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COERCIVE ESTIMATE FOR DEGENERATE ELLIPTIC PARABOLIC EQUATIONS

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Abstract. In this work a class of degenerate elliptic-parabolic equations of the second order of non-divergent structure is considered. For solutions of boundary value problems of these equations the coercive estimation in appropriate Sobolev space is established.

1. Introduction

Let Ω be a bounded domain in R^n with a boundary $\partial\Omega$, $\partial\Omega \subset C^2$, Q_T be a cylinder $\Omega \times (0,T)$, where $T \in (0,\infty)$. Let's consider in Q_T the boundary value problem

$$Lu = \sum_{i,j=1}^{n} a_{ij}\left(x,t\right) u_{ij} + \psi\left(x,t\right) u_{tt} - u_t = f\left(x,t\right), \tag{1.1}$$

$$u/_{\Gamma(Q_T)} = 0, \tag{1.2}$$

where for $i,j = \overline{1,n}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $u_i = \frac{\partial u}{\partial x_i}$, $\Gamma\left(Q_T\right) = \left(\partial\Omega \times [0,T]\right) \cup$

$\left(\Omega \times \{(x,t) : t = 0\}\right)$ is parabolic boundary of Q_T, and

$$\psi\left(x,t\right) = \omega\left(x\right) \lambda\left(t\right) \varphi\left(T-t\right), \tag{1.3}$$

where $\omega\left(x\right) \in A_p$ satisfy the condition of Muckenhoupt (see [2]), $\lambda\left(t\right) \geq 0, \lambda\left(t\right) \in C^1\left[0,T\right], \varphi\left(z\right) \geq 0, \varphi'\left(z\right) \geq 0, \varphi\left(z\right) \in C^1\left[0,T\right], \varphi\left(0\right) = \varphi'\left(0\right) = 0,$

$$\varphi\left(z\right) \geq \beta\, z\, \varphi'\left(z\right),$$

where β is a positive constant.

Assume that for coefficients of the operator L the following conditions are fulfilled.

$\left\|a_{ij}\left(x,t\right)\right\|$ is a real symmetrical matrix with measurable elements defined on Q_T and for every $(x,t) \in Q_T$ and $\xi \in R^n$ the inequalities are true

$$\gamma\omega\left(x\right)\left|\xi\right|^2 \leq \sum_{i,j=1}^{n} a_{ij}\left(x,t\right) \xi_i \xi_j \leq \gamma^{-1}\omega\left(x\right)\left|\xi\right|^2, \tag{1.4}$$

where γ is a constant from a semiinterval $[0,1]$.

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The purpose of this work is to obtain a coercive estimation for the problem (1.1)-(1.2) in appropriate Sobolev space.

The obtained estimation can be used when proving a unique strong (almost everywhere) solvability of the first boundary-value problem (1.1)-(1.2) for every $f(x,t) \in L_2(Q_T)$.

The theory of degenerated elliptic-parabolic equations ascend to classical work by Keldysh [7] in which the correct statements of boundary-value problems for the equations of a kind (1.1) with one space variable were found G.Fichera [3] has established a weak solvability of the first boundary-value problem for a wide class second order equations with the nonnegative characteristic form (see also [8]). As to a strong solvability of the first boundary-value problem for elliptic-parabolic equations in the nondivergent form with smooth coefficients, we shall note in this connection the works [4-6]. The similar result for the equations of a kind (1.1) is case coefficients satisfy to Cordes condition is obtained in [1].

The paper organizing by following. In Section 2 we present some definitions and preliminary results. In Section 3 we give main results.

2. Definitions and preliminary results

For $R > 0, x^0 \in \Omega$ we denote a ball $\{x : |x - x^0 < R|\}$ by $B_R(x^0)$ and the set $B_R(x^0) \times (0,T)$ by $Q_T^R(x^0)$. Let $\bar{B}_R(x^0) \subset \Omega$. We say that $u(x,t) \in A(Q_T^R(x^0))$ if $u(x,t) \in (\bar{Q}_T^R(x^0))$, $u/_{t=0} = 0$ and $\sup p\, u \in \bar{Q}_T^{\rho}(x^0)$ for some $\rho \in (0,R)$. We say that $u(x,t) \in A_1(Q_T^R(x^0))$ if $u(x,t) \in (\bar{Q}_T^R(x^0))$, $u/_{t=0} = 0$. Finally, $u(x,t) \in B(Q_T^R(x^0))$ if $u(x,t) \in A(\bar{Q}_T^R(x^0))$ and $u/_{t=T} = u_t/_{t=T} = 0$. Everywhere further a notation $C(\cdot)$ shows that a positive constants C depends only on the contents of brackets.

