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ON SOLVABILITY OF AN EXTERNAL PROBLEM WITH IMPEDANCE BOUNDARY CONDITION FOR HELMHOLTZ EQUATION BY INTEGRAL EQUATIONS METHOD

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Abstract. In the paper we study the solvability of an external problem with an impedance boundary condition for the Helmholtz equation by the method of weakly singular integral equations.

1. Introduction

It is known that in theory of acoustic waves, external boundary value problems for the Helmholtz equation is of great importance. Existence of the solutions of Dirichlet and Neumann external boundary value problems for the Helmholtz equation by the method of weakly singular integral equations was considered in the papers [1,3-7]. In the monograph [2] the solvability of an external problem with an impedance boundary condition for the Helmholtz equation by the method of singular integral equations is given. It should be noted that the solution of a singular integral equation is much complicated than the solution of a weakly singular integral equation (since a singular integral operator is not compact, while a weakly singular integral operator is). Therefore, it is suitable to reduce an external problem with an impedance boundary condition for the Helmholtz equation to a weakly singular integral equation, and this paper is devoted to this matter.

Let $D \subset \mathbf{R}^3$ be a bounded domain with boundary $S \in \Lambda_\alpha$, where Λ_α is a class of Lyapunov surfaces with an index $0 < \alpha \leq 1$. Recall that an external problem with an impedance boundary condition for the Helmholtz equation is to find the function u twice continuously-differentiable on $\mathbf{R}^3 \backslash \overline{D}$ and continuous on S , prossessing a normal derivative in the sense of uniform convergence and satisfying the Helmholtz equation $\Delta u + k^2 u = 0$ in $\mathbf{R}^3 \backslash \overline{D}$, the Sommerfeld radiation condition

$$\left(\frac{x}{|x|}, grad\ \ u\left(x\right) \right) - iku\left(x\right) = o\left(\frac{1}{|x|} \right), \quad |x| \to \infty,$$

uniformly along all directions $x/\left|x\right|$, and the boundary condition

$$\frac{\partial u\left(x\right)}{\partial \vec{n}\left(x\right)} + f\left(x\right) u\left(x\right) = g\left(x\right) \quad \text{on} \quad S, \tag{1.1}$$

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where k is a wave number, moreover Imk ≥ 0, $\vec{n}(x)$ is a unit external normal at the point $x \in S$, while f and g are the given continuous functions on S, and

$$\operatorname{Im}\left(\overline{k}f\right) \geq 0 \quad \text{on} \quad S. \tag{1.2}$$

It should be pointed that, in particular, for $f \equiv 0$ we get Neumann's external boundary value problem for the Helmholtz equation, while for $f \equiv const \neq 0$ a mixed boundary value problem for the Helmholtz equation.

2. Main result

Let $\nu(x, \varphi)$ be a simple layer acoustic potential, while $w(x, \varphi)$ a double layer acoustic potential, i.e.

$$\nu(x, \varphi) = \int_S \Phi_k(x, y)\, \varphi(y)\, dS, \quad w(x, \varphi) = \int_S \frac{\partial \Phi_k(x, y)}{\partial \vec{n}(y)} \varphi(y)\, dS_y,$$

where $\Phi_k(x, y) = e^{ik|x-y|} / (4\pi |x-y|)$, $x, y \in \mathbf{R}^3$, $x \neq y$.

We will look for the solution of an external problem with an impedance boundary condition for the Helmholtz equation in the form

$$u(x) = \nu(x, \varphi) + i\eta w(x, \nu_0), x \in \mathbf{R}^3 \backslash \overline{D},$$

where η is a real number, and if Im$k > 0$, then $\eta = 0$, if Im$k = 0$, then $\eta \neq 0$, and $\nu_0(x, \varphi)$ is a simple layer potential for the Laplace equation, i.e.

$$\nu_0(x, \varphi) = \nu(x, \varphi)|_{k=0} = \int_S \Phi_0(x, y)\, \varphi(y)\, dS_y.$$

