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NECESSARY OPTIMALITY CONDITIONS OF QUASI-SINGULAR CONTROLS IN OPTIMAL CONTROL

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Abstract. An optimal control problem described by a system of Volterra type integro-differential equations is considered. Assuming the control domain to be convex, the necessary optimality condition in the form of a linearized maximum condition is proved. Investigating special increments in the quality criterion, different necessary optimality conditions of quasi-singular controls are established.

1. Introduction

The fundamental result of theory of necessary optimality conditions-the Pontryagin maximum principle [23] to present time was proved for different problems of optimal control of ordinary dynamical systems (see e.g. [2,3,10,12,13,16,19,22, 23,24,25,28,29]). But there are cases when the Pontryagin maximum principle or its corollaries degenerate and become inefficient. Such cases,are called singular cases and appropriate controls, singular controls [1,5,6,25]. Singular controls arise in many practical problems from rocket dynamics, space navigation, etc [see e.g. 5,22]. Furthermore, in many cases the number of controls selected by the maximum principle or by its corollaries, is rather great. Thus, these is a need to obtain necessary optimality conditions of second order that admit to narrow essentially the set of controls suspicious on optimality.

Different necessary optimality conditions of singular in this or other sense controls described by ordinary differential equations were obtained in [1,3,5,6,14,18, 20,22] and others in different ways.

In spite of the fact that in continuous optimal control problems the linearized maximum principle at some assumptions is the corollary of the maximal principle, there are cases when the admissible control without degeneration satisfies the maximum condition, but along it the linearized maximal principle degenerates. The case of degeneration of the linearized maximum condition [5] is called a quasi-singular case. It is clear that necessary optimality conditions of quasi-singular controls admit to study optimality of the controls that satisfy the maximum condition without degeneration, as well.

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It is clear that the controls singular in the sense of the Pontryagin maximal principle at appropriate conditions will be quasi-singular as well. And the contrary one, generally speaking, is not true, i.e. a quasi-singular control may also be not singular in the sense of the Pontryagins maximum principle. Therefore, necessary optimality conditions of quasi-singular controls admit to reveal also the controls that satisfy the Pontryagin maximum condition without degeneration. Some necessary optimality conditions as the Pontryagin maximum principle and linearized maximum condition in the processes described by the system of integro-differential equations of Volterra type have been obtained in [9,17,21,30,31] and others.

In the present paper, we consider an optimal control problem described by a system of integro-differential equations of Volterra type. A number of necessary optimality conditions of quasi-singular controls, i.e. of controls along of which the linearized maximal principle degenerates, were established.

2. Problem statement

Assume that the controlled process on a fixed interval of time $T = [t_0, t_1]$ is described by the following system of integro-differential equations of Volterratype.

$$\dot{x}(t) = f(t, x(t), u(t)) + \int_{t_0}^{t} K(t, \tau, x(\tau), u(\tau))d\tau \tag{2.1}$$

with initial condition

$$x(t_0) = x_0, \tag{2.2}$$

Here $f(t, x, u)$, $K(t, \tau, x, u)$ are the given n-dimensional vector-functions continuousin $T \times R^n \times R^r$ and $T \times T \times R^n \times R^r$, respectively, together with partial derivatives withrespect to $(x, u)$to second order inclusively, t_0, t_1, x_0 are given, $u = u(t)$ is an r-dimensional piecewise - continuous (with finite number of discontinuity points) vector ofcontrol actions with the values from the given non-empty, bounded and convex set $U \subset R^r$, i.e.

$$u(t) \in U \subset R^r, \; t \in T. \tag{2.3}$$

Such control functions are called admissible.

It is assumed that to each admissible control $u(t)$ there corresponds a unique, continuous and piecewise-smooth solution $x(t)$ of problem (2.1)-(2.2).

If in place of admissible controls we take a class of measurable and bounded vector-functions, then the solution of problem (2.1)-(2.2) will belong to the class of absolutely-continuous vector functions.