Let us introduce the Banach spaces of functions $u(x,t)$ given on Q_T with finite norms

$$\|u\|_{W^1_{2,\omega}(Q_T)} = \left(\int_{Q_T} \omega(x)\left(u^2 + \sum_{i=1}^n u_{x_i}^2\right) dxdt\right)^{\frac{1}{2}},$$

$$\|u\|_{W^2_{2,\omega}(Q_T)} = \left(\int_{Q_T} \omega(x)\left(u^2 + \sum_{i=1}^n u_{x_i}^2 + \sum_{i,j=1}^n u_{x_i x_j}^2\right) dxdt\right)^{\frac{1}{2}},$$

$$\|u\|_{W_2^{2,1}(Q_T)} = \|u\|_{W^2_{2,\omega}(Q_T)} + \|u_t\|_{L_2(Q_T)},$$

$$\|u\|_{W^{2,2}_{2,\psi}(Q_T)} =$$

$$= \left(\int_{Q_T}\left[\omega(x)\left(u^2 + \sum_{i=1}^n u_{x_i}^2 + \sum_{i,j=1}^n u_{x_i x_j}^2\right) + \right.\right.$$

$$\left.\left. + u_t^2 + \psi^2(x,t)\, u_{tt}^2 + \psi(x,t)\sum_{i=1}^n u_{it}^2\right] dxdt\right)^{\frac{1}{2}},$$

$$\|u\|_{W^{1,1}_{2,\psi}(Q)} = \left(\int_{Q_T}\left[\omega(x)\left(u^2 + \sum_{i=1}^n u_{x_i}^2\right) + u_t^2 + \psi^2(x,t)\, u_{tt}^2\right] dxdt\right)^{\frac{1}{2}},$$

respectively $\overset{\circ}{W}{}^{1,1}_{2,\psi}(Q_T)$ is a subspace of space $W^{1,1}_{2,\psi}(Q_T)$ that has a set of all functions from $C^\infty(Q_T)$, vanishing on parabolic boundary $\Gamma(Q_T)$.

Let's consider a model operator

$$Z_0 = \omega(x)\Delta + \psi(x,t)\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial t},$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x^2}$ is a Laplace operator.

Lemma 2.1. *If the function $\psi(x,t)$ is chosen as in (1.3) and the condition (1.4) are fulfilled then such $T_1(\psi(x,t),n)$ exist at $T \le T_1$ for any function $u(x.t) \in A\left(Q_T^R(x^0)\right)$ the estimate is true*

$$\int_{Q_T^R(x^0)}\left[\omega^2(x)\sum_{i,j=1}^n u_{ij}^2 + u_t^2 + \psi^2(x,t)u_{tt}^2 + \psi(x,t)\sum_{i=1}^n u_{it}^2\right]dxdt \le$$

$$(1+DS)\int_{Q_T^R(x^0)}(Z_0u)^2\,dxdt, \tag{2.1}$$

where $S = S(\psi,n)$ is some constant, $D = D(T) = q(T) + q_1(T)$, $q(T) = \sup\limits_{t\in[0,T]}\varphi'(t)$, $q_1(T) = \sup\limits_{t\in[0,T]}\varphi(t)$.

Proof. Let's denote for simplicity $B_R(x^0)$ and $Q_T^R(x^0)$ by B and Q respectively and $dxdt$ by dv. We have

$$I = \int_Q (Z_0u)^2\,dv = \int_Q (\omega(x)\Delta u + \psi(x,t)u_{tt} - u_t)^2\,dv =$$

$$= \int_Q \omega^2(x)(\Delta u)^2\,dv + \int_Q \psi^2(x,t)u_{tt}^2 dv + \int_Q u_t^2 dv +$$

$$+2\int_Q \omega(x)\psi(x,t)\Delta u u_{tt}dv - 2\int_Q \psi(x,t)u_tu_{tt}dv = i_1+i_2+i_3+i_4+i_5+i_6. \tag{2.2}$$

We will consider each summand separately. We have

$$i_1 = \int_Q \omega^2(x)(\Delta u)^2\,dx = \sum_{i,j=1}^n\int_Q \omega^2(x)u_{ii}u_{jj}dv.$$

Here, successively applying integration by parts with respect to variables x_i, x_j and taking into account that $\frac{\partial u}{\partial x_j}/_{\partial B} = 0$, we obtain

$$\sum_{i,j=1}^n\int_Q \omega^2(x)u_{ii}u_{jj}dv = -\sum_{i,j=1}^n\int_Q \omega^2(x)u_{iij}u_jdv - 2\sum_{i,j=1}^n\int_Q \omega(x)\omega_{x_j}(x)u_{iij}u_jdv =$$

$$= \sum_{i,j=1}^n\int_Q \omega(x)\omega_{x_j}(x)u_{ij}^2u_{ij}^2dv.$$

The integrals i_2, i_3 do not change. In the integral i_4, integrating by parts with respect to a variable t and x_i and taking into account that $\psi(x,t) = u_{ii}/_{t=0} = 0$, we get

$$i_4 = 2\int_Q \omega(x)\psi(x,t)\Delta u u_{tt}dv = 2\sum_{i=1}^n\int_Q \omega(x)\psi(x,t)u_{ii}u_{tt}dv =$$

$$= -2\sum_{i=1}^{n}\int_Q (\psi(x,t)\,\omega(x)\,u_{ii})\,u_t dv = -2\sum_{i=1}^{n}\int_Q \omega(x)\,\psi_t(x,t)\,u_{ii}u_t dv-$$

$$= -2\sum_{i=1}^{n}\int_Q \omega(x)\,\psi(x,t)\,u_{iit}u_t dv = -2\sum_{i=1}^{n}\int_Q \omega(x)\,\psi_t(x,t)\,u_{ii}u_t dv+$$

$$+2\sum_{i=1}^{n}\int_Q \omega(x)\,\psi_i(x,t)\,u_t u_{it} dv + 2\sum_{i=1}^{n}\int_Q \omega(x)\,\psi(x,t)\,u_{it}^2 dv. \qquad (2.3)$$