It is known that (see [2]) the function $u(x)$ satisfies the Helmholtz equation and the Sommerfeld radiation condition at infinity, the functions $\Phi_0(x, y)$ and $\nu_0(x, y)$ satisfy the Laplace equation. Applying Green's second formula, we get

$$\int_S \nu_0(y, \varphi) \frac{\partial \Phi_0(x, y)}{\partial \vec{n}(y)} dS_y = \int_S \Phi_0(x, y) \frac{\partial \nu_0(y, \varphi)}{\partial \vec{n}(y)} dS_y, \quad x \in \mathbf{R}^3 \backslash \overline{D},$$

where the normal derivative on S is understood in the sense

$$\frac{\partial \nu_0(y, \varphi)}{\partial \vec{n}(y)} = \lim_{\substack{h \to 0 \\ h > 0}} \frac{\partial \nu_0(y - h\vec{n}(y), \varphi)}{\partial \vec{n}(y)}, \quad y \in S.$$

Then, taking into account the limit value of the normal derivative of a simple layer potential, we have:

$$w(x, \nu_0) = \int_S \frac{(\Phi_k(x, y) - \Phi_0(x, y))}{\partial \vec{n}(y)} \nu_0(y, \varphi)\, dS_y +$$

$$+ \int_S \Phi_0(x, y) \left(\int_S \frac{\partial \Phi_0(y, t)}{\partial \vec{n}(y)} \varphi(t)\, dS_t \right) dS_y +$$

$$+ \frac{1}{2} \int_S \Phi_0(x, y)\, \varphi(y)\, dS_y, \; x \in \mathbf{R}^3 \backslash \overline{D},$$

and it means

$$u(x)=\nu(x,\varphi)+i\eta\left[\int_S \frac{\partial\left(\Phi_k(x,y)-\Phi_0(x,y)\right)}{\partial\vec{n}(y)}\left(\int_S \Phi_0(y,t)\,\varphi(t)\,dS_t\right)dS_y+\right.$$

$$+\int_S \Phi_0(x,y)\left(\int_S \frac{\partial\Phi_0(y,t)}{\partial\vec{n}(y)}\varphi(t)\,dS_t\right)dS_y+$$

$$\left.+\frac{1}{2}\int_S \Phi_0(x,y)\,\varphi(y)\,dS_y\right],\, x\in\mathbf{R}^3\backslash\overline{D}. \qquad (2.1)$$

It is easy to calculate

$$\frac{\partial\left(\Phi_k(x,y)-\Phi_0(x,y)\right)}{\partial\vec{n}(y)}=\frac{\left(\overrightarrow{yx},\vec{n}(y)\right)\left((1-ik|x-y|)\,e^{ik|x-y|}-1\right)}{4\pi|x-y|^3},$$

$$\frac{\partial}{\partial\vec{n}(y)}\left(\frac{\partial\left(\Phi_k(x,y)-\Phi_0(x,y)\right)}{\partial\vec{n}(y)}\right)=\frac{K(x,y)}{4\pi|x-y|^5},$$

where

$$K(x,y)=\left(\overrightarrow{yx},\vec{n}(x)\right)\times$$
$$\times\left(\overrightarrow{xy},\vec{n}(y)\right)\left[\left(3-3ik|x-y|-k^2|x-y|^2\right)e^{ik|x-y|}-3\right]+$$
$$+\left(\vec{n}(y),\vec{n}(x)\right)\left[(1-ik|x-y|)\,e^{ik|x-y|}-1\right]|x-y|^2.$$

Since

$$\left|(1-ik|x-y|)\,e^{ik|x-y|}-1\right|\le M|x-y|^2,\quad \forall x,y\in S,$$

then

$$\left|\frac{\partial\left(\Phi_k(x,y)-\Phi_0(x,y)\right)}{\partial\vec{n}(y)}\right|\le M|x-y|^{\alpha},$$

$$\left|\frac{\partial}{\partial\vec{n}(x)}\left(\frac{\partial\left(\Phi_k(x,y)-\Phi_0(x,y)\right)}{\partial\vec{n}(y)}\right)\right|\le\frac{M}{|x-y|}.$$

Here and in what follows, M denotes positive constants different at different inequalities.