The existence and uniqueness of the solution of Cauchy problem (2.1)-(2.2) may be proved by the known methods (see e.g. [29,30,31,32]).

On the solutions of problem (2.1)-(2.2) generated by all possible admissible controls define the multi-point functional

$$S(u) = \varphi(x(T_1), x(T_2), ..., x(T_k)). \tag{2.4}$$

Here $\varphi(a_1, a_2, ..., a_k)$ is the given twice continuously differentiable in R^{nk} scalar-function, $T_i \in (t_0, t_1]$, $i = \overline{1,n}$ are the given points, moreover, $t_0 < T_1 < T_2 < ... < T_k \leq t_1$.

The admissible control $u(t)$ delivering minimum to functional (2.4) at restrictions (2.1)-(2.3) is said to be an optimal control, and the appropriate process $(u(t), x(t))$ an optimal process.

3. Calculation of the special increment of the quality functional

Let $(u(t),\ x(t))$ be a fixed admissible process. Because of convexity of the control domain U "the perturbed" control may be defined by the formula

$$u_\mu(t) = u(t) + \mu(v(t) - u(t)), \quad t \in T. \tag{3.1}$$

Here $\mu \in [0, 1]$ is an arbitrary number, $v(t) \in U,\ t \in T$ is an arbitrary admissible control.

Denote by $\Delta x_\mu(t)$ the special increment of the trajectory $x(t)$ responding to the special increment

$$\Delta u_\mu(t) = \mu \left[v(t) - u(t)\right]$$

of the control $u(t)$.

It is clear that $\Delta x_\mu(t)$ is the solution of the problem

$$\begin{aligned}\Delta x_\mu(t) = [f(t, x(t) + \Delta x_\mu(t), u(t) + \Delta u_\mu(t)) - f(t, x(t), u(t)] + \\ + \int_{t_0}^{t} [K(t, \tau, x(\tau) + \Delta x_\mu(\tau), u(\tau) + \Delta u_\mu(\tau)) -] \\ - K(t, \tau, x(\tau), u(\tau))]\, d\tau, \quad t \in T,\end{aligned} \tag{3.2}$$

$$\Delta x_\mu(t_0) = 0. \tag{3.3}$$

Using (3.2)-(3.3), by the scheme similar to the scheme for example from [5], we prove

Lemma 3.1. *The special increment $\Delta x_\mu(t)$ of the trajectory $x(t)$ admits the representation*

$$\Delta x_\mu(t) = \mu \ell(t) + o\left(\mu\right), \tag{3.4}$$

where $\ell(t)$ is the solution of the variations equation

$$\begin{aligned}\ell(t) = f_x(t, x(t), u(t))\ell(t) + f_u(t, x(t), u(t))(v(t))(v(t) - u(t)) + \\ + \int_{t_0}^{t} [K_x(t, \tau, x(\tau), u(\tau))\ell(\tau) + K_u(t, \tau, x(\tau), u(\tau))(v(\tau)) - u(\tau))]\, d\tau,\end{aligned} \tag{3.5}$$

$$\ell(t_0) = 0. \tag{3.6}$$

Introduce the analogue of the Hamilton-Pontryagin function

$$H(t,x(t),u(t),\psi(t)) = \psi'(t)f(t,x(t),u(t)) + \int_t^{t_1} \psi'(\tau)K(\tau,t,x(t),u(t))d\tau,$$

where $\psi = \psi(t)$ is an n-dimensional vector-function of adjoint variables being the solution of the adjoint equation (linear non-homogeneous integral equation of Volterra type)

$$\psi(t) = \int_t^{t_1} H_x(\tau,x(\tau),u(\tau),\psi(\tau))d\tau - \sum_{i=1}^{k} \alpha_i(t)\frac{\partial\varphi(x(T_1),...,x(T_k))}{\partial a_i} \tag{3.7}$$

Here $\alpha_i(t)$ is a characteristic function from the interval $[t_0, T_i]$.