Recollecting that $\psi(x,t) = u_{ii}/_{t=0} = 0$ we rewrite (2.3) in the following form

$$i_4 = -2\sum_{i=1}^{n}\int_Q \omega^2(x)\,\lambda'(t)\,\varphi(T-t)\,u_{ii}u_t dv + 2\sum_{i=1}^{n}\int_Q \omega^2(x)\,\lambda(t)\,\varphi'(T-t)\,u_{ii}u_t dv+$$

$$+2\sum_{i=1}^{n}\int_Q \omega_{x_i}(x)\,\lambda(t)\,\varphi(T-t)\,u_t u_{it} dv + 2\sum_{i=1}^{n}\int_Q \omega^2(x)\,\lambda(t)\,\varphi(T-t)\,u_{it}^2 dv. \qquad (2.4)$$

For convenience we denote the expression

$$\omega(x)\left[\left|\lambda'(t)\right|\,\varphi(T-t) + \lambda(t)\,\varphi'(T-t)\right] = A.$$

Then first two terms in the right hand side of i_4 can be estimated as follows

$$-2\sum_{i=1}^{n}\int_Q \omega^2(x)\,\lambda'(t)\,\varphi(T-t)\,u_{ii}u_t dv + 2\sum_{i=1}^{n}\int_Q \omega^2(x)\,\lambda(t)\,\varphi'(T-t)\,u_{ii}u_t dv \ge$$

$$-2\sum_{i=1}^{n}\int_Q A u_{ii}u_t dv \ge \sum_{i=1}^{n}\int_Q A u_{ii}^2 dv - \sum_{i=1}^{n}\int_Q A u_t^2 dv \ge$$

$$-\sum_{i,j=1}^{n}\int_Q A u_{ij}^2 dv - n\int_Q A u_t^2 dv. \qquad (2.5)$$

We assume that $|\omega_{x_i}(x)| \le C_1\sqrt{\omega(x)}$, $i=\overline{1,n}$, $C_1-$is some constant. Then the third term in (2.3) can be estimated from below by

$$-2C_1\sum_{i=1}^{n}\int_Q \sqrt{\omega(x)}\lambda(t)\,\varphi(T-t)\,|u_t|\,|u_{it}|\,dv,$$

and taking into account that for any $\varepsilon > 0$ and arbitrary a and b the inequality

$$2\,|ab| \le \varepsilon a^2 + \frac{1}{\varepsilon}b^2,$$

is true, we conclude that

$$2\sum_{i=1}^{n}\int_Q \omega_{x_i}(x)\,\lambda(t)\,\varphi(T-t)\,u_t u_{it} dv$$

$$\ge -2C_1\sum_{i=1}^{n}\int_Q \sqrt{\omega(x)}\lambda(t)\,\varphi(T-t)\,|u_t|\,|u_{it}|\,dv \ge$$

$$-C_1\varepsilon\sum_{i=1}^{n}\int_Q \omega(x)\,\lambda(t)\,\varphi(T-t)\,u_{it}^2 - \frac{C_1 n}{\varepsilon}\int_Q \lambda(t)\,\varphi(T-t)\,u_t^2 dv \ge$$

$$-C_1\varepsilon\sum_{i=1}^{n}\omega(x)\lambda(t)\varphi(T-t)u_{it}^2dv-\frac{C_1nq_1(T)k}{\varepsilon}\int_Q u_t^2dv. \qquad (2.6)$$

where $k=\sup_{t\in[0,T]}\lambda(t)$ and $\varepsilon>0$ will be chosen later. Then from (2.3)-(2.6) we obtain an estimate for i_4

$$i_4\geq-\sum_{i,j=1}^{n}\int_Q Au_{ij}^2dv-n\int_Q Au_t^2dv-C_1\varepsilon\sum_{i=1}^{n}\int_Q\omega(x)\lambda(t)\varphi(T-t)u_{it}^2dv-$$

$$\frac{C_1nq_1k}{\varepsilon}\int_Q u_t^2dv+2\sum_{i=1}^{n}\omega(x)\lambda(t)\varphi(T-t)u_{it}^2dv\geq$$

$$-\sum_{i,j=1}^{n}\int_Q Au_{ij}^2dv-n\int_Q Au_t^2dv+(2-C_1\varepsilon)\sum_{i=1}^{n}\int_Q\omega(x)\lambda(t)\varphi(T-t)u_{tt}^2dv$$

$$-\frac{C_1nq_1(T)k}{\varepsilon}\int_Q u_t^2dv.$$

Then

$$i_5=-2\int_Q\omega(x)\Delta uu_tdv=-2\sum_{i=1}^{n}\omega(x)u_{ii}u_tdv.$$

Integrating by parts with respect to a variable x_i and taking into account that $u_i/_{t=0}=0$, we obtain

$$i_5=-2\int_Q\left[\omega_{x_i}(x)u_iu_{it}+\omega u_iu_{it}\right]dv=$$

$$\sum_{i=1}^{n}\int_Q\omega(x)\left(u_i^2\right)_tdv=\sum_{i=1}^{n}\int_B\omega(x)u_t^2(x,T)dx\geq0.$$

