0(x)$ of equation (2.4) should satisfy the following conditions:

$$\int_{-\infty}^{+\infty} \Phi(x) f_0(x)\,dx = 0, \ \int_{-\infty}^{+\infty} x\Phi(x) f_0(x)\,dx = 0,$$

$$\int_{-\infty}^{+\infty} a(x) f_0(x)\,dx = 0. \tag{2.9}$$

Now from (2.9) it follows that if the conditions

$$\int_{-\infty}^{+\infty} x^2\left|\Phi(x) f_0(x)\right|dx < +\infty, \int_{-\infty}^{+\infty} \left|xa(x) f_0(x)\right|dx < +\infty, \tag{2.10}$$

are fulfilled, then

$$\varphi_0(p) = O\left(p^2\right), \quad p\to 0. \tag{2.11}$$

From the conditions a)-c) it follows that if the solution $f_0(x)$ of equation (2.4) satisfies the condition

$$\sup_{-\infty<x<+\infty}\left(1+x^2\right)\left|f_0(x)\right| < +\infty, \tag{2.12}$$

then estimations (2.10) and (2.11) are valid. Now prove inequality (2.12). Using condition (2.9), rewrite equation (2.4) in the form:

$$f_0(x) = \int_{-\infty}^{+\infty}\left\{\left[\frac{1}{2}|x-y| - \frac{1}{2}x + \frac{1}{2}y\right]\Phi(y) + i\left[sgn(x-y)-1\right]a(y)\right\} f_0(y)\,dy. \quad (2.13)$$

Multiply equation (2.13) by the function $1+x^2$ and assume $l_0(x) = \left(1+x^2\right) f_0(x)$. Then $l_0(x)$ satisfies the equation

$$l_0(x) = \int_{-\infty}^{+\infty}\left\{\left[\frac{1}{2}|x-y| - \frac{1}{2}x + \frac{1}{2}y\right]\Phi(y) + i\left[sgn(x-y)-1\right]a(y)\right\}\frac{1+x^2}{1+y^2} l_0(y)\,dy.$$

After elementary transformations we get the equality

$$l_0(x) = \int_{-\infty}^{+\infty}\left\{\left[\frac{1}{2}|x-y| - \frac{1}{2}x + \frac{1}{2}y\right]\Phi(y) + i\left[sgn(x-y)-1\right]a(y)\right\}\frac{1+x^2}{1+y^2} l_0(y)\,dy+$$
$$\int_{-x}^{x}\left\{\left[\frac{1}{2}|x-y| - \frac{1}{2}x + \frac{1}{2}y\right]\Phi(y) + i\left[sgn(x-y)-1\right]a(y)\right\}\frac{1+x^2}{1+y^2} l_0(y)\,dy+$$
$$\int_{x}^{+\infty}\left\{\left[\frac{1}{2}|x-y| - \frac{1}{2}x + \frac{1}{2}y\right]\Phi(y) + i\left[sgn(x-y)-1\right]a(y)\right\}\frac{1+x^2}{1+y^2} l_0(y)\,dy =$$
$$\int_{-\infty}^{-x}\left\{\left[\frac{1}{2}|x-y| - \frac{1}{2}x + \frac{1}{2}y\right]\Phi(y) + i\left[sgn(x-y)-1\right]a(y)\right\}\frac{1+x^2}{1+y^2} l_0(y)\,dy+$$
$$\int_{x}^{+\infty}\left\{\left[\frac{1}{2}|x-y| - \frac{1}{2}x + \frac{1}{2}y\right]\Phi(y) + i\left[sgn(x-y)-1\right]a(y)\right\}\frac{1+x^2}{1+y^2} l_0(y)\,dy =$$
$$L^{(1)}l_0(x) + L^{(2)}l_0(x),$$

where

$$L^{(1)}l_0(x) = \int_{-\infty}^{-x}\left\{\left[\frac{1}{2}|x-y| - \frac{1}{2}x + \frac{1}{2}y\right]\Phi(y) + \right.$$
$$\left. +i\left[sgn(x-y)-1\right]a(y)\right\}\frac{1+x^2}{1+y^2} l_0(y)\,dy,$$
$$L^{(2)}l_0(x) = \int_{x}^{+\infty}\left\{\left[\frac{1}{2}|x-y| - \frac{1}{2}x + \frac{1}{2}y\right]\Phi(y) + \right.$$
$$\left. +i\left[sgn(x-y)-1\right]a(y)\right\}\frac{1+x^2}{1+y^2} l_0(y)\,dy.$$

If we take into account the inequality $\frac{1+x^2}{1+y^2} \leq 1$ in the representation

$$L^{(2)}l_0(x) = \int_{x}^{+\infty}\left[(y-x)\Phi(y) - 2ia(y)\right]\frac{1+x^2}{1+y^2} l_0(y)\,dy,$$

from the conditions a)-c) it follows that the operator $L^{(2)}$ is bounded in $C(R_1)$. The boundedness of the operator $L^{(1)}$ in $C(R_1)$ is proved in the same way. Further, as in [3], we can show that the linear integral

$$L_0 f(x) = \int_{x}^{+\infty} L_0(x,y) f(y)\,dy$$

with the kernel

$$L_0(x,y) = \left\{ \left[\frac{1}{2}|x-y| - \frac{1}{2}x + \frac{1}{2}y \right] \Phi(y) + i\left[sgn(x-y) - 1 \right] a(y) \right\} \frac{1+x^2}{1+y^2}$$

is completely continuous in $C(R_1)$.

Assume $L_0^{(n)} = L_0\chi_n$, where χ_n is the operator of multiplication by the characteristic function of the section $[-n,n]$ and $l_0(x) = L_0^{(n)} l_0(x)$. According to general theory of compact operators (see [13, p. 41] or [12]) there exists a sequence of numbers $\{\gamma_n\}$ converging to a unit such that the sequence $l_n^{(0)}(x) = -\gamma_n L_0^{(n)} l_n^{(0)}(x)$ converges to $l_0(x) = (1+x^2) f_0(x)$ as $n \to \infty$ in uniform topology of the space $C(R_1)$. Thus, inequality (2.12) is proved. Taking into account equalities (2.6), (2.7) and formula (2.11), we get

$$\int_{-\infty}^{+\infty} \frac{1}{p^4} |\varphi_0(p)|^2 dp = \lim_{\delta \to 0} \int_{|p| \geq \delta} \frac{1}{p^4} |\varphi_0(p)|^2 dp = 0,$$

whence $\varphi_0(p) = 0$. Hence it follows the equality $f_0(x) = 0$. This contradicts the fact that $\|f_0(x)\|_{C(R_1)} = 1$. □

From Theorems 2.1 and 2.2 it follows that the following theorem is valid.

Theorem 2.3. *If conditions a)-c) are fulfilled, then the negative part of the spectrum of the self-adjoint operator H consists of a finitely many non-negative eigenvalues of finite multiplicity.*

Remark 2.2. For $a(x) = 0$ this result agrees well with the well known result for the Sturm-Liouville operator (see [9, p. 264]).

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