Using (3.1), (3.4), the special increment of the quality creation (2.4) may be represented as follows

$$\begin{gathered}S(u+\Delta u_\mu(t)) - S(u) = -\mu \int_{t_0}^{t_1} H_u'(t,x(t),u(t),\psi(t))(v(t)-u(t))dt+ \\ +\frac{\mu^2}{2}\left[\sum_{i,j=1}^{k} \ell'(T_i)\frac{\partial^2\varphi(x(T_1),x(T_2),...,x(T_k))}{\partial a_i \partial a_j}\ell(T_j)- \right. \\ -\int_{t_0}^{t_1}\big[\ell'(t)H_{xx}(t,x(t),u(t),\psi(t)\ell(t) + 2(v(t)- \\ -u(t))'H_{ux}(t,x(t),u(t),\psi(t))\ell(t)+ \\ \left. +(v(t)-u(t)'H_{uu}(t,x(t),u(t),\psi(t))(v(t)-u(t))\big]\,dt\right] + o(\mu^2).\end{gathered} \tag{3.8}$$

4. Integral necessary optimality conditions of quasisingular controls

From the expansion (3.8) of the quality functional (2.4) it follows that along the optimal control $u(t)$

$$\int_{t_0}^{t_1} H_u'(\theta,x(\theta),u(\theta),\psi(\theta))(v(t)-u(t))dt \le 0. \tag{4.1}$$

Here and in the sequel, $\theta \in [t_0, t_1]$ is an arbitrary continuity point of the control $u(t)$.

Relation (4.1) is the analogue of the linearized integral maximum condition (see e.g. [4,9]) for the problem under consideration and is a first order necessary optimality condition.

Using for example the lemma from [26, p. 8], we prove the equivalence of optimality condition (4.1) to the following:

$$H_u'(\theta,x(\theta),u(\theta),\psi(\theta)(w-u(\theta)) \le 0, \tag{4.2}$$

for all $\theta \in [t_0, t_1)$ and $w \in U$.

We can show that necessary optimality conditions (4.1) and (4.2) are equivalent.

Relation (4.2) is the analogue of the pointwise linearized maximum condition.

Study the case of degeneration of the pointwise linearized maximum condition (4.2).

Definition 4.1. Call the admissible control $u(t)$ a quasisingular control in problem (2.1)-(2.4) if for all $\theta \in [t_0, t_1)$ and $w \in U$

$$H_u'(\theta, x(\theta), u(\theta), \psi(\theta))(w - u(\theta)) = 0. \tag{4.3}$$

The case when relation (4.3) is fulfilled, is called a quasisingular case. It is clear that in quasisingular case the linearized necessary optimality conditions lose substantial value.

Allowing for (4.3), from expansion (3.8) we get that for the optimality of the quasi-singular control$u(t)$in problem (2.1)-(2.4) it is necessary that the inequality

$$\sum_{i,j=1}^{k} \ell'(T_i) \frac{\partial^2 \varphi(x(T_1), x(T_2), ..., x(T_k))}{\partial a_i \partial a_j} \ell(T_i) -$$
$$- \int_{t_0}^{t_1} \left[\ell'(t) H_{xx}(t, x(t), u(t), \psi(t) \ell(t) + \right.$$
$$+2(v(t) - u(t))' H_{ux}(t, x(t), u(t), \psi(t)) \ell(t) + (v(t) - u(t))' \times$$
$$\times H_{uu}(t, x(t), u(t), \psi(t))(v(t) - u(t)) \big] \, dt \geq 0 \tag{4.4}$$

to, be fulfilled for all $v(t) \in U$, $t \in T$.

Inequality (4.4) is an implicit necessary optimality condition of quasi-singular controls. Therefore its practical usefulness is not great. But using it, we can get a number of necessary optimality conditions explicitly expressed directly by the parameters of problem (2.1)-(2.4).