And finally

$$i_6=-2\int_Q\psi(x,t)u_{tt}u_tdv=-\int_Q\psi(x,t)\left(u_t^2\right)_tdv=$$

$$=-\int_Q\omega(x)\lambda(t)\varphi(t-T)\left(u_t^2\right)_tdv.$$

Integrating by parts with respect to t and taking into account that $\varphi(T-t)/_{t=T}=0$, we get

$$i_6\geq\int_Q\omega(x)\left(\lambda(t)\varphi(T-t)\right)_tu_t^2dv\geq\int_Q\omega(x)\lambda'(t)\varphi(T-t)u_t^2dv-$$

$$-\int_Q\omega(x)\lambda(t)\varphi'(T-t)u_t^2dv\geq-\int_Q Au_t^2dv.$$

So

$$I\geq\sum_{i,j=1}^{n}\int_Q\omega^2(x)u_{ij}^2dv+\int_Q\psi^2(x,t)u_{tt}^2dv+$$

$$+\int_Q u_t^2dv+(2-C_1\varepsilon)\sum_{i=1}^{n}\int_Q\varphi(x,t)u_{it}^2dv-$$

$$-\frac{C_1 n q_1(T)k}{\varepsilon}\int_Q u_t^2 dv-\sum_{i,j=1}^{n}\int_Q A u_{ij}^2 dv-n\int_Q A u_i^2 dv-\int_Q A u_t^2 dv\geq$$

$$\geq\left[1-S_1 D(T)\right]\sum_{i,j=1}^{n}\int_Q \omega(x)\, u_{ij}^2 dv+$$

$$+\int_Q \psi^2(x,t)\, u_{tt}^2 dv+\left[1-S_2 D(T)-\frac{C_1 n q_1(T)k}{\varepsilon}\right]\times$$

$$\times\int_Q u_t^2 dv+(2-C_1\varepsilon)\sum_{i=1}^{n}\int_Q \psi(x,t)\, u_{it}^2 dv.$$

Assume that $\varepsilon=\frac{1}{C_1}$, then

$$I\geq\left[1-S_3 D(t)\right]\sum_{i,j=1}^{n}\int_Q \omega(x)\, u_{ij}^2 dv+$$

$$+\int_Q \psi^2(x,t)\, u_{tt}^2 dv+\left[1-S_4 D(t)-C_1^2 n q_1 k\right]\int_Q u_t^2 dv+\sum_{i=1}^{n}\int_Q \psi(x,t)\, u_{it}^2 dv\geq$$

$$\geq\left[1-SD(T)\right]\sum_{i,j=1}^{n}\int_Q \omega^2(x)\, u_{ij}^2 dv+$$

$$+\int_Q \psi^2(x,t)\, u_{tt}^2 dv+\int_Q u_t^2 dv+\sum_{i=1}^{n}\int_Q \psi(x,t)\, u_{it}^2 dv,$$

where S_1, S_2, S_3, S_4, S are some constants dependent on ψ and n.

Therefore

$$I\geq\left[1-SD(t)\right]$$

$$\left(\sum_{i,j=1}^{n}\int_Q \omega(x)\, u_{ij}^2 dv+\int_Q \psi^2(x,t)\, u_{tt}^2 dv+\int_Q u_t^2 dv+\sum_{i=1}^{n}\int_Q \psi(x,t)\, u_{it}^2 dv\right). \tag{2.7}$$

Considering T_1 so small that $S\cdot D(T_1)\leq\frac{1}{2}$, and taking into account that at $T\leq T_1$

$$\frac{1}{1-S\cdot D(T)}=1+\frac{S\cdot D(T)}{1-S\cdot D(T)}\leq 1+2S\cdot D(T),$$

we conclude that

$$\int_Q\left(\sum_{i,j=1}^{n}\omega^2(x)\, u_{ij}^2+u_t^2+\psi^2 u_{tt}^2+\psi\sum_{i=1}^{n}u_{it}^2 dv\right)\leq\left[1+2S\cdot D(T)\right]\int_Q (Z_0 u)^2 dv.$$

Lemma is proved.

3. Coercive estimates for generalized solutions and main result

Lemma 3.1. *If the conditions of the previous lemma are fulfilled, then for any function* $u(x,t) \in A\left(Q_T^R\left(x^0\right)\right)$ *at* $T \le T_2(\psi, n, \delta)$ *the following estimate is true*

$$I = \int_{Q_T^R(x^0)} \left(\omega^2(x) \sum_{i.j=1}^{n} u_{ij}^2 + u_t^2 + \psi^2(x,t) u_{tt}^2 + \psi(x,t) \sum_{i=1}^{n} u_{it}^2 \right) dxdt \le$$

$$C_2 \int_{Q_T^R(x^0)} (Zu)^2 \, dxdt, \tag{3.1}$$

where $C_2 = C_2(\psi, n, \delta)$.

