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A VARIATION OF THE L^p UNCERTAINTY PRINCIPLES FOR THE FOURIER TRANSFORM

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Abstract. We obtain several analogs of Heisenberg-Pauli-Weyl-type inequality, Donoho-Stark-type inequality and Matolcsi-Szücs-type inequality for L^p-functions.

1. Introduction

In this paper, we consider $\mathbb{R}^d$ with the Euclidean inner product $\langle .,. \rangle$ and norm $|y| := \sqrt{\langle y, y\rangle}$. We denote by μ the measure on $\mathbb{R}^d$ given by $\mathrm{d}\mu(y) := (2\pi)^{-d/2}\mathrm{d}y$; and by $L^p(\mu)$, $1 \le p \le \infty$, the space of measurable functions f on $\mathbb{R}^d$, such that

$$\|f\|_{L^p(\mu)} := \Big(\int_{\mathbb{R}^d} |f(y)|^p \mathrm{d}\mu(y)\Big)^{1/p} < \infty, \quad 1 \le p < \infty,$$
$$\|f\|_{L^\infty(\mu)} := \operatorname*{ess\,sup}_{y\in\mathbb{R}^d} |f(y)| < \infty.$$

For $f \in L^1(\mu)$ the Fourier transform is defined by

$$\mathcal{F}(f)(x) := \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} f(y)\mathrm{d}\mu(y), \quad x \in \mathbb{R}^d.$$

Many uncertainty principles have already been proved for the Fourier transform: Heisenberg-Pauli-Weyl inequality [2, 8], Cowling-Price's inequality [2], local uncertainty inequality [4, 13, 14], Donoho-Stark's inequality [3] and Matolcsi-Szücs inequality [1, 10]. Laeng and Morpurgo [9], and Morpurgo [11] obtained Heisenberg inequality involving a combination of L^1-norms and L^2-norms. Folland and Sitaram [5], next Nemri and Soltani [12, 16, 17] proved general forms of the Heisenberg-Pauli-Weyl inequality and the Donoho-Stark's inequality.

In this paper, we shall use Ghobber's techniques [6], Nash-type inequalities and Clarkson-type inequalities in the Fourier analysis to establish uncertainty inequalities of Heisenberg-type on $L^1 \cap L^p(\mu)$ for $1 < p \le 2$, on $L^2 \cap L^p(\mu)$ for $1 < p < 2$, and on $L^{p_1} \cap L^{p_2}(\mu)$ for $1 < p_1 < p_2 \le 2$. Next, building on the techniques of Donoho and Stark [3] and Soltani [15], we show uncertainty principles and bandlimited principles of concentration-type on $L^1 \cap L^p(\mu)$ for $1 < p \le 2$, and on $L^{p_1} \cap L^{p_2}(\mu)$ for $1 < p_1 < p_2 \le 2$. Finally, based on the ideas

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of Ghobber and Jaming [7] we establish uncertainty principles of Matolcsi-Szücs-type on $L^1 \cap L^p(\mu)$ for $1 < p \leq 2$, and on $L^{p_1} \cap L^{p_2}(\mu)$ for $1 < p_1 \leq p_2 \leq 2$.

This paper is organized as follows. In Section 2 we give uncertainty inequality of Heisenberg-type on $L^1 \cap L^p(\mu)$ for $1 < p \leq 2$. In Section 3 we present uncertainty inequality of Heisenberg-type on $L^2 \cap L^p(\mu)$ for $1 < p < 2$. In Section 4 we establish uncertainty inequality of Heisenberg-type on $L^{p_1} \cap L^{p_2}(\mu)$ for $1 < p_1 < p_2 \leq 2$. In Section 5 we show uncertainty inequality of Donoho-Stark-type on $L^1 \cap L^p(\mu)$ for $1 < p \leq 2$, and on $L^{p_1} \cap L^{p_2}(\mu)$ for $1 < p_1 < p_2 \leq 2$. In Section 6 we state an $L^{p_1} \cap L^{p_2}(\mu)$ bandlimited inequality of concentration-type. The last section is devoted to follow uncertainty principles of Matolcsi-Szücs-type on $L^1 \cap L^p(\mu)$ for $1 < p \leq 2$, and on $L^{p_1} \cap L^{p_2}(\mu)$ for $1 < p_1 \leq p_2 \leq 2$.

2. Heisenberg principle on $L^p \cap L^1(\mu)$

The Fourier transform of a function f in $L^1(\mu)$, is defined by

$$\mathcal{F}(f)(x) := \int_{\mathbb{R}^d} e^{-i\langle x,y\rangle} f(y) \mathrm{d}\mu(y), \quad x \in \mathbb{R}^d.$$

Some of the properties of Fourier transform $\mathcal{F}$ are collected bellow (see [18, 19]).