Therefore,

$$\frac{\partial}{\partial\vec{n}(x)}\left(\int_S \frac{\partial\left(\Phi_k(x,y)-\Phi_0(x,y)\right)}{\partial\vec{n}(y)}\left(\int_S \Phi_0(y,t)\,\varphi(t)\,dS_t\right)dS_y\right)=$$

$$=\int_S \frac{\partial}{\partial\vec{n}(x)}\left(\frac{\partial\left(\Phi_k(x,y)-\Phi_0(x,y)\right)}{\partial\vec{n}(y)}\right)\left(\int_S \Phi_0(y,t)\,\varphi(t)\,dS_t\right)dS_y,\quad x\in S.$$

Furthermore, taking into account in equality (2.1) the limit value of the normal derivative of a simple layer potential, we get

$$\frac{\partial u^+(x)}{\partial\vec{n}(x)}=-\frac{2+i\eta}{4}\varphi(x)+\int_S \frac{\partial\Phi_k(x,y)}{\partial\vec{n}(x)}\varphi(y)\,dS_y+$$

$$+i\eta\left[\int\limits_S \frac{\partial}{\partial\vec{n}(x)}\left(\frac{\partial\left(\Phi_k(x,y)-\Phi_0(x,y)\right)}{\partial\vec{n}(y)}\right)\left(\int\limits_S \Phi_0(y,t)\,\varphi(t)\,dS_t\right)dS_y+\right.$$

$$\left.+\int\limits_S \frac{\partial\Phi_0(x,y)}{\partial\vec{n}(x)}\left(\int\limits_S \frac{\partial\Phi_0(y,t)}{\partial\vec{n}(y)}\varphi(t)\,dS_t\right)dS_y\right]. \qquad (2.2)$$

Finally, considering the limit value of double layer potential, we find

$$u^+(x)=\lim_{\substack{t\to x\\ t\in\mathbf{R}^3\setminus\overline{D}}} u(t)=\int\limits_S \Phi_k(x,y)\,\varphi(y)\,dS_y+$$

$$+i\eta\left[\int\limits_S \frac{\partial\Phi_0(x,y)}{\partial\vec{n}(y)}\left(\int\limits_S \Phi_0(y,t)\,\varphi(t)\,dS_t\right)dS_y+\right.$$

$$\left.+\frac{1}{2}\int\limits_S \Phi_0(x,y)\,\varphi(y)\,dS_y\right],\quad x\in S. \qquad (2.3)$$

As a result, taking into account (2.2) and (2.3) in boundary condition (1.1), we get a boundary integral equation (BIE)

$$-\frac{2+i\eta}{4}\varphi(x)+\int\limits_S \frac{\partial\Phi_k(x,y)}{\partial\vec{n}(x)}\varphi(y)\,dS_y+$$

$$+i\eta\int\limits_S \frac{\partial}{\partial\vec{n}(x)}\left(\frac{\partial\left(\Phi_k(x,y)-\Phi_0(x,y)\right)}{\partial\vec{n}(y)}\right)\left(\int\limits_S \Phi_0(y,t)\,\varphi(t)\,dS_t\right)dS_y+$$

$$+i\eta\int\limits_S \frac{\partial\Phi_0(x,y)}{\partial\vec{n}(x)}\left(\int\limits_S \frac{\partial\Phi_0(y,t)}{\partial\vec{n}(y)}\varphi(t)\,dS_t\right)dS_y+f(x)\left[\int\limits_S \Phi_k(x,y)\,\varphi(y)\,dS_y+\right.$$

$$\left.+i\eta\int\limits_S \frac{\partial\Phi_k(x,y)}{\partial\vec{n}(y)}\left(\int\limits_S \Phi_0(y,t)\,\varphi(t)\,dS_t\right)dS_y+\frac{i\eta}{2}\int\limits_S \Phi_0(x,y)\,\varphi(y)\,dS_y\right]=$$