Proof. Again denote $Q_T^R\left(x^0\right)$ by Q for simplicity. Assuming $T \le T_1$ we get from the previous lemma

$$I^{1/2} \le C_3 \|Z_0 u\|_{L_2(Q)} \le C_3 \|Zu\|_{L_2(Q)} + C_3 \|(Z - Z_0) u\|_{L_2}. \tag{3.2}$$

Let $\delta = \sup\limits_{Q_T} \left(\sum_{i,j=1}^n (a_{ij}(x,t) - \delta_{ij})^2 \right)^{1/2}$, δ_{ij} is a Kronecker's symbol. In (3.2) we define $C_3 = \sqrt{1 + S \cdot D(T)}$. On the other hand

$$\|(Z - Z_0) u\|_{L_2(Q)} = \left(\left(\int_Q \sum_{i,j=1}^{n} (a_{ij}(x,t) - \delta_{ij}) u_{ij} \right)^2 dxdt \right)^{1/2} \le$$

$$\le \delta \left(\int_Q \sum_{i,j=1}^{n} \omega^2(x) u_{ij}^2 dxdt \right)^{1/2}.$$

Thus, from (3.2) we conclude

$$I^{1/2} \le C_3 \|Zu\|_{L_2(Q)} + \delta C_3 \left(\int_Q \sum_{i,j=1}^{n} \omega^2(x) u_{ij}^2 dxdt \right)^{1/2}. \tag{3.3}$$

As $q(T) \to 0$, $q_1(T) \to 0$, as $T \to 0$ (as so $\sup\limits_{t \in [0,T]} \varphi(T-t) = q_1(T) \Rightarrow \lim\limits_{T \to 0} q_1(T) = 0$, $\sup\limits_{t \in [0,T]} \varphi'(T-t) = q(T) \Rightarrow \lim\limits_{T \to 0} q(T) = 0$) then $D(T)$ tends to zero as $T \to 0$.

Taking into account the above-stated and the fact that at $\delta < 1$ such $T'(\psi, \delta, n)$ exists that $\delta(1 + 2S \cdot D(T))^{\frac{1}{2}} \le \frac{1+\delta}{2} < 1$ and assuming that $T_2 = \min\{T_1, T'\}$. We get the needed estimate (3.1) at $T \le T_2$ from (3.3).

The lemma is proved.

Everywhere further, without losing generality, we will consider $R \le 1$.

Lemma 3.2. *If the conditions* (1.4), (2.1) *are fulfilled for coefficients of the operator* Z, *then at* $T \le T_2$ *the following estimate is true for any function* $u(x,t) \in A\left(Q_T^R\left(x^0\right)\right)$

$$\|u\|_{W_{2,\psi}^{2,2}(Q_T^R)} \le C_4(\psi, \delta, n) \|Zu\|_{L_2(Q_T^R)}. \tag{3.4}$$

The proof follows from lemma 3.1 and Friedrich's inequality

Lemma 3.3. *If the conditions* (1.4), (2.1) *are fulfilled for coefficients of the operator* Z*, then at* $T \leq T_2$ *and at any* $\varepsilon > 0$ *the following estimate is true for any function* $u(x,t) \in A_1\left(Q_T^R\left(x^0\right)\right)$

$$\|u\|_{W_{2,\psi}^{2,2}\left(Q_T^{R/2}(x^0)\right)} \leq C_8 \|Zu\|_{L_2\left(Q_T^R(x^0)\right)} +$$

$$\varepsilon \|u\|_{W_{2,\psi}^{2,2}\left(Q_T^R(x^0)\right)} + \frac{C_9(\psi,\delta,n)}{\varepsilon R^2} \|u\|_{L_2\left(Q_T^R(x^0)\right)}. \tag{3.5}$$

Proof. Let Q have the same meaning as before, and $Q_1 = B_{R/2}\left(x^0\right) x\,(0,T)$. Let's consider a function $\eta(x) \in C_0^\infty\left(B_R\left(x^0\right)\right)$, such that $\eta(x) = 1$ at $x \in B_{R/2}\left(x^0\right)$, $0 \leq \eta(x) \leq 1$ and

$$|\eta_{x_i}| \leq \frac{C_{10}(n)}{R}, \left|\eta_{x_i x_j}\right| \leq \frac{C_{11}}{R^2},\ i,j = \overline{1,n}. \tag{3.6}$$

Applying lemma 3.2 to a function $u(x,t) \cdot \eta(x)$ we get

$$\|u\|_{W_{2,\psi}^{2,2}(Q_1)} = \|u\eta\|_{W_{2,\psi}^{2,2}(Q_1)} \leq \|u\eta\|_{W_{2,\psi}^{2,2}(Q)} \leq C_8 \|Z(u\eta)\|_{L_2(Q)}. \tag{3.7}$$

But on the other hand

$$Z(u\eta) = \eta Zu + uZ\eta + 2\sum_{i,j=1}^n a_{ij}(x,t)\, u_i u_j.$$

Therefore, using (3.6) and (3.7) we conclude

$$\|u\|_{W_{2,\psi}^{2,2}(Q_1)} \leq C_8 \|Zu\|_{L_2(Q)} + \frac{C_9(\psi,\delta,n)}{R^2} \|u\|_{L_2(Q)} + \frac{C_{10}(\delta,n)}{R} \sum_{i=1}^n \|\omega(x)\, u_i\|_{L_2(Q)}.$$