The solution of problem (3.5)-(3.6) based on the formula on integral representation of solutions of linear non-homogeneous integro-differential equations of Volterra type (see e.g. [33]) admits the representation

$$\delta x(t) = \int_{t_0}^{t} Q(t, \tau)(v(\tau) - u(\tau)) d\tau, \tag{4.5}$$

Where, $Q(t, \tau)$ is $(n \times n)$ matrix function defined by the formula

$$Q(t, \tau) = F(t, \tau) f_u(\tau, x(\tau), u(\tau)) + \int_{\tau}^{t} F(t, s) K_u(s, \tau, x(\tau), u(\tau)) ds.$$

Here $F(t, \tau)$ is the analogue of the Cauchy matrix being the solution of the problem

$$F_\tau(t, \tau) = -F(t, \tau) f_x(\tau, x(\tau), u(\tau)) - \int_{\tau}^{t} F(t, s) K_x(s, \tau, x(\tau), u(\tau)) d\tau, \tag{4.6}$$

$F(t, t) = E(E - (n \times n)$-is unit matrix).
Let by definition

$$M(\tau, s) = - \sum_{i,j=1}^{k} \alpha_i(\tau) \alpha_j(s) Q'(T_i, \tau) \frac{\partial^2 \varphi(x(T_1), x(T_2), ..., x(T_k))}{\partial a_i \partial a_j} Q(T_j, s) +$$

$$+\int_{\max(\tau,s)}^{t_1} Q'(t,\tau)H_{xx}(t,x(t),u(t),\psi(t))Q(t,s)dt. \qquad (4.7)$$

Theorem 4.1. *(Integral necessary optimality condition, of quasi-singular controls). For the optimality of the quasi-singular control $u(t)$in problem* (2.1)-(2.4) *it is necessary that the inequality*

$$\int_{t_0}^{t_1}\int_{t_0}^{t_1}(v(\tau)-u(\tau))'M(\tau,s)(v(s)-u(s))dsd\tau+$$
$$+\int_{t_0}^{t_1}(v(t)-u(t))'H_{uu}(t,x(t),u(t),\psi(t))(v(t)-u(t)dt+$$
$$+2\int_{t_0}^{t_1}\left[\int_{t}^{t_1}(v(\tau)-u(\tau))'H_{ux}(\tau,x(\tau),u(\tau),\psi(\tau))Q(\tau,t)d\tau\right]$$
$$(v(t)-u(t))dt\le 0 \qquad (4.8)$$

to be fulfilled for all $v(t)\in U$, $t\in T$.

Proof. Using representation (4.5), we obtain

$$\sum_{i,j=1}^{k}\ell'(T_i)\frac{\partial\varphi^2(x(T_1),x(T_2),...,x(T_k))}{\partial a_i\partial a_j}\ell'(T_j)=$$
$$=\int_{t_0}^{t_1}\int_{i,j=1}^{t_1}\alpha_i(\tau)a_j(s)(v(\tau)-u(\tau))'Q'(T_i,\tau)\times$$
$$\times\frac{\partial^2\varphi(x(T_1),x(T_2),...,x(T_k))}{\partial a_i\partial a_j}Q(T_j,s)(v(s)-u(s))dsd\tau. \qquad (4.9)$$

Using the Foubini theorem, we have

$$\int_{t_0}^{t_1}(v(t)-u(t))'H_{ux}(t,x(t),u(t),\psi(t))\ell(t)dt=$$
$$=\int_{t_0}^{t_1}\left[\int_{t}^{t_1}(v(\tau)-u(\tau))'H_{ux}(\tau,x(\tau),u(\tau),\psi(\tau))Q(\tau,t)d\tau\right]$$
$$(v(t)-u(t))dt. \qquad (4.10)$$