$$=g(x),\quad x\in S,$$

that may be rewritten in the operator form

$$\varphi+A\varphi=\psi, \qquad (2.4)$$

where

$$\psi=-4\,(2+i\eta)^{-1}\,g,$$

$$A=-2\,(2+i\eta)^{-1}\left(2K+2i\eta\,(T+G)+f\,(2L+2i\eta F+i\eta L_0)\right),$$

$$(L\varphi)(x)=\int\limits_S \Phi_0(x,y)\,\varphi(y)\,dS_y,\ (L_0\varphi)(x)=\int\limits_S \Phi_0(x,y)\,\varphi(y)\,dS_y,$$

$$(K\varphi)(x)=\int\limits_S \frac{\partial\Phi_k(x,y)}{\partial\vec{n}(y)}\varphi(y)\,dS_y,$$

$$(F\varphi)(x)=\int\limits_S \frac{\partial\Phi_k(x,y)}{\partial\vec{n}(y)}\left(\int\limits_S \Phi_0(y,t)\,\varphi(t)\,dS_t\right)dS_y,$$

$$(G\varphi)(x)=\int\limits_S \frac{\partial\Phi_0(x,y)}{\partial\vec{n}(x)}\left(\int\limits_S \frac{\partial\Phi_0(y,t)}{\partial\vec{n}(y)}\varphi(t)\,dS_t\right)dS_y,$$

$$(T\varphi)(x)=\int\limits_S \frac{\partial}{\partial\vec{n}(x)}\left(\frac{\partial\left(\Phi_k(x,y)-\Phi_0(x,y)\right)}{\partial\vec{n}(y)}\right)\times$$

$$\times\left(\int\limits_S \Phi_0(y,t)\,\varphi(t)\,dS_t\right)dS_y,\quad x\in S.$$

Theorem 2.1. *Integral equation (2.4) for an external problem with an impedance boundary condition has a unique solution for all wave numbers with* $\mathrm{Im}k\geq 0$ *and for any values of impedances* f *satisfying condition (1.2).*

Proof. At first consider the case $\mathrm{Im}k>0$. Then $\eta=0$, it means that equation (2.4) takes the form

$$\varphi-K\varphi-fL\varphi=-2g. \tag{2.5}$$

In this case, the wave member k is not an eigen value of the Dirichlet internal problem, and therefore equation (2.5) has a unique solution (see [2]).

Now consider the case $\mathrm{Im}k=0$ (therewith $\eta\neq 0$ and the number k^2 is real). Since the operator A is compact, then by virtue of Riesz-Fredholm theory it suffices to show that the homogeneous equation

$$\varphi+A\varphi=0 \tag{2.6}$$

has only a trivial solution $\varphi=0$. Let $\varphi\in C(S)$ be the solution of equation (2.6). Since the function u is the solution of the homogeneous external problem with an impedance boundary condition, then $u=0$ in $\mathbf{R}^3\backslash D$ (see [2]). By the theorem for a step of simple and double layer potentials and by the theorem for a jump of the normal derivative of simple and double layer potentials (see [2]), we get

$$u^+(x)-u^-(x)=i\eta\nu_0(x,\varphi),\ \frac{\partial u^+(x)}{\partial\vec{n}(x)}-\frac{\partial u^-(x)}{\partial\vec{n}(x)}=-\varphi(x),\ x\in S,$$

and this means

$$u^-(x)=-i\eta\nu_0(x,\varphi),\ \frac{\partial u^-(x)}{\partial\vec{n}(x)}=\varphi(x),\ x\in S, \tag{2.7}$$

where

$$u^-(x)=\lim_{\substack{t\to x\\ t\in D}}u(t),\ \frac{\partial u^-(x)}{\partial\vec{n}(x)}=\lim_{\substack{h\to 0\\ h>0}}\frac{\partial u(x-h\vec{n}(x))}{\partial\vec{n}(x)}.$$