The last estimate together with (3.7) gives

$$\|u\|_{W_{2,\psi}^{2,2}(Q_1)} \leq C_8 \|Zu\|_{L_2(Q)} + \frac{C_{11}(\psi,\delta,n)}{R^2} \|u\|_{W_{2,\omega}^1(Q)}. \tag{3.8}$$

By interpolation inequality for any $\varepsilon_1 > 0$

$$\|u\|_{W_{2,\omega}^1(Q)} \leq \varepsilon_1 \|u\|_{W_{2,\omega}^2(Q)} + \frac{C_{12}(n)}{\varepsilon_1} \|u\|_{L_2(Q)}. \tag{3.9}$$

Let's fix an arbitrary $\varepsilon > 0$ and assume that $\varepsilon_1 = \frac{\varepsilon R^2}{c_{11}}$. Then from (3.8)-(3.9) taking into account

$$\|u\|_{W_{2,\omega}^2(Q)} \leq \|u\|_{W_{2,\psi}^{2,2}(Q)},$$

we obtain

$$\|u\|_{W_{2,\psi}^{2,2}(Q_1)} \leq C_8 \|Zu\|_{L_2(Q)} + \varepsilon \|u\|_{W_{2,\psi}^{2,2}(Q_1)} + \frac{C_{11}C_{12}}{\varepsilon R} \|u\|_{L_2(Q)},$$

and whence needed estimate (3.5) follows with $c_9 = c_{11} \cdot c_{12}$. Lemma is proved.

Let's denote the set $\{x : x \in \Omega, dist\,(x,\partial\Omega) > \rho\}$ for $\rho > 0$ by Ω_ρ and let $Q_T(\rho) = \Omega_\rho \times (0,T)$.

Corollary 3.1. *If coefficients of the operator* Z *satisfy the conditions* (1.4), (2.1), *then at* $T \leq T_2$ *and any* $\varepsilon > 0$ *the following estimate is true for any function* $u(x,t) \in C^\infty\left(\bar{Q}_T\right)$, $u/_{t=0} = 0$.

$$\|u\|_{W_{2,\psi}^{2,2}\left(Q_{T(\rho)}\right)} \leq C_{13}(\psi,\delta,n,\rho,\Omega) \|Zu\|_{L_2(Q_T)} +$$

$$+\varepsilon \|u\|_{W^{2,2}_{2,\psi}(Q_T)} + C_{14}(\psi,\delta,n,\rho,\Omega)\|u\|_{L_2(Q_T)}.$$

Lemma 3.4. *If coefficients of the operator Z satisfy the conditions* (1.4), (2.1), *then such that $\rho_1(n,\sigma,\Omega)$ exists, if $T \le T_2$ for any $\delta > 0$ the following estimate is true for any function $u(x,t) \in C^\infty(\bar{Q}_T)$, $u/_{\Gamma(Q_T)} = 0$*

$$\|u\|_{W^{2,2}_{2,\psi}(Q^1_T(\rho_1))} \le C_{15}(\psi,\delta,n,\rho_1,\Omega)\|Zu\|_{L_2(Q_T)}$$

$$+\varepsilon \|u\|_{W^{2,2}_{2,\psi}(Q_T)} + \frac{C_{16}(\psi,\delta,n,\rho_1,\Omega)}{\varepsilon}\|u\|_{L_2(Q_T)}, \tag{3.10}$$

where $Q^1_T(\rho_1) = Q_T \backslash Q_T(\rho_1)$.

Proof. Let's fix arbitrary point $x^0 \in \partial\Omega$ and $\varepsilon > 0$. Let's make an orthogonal transformation of coordinates $x \to y$ so that the tangent hyperplane to $\partial\tilde{\Omega}$ at the point y^0 is perpendicular to y_n axis (Here $\tilde{\Omega}$,$\partial\tilde{\Omega}$ and y^0 are images of Ω, $\partial\Omega$ and x^0 under such a transformation). By condition $\partial\Omega \subset C^2$ for simplicity we take that the equation of $\partial\tilde{\Omega}$ is given by $y_n = (y_1,...,y_{n-1})$ in its intersection with some neighbourhood O_h of the point y^0, and $\nu \in C^2$, and the part of $\tilde{\Omega}$ adjacent to $\partial\tilde{\Omega} \bigcap O_h$ is situated on the set $\{y : y_n > 0\}$.

Let's make another transformation of coordinates $y \to z$ in the following way $z_1 = y_1, ... z_{n-1} = y_{n-1}$, $z_n = y_n - \nu(y_1,...,y_{n-1})$. Denote by $\tilde{a}_{ij}(z,t)$ the images of $\tilde{a}_{ij}(y)$ under such transformation of space coordinates, $i,j = \overline{1,n}$, and by z^0 an image of the point y^0. We have

$$\bar{a}_{ij}(z,t) = \sum_{k,l=1}^{n} \tilde{a}_{kl}(y,t)\frac{\partial z_i}{\partial y_k}\frac{\partial z_j}{\partial y_l}, i,j = \overline{1,n}, k,l = \overline{1,n}.$$

It is clear that

$$\frac{\partial y_i}{\partial z_k} = \begin{cases} \delta_{ik}, \ for\ i < n, \\ -\nu y_k\,, \ for\ i = n,\ k < n, \\ 1\,, \ for\ i = k = n. \end{cases}$$

Let $1 \le i,j \le n-1$. Then $\bar{a}_{ij}(z,t) = \sum_{k,l=1}^{n} \tilde{a}_{kl}(y,t)\delta_{ik}\delta_{jl} = \tilde{a}_{ij}(y,t)$.