Finally, similar to the papers [14,16] we get

$$\int_{t_0}^{t_1}\ell(t)H_{xx}(t,x(t),u(t),\psi(t)\ell(t)dt-$$
$$-\int_{t_0}^{t_1}\left(\int_{t_0}^{t}Q(t,\tau)(v(t)-u(\tau))d\tau\right)H_{xx}(t,x(t),u(t),\psi(t))\times$$
$$\times\left(\int_{t_0}^{t}Q(t,s)(v(s)-u(s))ds\right)dt=$$
$$=\int_{t_0}^{t_1}\int_{t_0}^{t_1}(v(\tau)-u(\tau))'\left\{\int_{\max(\tau,s)}^{t_1}Q'(t,\tau)H_{xx}(t,x(t),u(t),\psi(t))Q(t,s)dt\right\}$$
$$(v(s)-u(s))dsd\tau. \qquad (4.11)$$

Taking into account identities (4.9)-(4.11) and considering denotation (4.7) in inequality

(4.4), we arrive at relation (4.8).

□

The direct corollary of Theorem 4.1 is

Corollary 4.1. *For the optimality of the quasisingular control $u(t)$ in problem* (2.1)-(2.4)*it is necessary thatthe inequality*

$$(v-u(\theta))' H_{uu}(\theta, x(\theta), u(\theta))(v-u(\theta)) \leq 0 \tag{4.12}$$

to be fulfilled for all $v \in U$ and $\theta \in [t_0, t_1)$.

5. Multi-point necessary optimality conditions of second order quasisingular controls

Study the case of degeneration of optimality condition (4.12).

Definition 5.1. If for all $v \in U$ and $\theta \in [t_0, t_1)$

$$(v-u(\theta))' H_{uu}(\theta, x(\theta), u(\theta), \psi(\theta))(v-u(\theta)) = 0, \tag{5.1}$$

then$u(t)$is said to be a quasi-singular control of second order.

Using inequality (4.8) for quasisingular second order controls we can get point wise necessary optimality conditions.

Now derive necessary optimality conditions of quasi-singular controls.

Let $u(t)$ be a quasi-singular second order control.

Define the special variation of the control $u(t)$ by the formula

$$v(t;\varepsilon) = \sum_{i=1}^{m} \delta u(t, \varepsilon; \theta_i, \ell_i, v_i) \tag{5.2}$$

Here m is an arbitrary natural number, $\ell_i \geq 0$, $i = \overline{1,m}$ are arbitrary numbers, $v_i \in U$, $i = \overline{1,m}$ are arbitrary vectors, $\theta_i \in [t_0, t_1)$, $i = \overline{1,m}$ are arbitrary continuity points of the function $u(t)$ such that $t_0 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq ... \leq \theta_m < t_1$, $\varepsilon > 0$ is a rather small arbitrary number such that $\theta_m + \varepsilon < t_1$ and $\delta u(t, \varepsilon; \theta_i, \ell_i, v_i)$ is a special variation of the control $u(t)$ defined by the formula

$$\delta u(t, \varepsilon; \theta_i, \ell_i, v_i) = \begin{cases} v_i, \ t \in [\theta_i, \theta_i + \ell_i\varepsilon), \\ u(t), \ t \in T \setminus [\theta_i, \theta_i + \ell_i\varepsilon). \end{cases} \tag{5.3}$$

Summation of special variations (5.3) of the control $u(t)$ it defined as follows (see e.g.[7,11]).

lf $\theta_1 = \theta_2$, then the sum of variations $\delta u(t, \varepsilon; \theta_1, \ell_1, v_1)$ and $\delta u(t, \varepsilon; \theta_2, \ell_2, v_2)$ is understood as the variation of the control $u(t)$ of the form

$$v(t;\varepsilon) = \begin{cases} v_1, & t \in [\theta_1, \theta_1 + \ell_1\varepsilon), \\ v_2, & t \in [\theta_1 + \ell_1\varepsilon, \theta_1 + (\ell_1 + \ell_2)\varepsilon), \\ u(t), & t \in T \setminus [\theta_1, \theta_1 + (1 + \ell_2)\varepsilon). \end{cases}$$