Hence

$$u^-(x)=-i\eta\int\limits_S \Phi_0(x,y)\,\frac{\partial u^-(y)}{\partial\vec{n}(y)}dS_y,x\in S.$$

Applying the Green first formula and considering (2.7), we find:

$$i\eta\int\limits_{S}\varphi\left(x\right)\overline{\nu}_0\left(x,\varphi\right)dS_x=\int\limits_{S}\overline{u}^-\left(x\right)\frac{\partial u^-\left(x\right)}{\partial\vec{n}\left(x\right)}dS_x=$$

$$=\int\limits_{D}\left(\left|gradu\left(x\right)\right|^2-k^2\left|u\left(x\right)\right|^2\right)dx, \tag{2.8}$$

$$-i\eta\int\limits_{S}\overline{\varphi}\left(x\right)\nu_0\left(x,\varphi\right)dS_x=\int\limits_{S}u^-\left(x\right)\frac{\partial\overline{u}^-\left(x\right)}{\partial\vec{n}\left(x\right)}dS_x=$$

$$=\int\limits_{D}\left(\left|gradu\left(x\right)\right|^2-k^2\left|u\left(x\right)\right|^2\right)dx.$$

Hence

$$2i\eta\int\limits_{S}\left(\varphi\left(x\right)\left(\int\limits_{S}\Phi_0\left(x,y\right)\overline{\varphi}\left(y\right)dS_y\right)-\right.$$

$$\left.-\overline{\varphi}\left(x\right)\left(\int\limits_{S}\Phi_0\left(x,y\right)\varphi\left(y\right)dS_y\right)\right)dS_x=0,$$

this means

$$\int\limits_{S}\int\limits_{S}\Phi_0\left(x,y\right)\varphi\left(x\right)\overline{\varphi}\left(y\right)dS_ydS_x=\int\limits_{S}\int\limits_{S}\Phi_0\left(x,y\right)\overline{\varphi}\left(x\right)\varphi\left(y\right)dS_ydS_x=$$

$$=\overline{\left(\int\limits_{S}\int\limits_{S}\Phi_0\left(x,y\right)\varphi\left(x\right)\overline{\varphi}\left(y\right)dS_ydS_x\right)}.$$

The latter means that the expression

$$\int\limits_{S}\int\limits_{S}\Phi_0\left(x,y\right)\varphi\left(x\right)\overline{\varphi}\left(y\right)dS_ydS_x$$

is valid. Having taken the imaginary part of equation (2.8), we get

$$\int\limits_{S}\int\limits_{S}\Phi_0\left(x,y\right)\varphi\left(x\right)\overline{\varphi}\left(y\right)dS_ydS_x=0.$$

Hence we have $\varphi=0$, and this completes the proof of the theorem. □

References

[1] H. Brakhage, P. Werner, Uber das Dirichletsche Aussenraumproblem fiir die Helmholtzsche Schwingungsgleichung. *Arch. Math.*, **16** (1965), 325-329. (German)

[2] D. Colton, R. Kress, *Integral Equation Methods in Scattering Theory.* Wiley, New York, (1983); Mir, Moscow, (1987).

[3] V.D. Kupradze, *Boundary Problems of the Theory of Vibrations and Integral Equations.* Gosudarstv. Izdat. Tehn.-Teor. Ltt., Moscow-Leningrad, (1950). (Russian)

[4] R. Leis, Zur Dirichletschen Randwertaufgabe des Aussenraums der Schwingungsgleichung. *Math. Zeit.*, **90** (1965), 205-211. (German)

[5] S. Muller, Zur Methode der Strahlungskapazittit von. *H. Weyl. Math. Zeit.*, **56** (1952), 80-83. (German)

[6] I.N. Vekua, On Metaharmonic Functions. *Trudy Tbilis. Mat. Inst.*, **12** (1943), 105-174. (Russian)

[7] H. Weyl, Kapazitat von Strahlungsfeldern. *Math. Zeit.*, **55** (1952), 187-198 (German).

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