As $\nu_{y_i}(y_0) = 0$, $i = \overline{1,n-1}$, for any $\varepsilon_2 > 0$ there exists such $h = h(y^0)$, that $|\nu_{y_i}(z)| < \varepsilon_2$ for $z \in \bar{\Omega} \bigcap O_h(z^0)$.

Thus $\bar{a}_{ij}(z,t) = \tilde{a}_{ij}(y,t)$ at $1 \le i,j \le n-1$.

$\bar{a}_{ij}(z,t) = \tilde{a}_{nj}(y,t) - \sum_{k=1}^{n-1} \tilde{a}_{kj}(y,t)\,\nu_{y_k}$ at $i = n$, $1 \le j \le n-1$.

$\bar{a}_{ij}(z,t) = \tilde{a}_{nn}(y,t) + \sum_{k,l=1}^{n-1} \tilde{a}_{kl}(y,t)\ \nu_{y_k}\nu_{y_l} - 2\sum_{k=1}^{n-1} \tilde{a}_{kn}(y,t)\ \nu_{y_k}$ at $i = j = n$.

Now let $\bar{u}(x,t)$ be an image of a function $u(x,t)$ after the transformations of space coordinates $x \to y$ and $y \to x$. It's clear that $C^+_h(y^0) = (\bar{\Omega} \bigcap B_h(y^0)) \times (0,T)$ represents a cylinder with the base $B_h(y^0) = \{z : |z - z^0| < h(y^0),\ z_h \ge 0\}$. For any $t \in (0,T)$ we continue the function $\bar{u}(z,t)$ in an odd way and the coefficients of the operator $\bar{Z}$ in an even way throught the hyperplaine $Z_n = 0$ into a semiball $B^-_h(y^0) = \{z : |z - z^0| < h(y^0), z_n < 0\}$ and denote the function and operator continued again by $\bar{u}(z,t)$ and $\bar{Z}$, respectively. Then if $C_h(y^0) = (B^+_h(y^0) \bigcup B^-_h(y^0)) \times (0,T)$, then $\bar{u}(z,t) \in W^{2,2}_{2,\psi}(C_h(y^0))$. By lemma 3.3 for any $\varepsilon_3 > 0$

$$\|\bar{u}\|_{W^{2,2}_{2,\psi}\left(C_{\frac{h}{2}}(y^0)\right)} \le C_{17}(\psi,\delta,n,y^0,\nu)\|\bar{Z}u\|_{L_2(C_h(y^0))} + \varepsilon_3\|\bar{u}\|_{W^{2,2}_{2,\psi}(C_h(y^0))}$$

$$+\frac{C_{18}\left(\psi,\delta,n,y^0,\nu\right)}{\varepsilon_3}\left\|\bar{u}\right\|_{L_2(C_h(y^0))}, \qquad (3.11)$$

where $C_{\frac{h}{2}}\left(y^0\right)=B_{\frac{h}{2}}\left(z^0\right)\times(0,T)$.

But each of every norm in the last inequality due to our continuation in doubled square of a norm taken with respect to the cylinder $C^{+}_{\frac{h}{2}}\left(y^0\right)$ (in the left hand side) and cylinder $C^{+}_{h}\left(y^0\right)$ (in the right hand side) respectively. Hence

$$\left\|\bar{u}\right\|_{W^{2,2}_{2,\psi}\left(C^{+}_{\frac{h}{2}}(y^0)\right)}\leq C_{19}\left\|\bar{Z}u\right\|_{L_2\left(C^{+}_{h}(y^0)\right)}+\varepsilon_3\left\|\bar{u}\right\|_{W^{2,2}_{2,\psi}\left(C^{+}_{h}(y^0)\right)}+\frac{c_{20}}{\varepsilon_3}\left\|\bar{u}\right\|_{L_2\left(C^{+}_{h}(y^0)\right)}.$$

Let $\Gamma_{\frac{h}{2}}\left(x^0\right)$ be a preimage of $C^{+}_{\frac{h}{2}}\left(y^0\right)$ in x variables. Then

$$\left\|\bar{u}\right\|_{W^{2,2}_{2,\psi}\left(\Gamma_{\frac{h}{2}}(x^0)\right)}\leq C_{21}\left\|Zu\right\|_{L_2(Q)}+C_{22}\varepsilon_3\left\|u\right\|_{W^{2,2}_{2,\psi}(Q)}+\frac{c_{23}}{\varepsilon_3}\left\|u\right\|_{L_2(Q)}, \qquad (3.12)$$

where constants depend only δ,n,Ω. Let's cover $\partial\tilde{\Omega}$ with a system of balls $\left\{B_{\frac{h_\nu}{2}}\left(z^\nu\right)\right\}$ and choose a finite subcover $\{B_1,...,B_N\}$ from it . It's obvious that N depends only on $\partial\Omega,n$. Let $\Gamma_1,...,\Gamma_N$ be preimage of cylinders $[B_1\times(0,T)]\,,...,$ $[B_N\times(0,T)]$ in (x,t) variables. Taking a sequence of both sides of (3.12) and adding up the obtained inequalities with respect to i from 1 to N, we get