But if $\theta_1 \neq \theta_2$ $(\theta_1 < \theta_2)$ then the sum of variations $\delta u(t, \varepsilon; \theta_1, \ell_1, v_1)$ and $\delta u(t, \varepsilon; \theta_2, \ell_2, v_2)$ is understood as the variation of the control $u(t)$ of the form

$$v(t;\varepsilon)=\begin{cases} v_1, & t\in[\theta_1,\theta_1+\ell_1\varepsilon)\,, \\ v_2, & t\in[\theta_2,\theta_1+\ell_2\varepsilon)\,, \\ u(t), & t\in T\setminus\bigcup_{i=1}^{2}[\theta_i,\theta_i+\ell_i\varepsilon)\,. \end{cases}$$

In the similar way, summation of needle-shaped variations (5.3) of the control $u(t)$ is extended on any finite number of variations of the control $u(t)$.

Taking into attention (5.2) in inequality (4.8), after some transformations we get

$$\varepsilon^2\left\{\sum_{j=1}^{m}\ell_i\ell_j(v_i-u(\theta_i))'M(\theta_i,\theta_j)(v_j-u(\theta_j))+\right.$$
$$+\sum_{i=1}^{m}\ell_i(v_i-u(\theta_i))'H_{ux}(\theta_i,x(\theta_i),u(\theta_i)\times$$
$$\left.\times\left[\ell_iQ(\theta_i,\theta_i)(v_i-u(\theta_i))+2\sum_{j=1}^{i-1}\ell_j(v_j-u(\theta_j))_jQ(\theta_i,\theta_j)\right]\right\}+0(\varepsilon^2)\le 0. \quad (5.4)$$

The following theorem follows from inequality (5.4) because of arbitrariness of

$$\varepsilon>0.$$

Theorem 5.1. *For the optimality of the quasi-singular second order control*$u(t)$ *in the problem under consideration, it is necessary that for any natural number* m *the inequality*

$$\sum_{i,j=1}^{m}\ell_i\ell_j(v_i-u(\theta))'M(\theta_i,\theta_j)(v_j-u(\theta_j))+$$
$$+\sum_{i=1}^{m}\ell_i(v_i-u(\theta_i))'H_{ux}(\theta_i,x(\theta_i),u(\theta_i)),\psi(\theta_i)\times$$
$$\times\left[\ell_iQ(\theta_i,\theta_i)(v_i-u(\theta_i))'+2\sum_{j=1}^{i-1}\ell_jQ(\theta_i,\theta_j)(v_j-u(\theta_i))\right]\le 0 \qquad (5.5)$$

to be fulfilled for all

$$v_i\in U,\ i=\overline{1,m},\ \ell_i\ge 0,\ i=\overline{1,m},\ \theta\in[t_0,t_1)\,,\ i=\overline{1,m}\,(t_0\le\theta_1\le...\le\theta_m<t_1).$$

Necessary optimality condition (5.5) belongs to the class of multi-point necessary optimality conditions for quasi-singular controls and admits to narrow essentially the set of controls suspicious on optimality [11,14,15,16,18,20, 21, 27].

The direct corollary of theorem 5.1 is

Theorem 5.2. *For the optimality of the quasisingular second order control* $u(t)$*it is necessary that the inequality*

$$(v-u(\theta))'\left[M(\theta,\theta)+H_{ux}(\theta,x(\theta),u(\theta),\psi(\theta))Q(\theta,\theta)\right](v-u(\theta))\le 0 \qquad (5.6)$$

to be fulfilled for all $v\in R^r,\ \theta\in[t_0,t_1)$.

Necessary optimality condition (5.6) is an analogue of the Gabasov-Kirillova condition from [8].

As it is seen, multi-point optimality condition (5.5) remains valid also in degeneration of the analogue of the Gabasov-Kirillova condition (5.6).

Conclusion

In the paper, we consider an optimal control problem described by a system of integro-differential equations assuming that the control domain is convex. An analogue of the linearized maximum condition is proved. The case of linearized maximum condition (quasi-singular case) is studied. Integral necessary optimality conditions are obtained.

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