$$\left\|u\right\|^2_{W^{2,2}_{2,\psi}(Q\backslash Q(\rho_1))}\leq 3N\left(C^2_{24}\left\|Zu\right\|^2_{L_2(Q)}+C^2_{25}\varepsilon^2_3\left\|u\right\|^2_{W^{2,2}_{2,\psi}(Q)}+\frac{C^2_{26}}{\varepsilon^2_3}\left\|u\right\|^2_{L_2(Q)}\right). \qquad (3.13)$$

where ρ_1 is such that $\bigcup_{i=1}^{N}\Gamma_i\supset Q\backslash Q\left(\rho_1\right)$. It is clear that $\rho_1=\rho_1\left(n,\delta,\partial\Omega\right)$. Now it's sufficient to put $C_{15}=3NC_{24}$, $C_{16}=3N^2C_{25}C_{26}$ and $\varepsilon_3=\frac{\varepsilon}{3NC_{25}}$ and the needed estimate (3.10) follows from (3.13).

Lemma is proved.

Lemma 3.5. *Under the conditions of Lemma 3.4 the following estimate is true for any* $u\left(x,t\right)\in W^{2,2}_{2,\psi}\left(Q_T\right)$ *at* $T\leq T_2$.

$$\left\|u\right\|_{W^{2,2}_{2,\psi}(Q_T)}\leq C_{27}\left(\psi,\delta,n,\rho,\Omega\right)\left\|Zu\right\|_{L_2(Q_T)}+C_{28}\left(\psi,\delta,n,\rho,\Omega\right)\left\|u\right\|_{L_2(Q_T)}. \qquad (3.14)$$

Proof. By lemma 3.3 and 3.4 for any $\varepsilon>0$

$$\left\|u\right\|^2_{W^{2,2}_{2,\psi}(Q_T(\rho_1))}\leq 3C^2_9\left\|Zu\right\|^2_{L_2(Q)}+3\varepsilon^2\left\|u\right\|^2_{W^{2,2}_{2,\psi}(Q)}+\frac{3C^2_{10}}{\varepsilon^2}\left\|u\right\|^2_{L_2(Q)},$$

and

$$\left\|u\right\|^2_{W^{2,2}_{2,\psi}(Q\backslash Q(\rho_1))}\leq 3C^2_{11}\left\|Zu\right\|^2_{L_2(Q)}+3\varepsilon^2\left\|u\right\|^2_{W^{2,2}_{2,\psi}(Q)}+\frac{3C^2_{12}}{\varepsilon^2}\left\|u\right\|^2_{L_2(Q)}.$$

Adding up these inequalities and denoting $C^2_9+C^2_{11}=C_{27}$, $C^2_{10}+C^2_{12}=C_{28}$, we obtain

$$\left\|u\right\|^2_{W^{2,2}_{2,\psi}(Q)}\leq 3C_{27}\left\|Zu\right\|^2_{L_2(Q)}+6\varepsilon^2\left\|u\right\|^2_{W^{2,2}_{2,\psi}(Q)}+\frac{3C_{28}}{\varepsilon^2}\left\|u\right\|^2_{L_2(Q)}.$$

Let's choose and fix $\varepsilon = \frac{1}{2\sqrt{3}}$. We get

$$\|u\|^2_{W^{2,2}_{2,\psi}(Q)} \le 6C_{27} \|Zu\|^2_{L^2_2(Q)} + 72C_{28} \|u\|^2_{L^2_2(Q)}.$$

Whence the needed estimate (3.14) follows with $C_{27} = \sqrt{6C_{27}}$, $C_{28} = \sqrt{6C_{28}}$. Lemma is proved.

Theorem 3.1. *If the conditions* (1.4), (2.1) *are fulfilled. Then such* $T_0 = T_0(\psi, \delta, n, \Omega)$ *exits that at* $T \le T_2$ *the following estimate is true for any function* $u(x,t) \in W^{2,2}_{2,\psi}(Q)$

$$\|u\|^2_{W^{2,2}_{2,\psi}(Q)} \le C_{29}(\psi, \delta, n, \Omega) \|Zu\|_{L_2(Q)}. \tag{3.15}$$

Proof. It's sufficient to prove estimate (3.15) for smooth functions $u(x,t)$ from $W^{2,2}_{2,\psi}(Q)$. We have for any $t \in (0,T)$ and $x \in \Omega$

$$u(x,t) = \int_0^t u_t(x,\tau)\, d\tau.$$

Using Cauchy-Bunyakowsky inequality we can write $u^2(x,t) = T \int_0^t u_t^2(x,\tau)\, d\tau$. Then

$$\int_Q u^2(x,t)\, dxdt \le T^2 \int_Q u_t^2(x,t)\, dxdt.$$

So

$$\|u\|^2_{L_2(Q)} \le T \|u_t\|_{L_2(Q)} \le T \|u\|_{W^{2,2}_{2,\psi}(Q)}.$$

Let $T_0 = \min\left\{T_2, \frac{1}{2C_{28}}\right\}$. Then from (3.14) at $T \le T_1$ the needed estimate (3.15) follows with $C_{29} = 2C_{27}$. Theorem is proved.

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