(a) $L^1 - L^\infty$-boundedness. For all $f \in L^1(\mu)$, $\mathcal{F}(f) \in L^\infty(\mu)$ and

$$\|\mathcal{F}(f)\|_{L^\infty(\mu)} \leq \|f\|_{L^1(\mu)}. \tag{2.1}$$

(b) Inversion theorem. Let $f \in L^1(\mu)$, such that $\mathcal{F}(f) \in L^1(\mu)$. Then

$$f(x) = \mathcal{F}(\mathcal{F}(f))(-x), \quad \text{a.e.} \quad x \in \mathbb{R}^d. \tag{2.2}$$

(c) Plancherel theorem. The Fourier transform $\mathcal{F}$ extends uniquely to an isometric isomorphism of $L^2(\mu)$ onto itself. In particular,

$$\|f\|_{L^2(\mu)} = \|\mathcal{F}(f)\|_{L^2(\mu)}. \tag{2.3}$$

Using relations (2.1) and (2.3) with Marcinkiewicz's interpolation theorem [18, 19], we deduce that for every $1 \leq p \leq 2$, and for every $f \in L^p(\mu)$, the function $\mathcal{F}(f)$ belongs to the space $L^q(\mu)$, $q = p/(p-1)$, and

$$\|\mathcal{F}(f)\|_{L^q(\mu)} \leq \|f\|_{L^p(\mu)}. \tag{2.4}$$

Theorem 2.1 (Nash-type inequality). *Let $s > 0$. If $1 < p \leq 2$, $q = p/(p-1)$ and $f \in L^1 \cap L^p(\mu)$, then*

$$\|\mathcal{F}(f)\|_{L^q(\mu)} \leq K_1(s,p) \|f\|_{L^1(\mu)}^{\frac{qs}{d+qs}} \| \, |y|^s \mathcal{F}(f)\|_{L^q(\mu)}^{\frac{d}{d+qs}},$$

where

$$K_1(s,p) = \frac{\left[\left(\frac{qs}{d}\right)^{\frac{d}{d+qs}} + \left(\frac{d}{qs}\right)^{\frac{qs}{d+qs}}\right]^{1/q}}{\left[2^{\frac{d}{2}} \Gamma(\frac{d}{2}+1)\right]^{\frac{s}{d+qs}}}.$$

Proof. Let $f \in L^1 \cap L^p(\mu)$, $1 < p \leq 2$, $q = p/(p-1)$ and $r > 0$. Then

$$\|\mathcal{F}(f)\|_{L^q(\mu)}^q = \|\chi_{B_r} \mathcal{F}(f)\|_{L^q(\mu_k)}^q + \|(1-\chi_{B_r})\mathcal{F}(f)\|_{L^q(\mu)}^q, \tag{2.5}$$

where $B_r = \{x \in \mathbb{R}^d : |x| < r\}$ and χ_{B_r} is the characteristic function of the set B_r.
Firstly,

$$\|(1-\chi_{B_r})\mathcal{F}(f)\|^q_{L^q(\mu)} \le r^{-qs}\| \, |y|^s\mathcal{F}(f)\|^q_{L^q(\mu)}. \tag{2.6}$$

By (2.1), we get

$$\|\chi_{B_r}\mathcal{F}(f)\|^q_{L^q(\mu_k)} \le \mu(B_r)\|\mathcal{F}(f)\|^q_{L^\infty(\mu)} \le \mu(B_r)\|f\|^q_{L^1(\mu)}.$$

On other hand we have

$$\mu(B_r) = \int_{\mathbb{R}^d} \chi_{B_r}(x)\mathrm{d}\mu(x) = c(d)r^d, \tag{2.7}$$

where

$$c(d) = \frac{1}{2^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)}. \tag{2.8}$$

Therefore,

$$\|\chi_{B_r}\mathcal{F}(f)\|^q_{L^q(\mu_k)} \le c(d)r^d\|f\|^q_{L^1(\mu)}, \tag{2.9}$$

Combining the relations (2.5), (2.6) and (2.9), we obtain

$$\|\mathcal{F}(f)\|^q_{L^q(\mu)} \le c(d)r^d\|f\|^q_{L^1(\mu)} + r^{-qs}\| \, |y|^s\mathcal{F}(f)\|^q_{L^q(\mu)}.$$

By choosing $r = \left(\frac{qs\| \, |y|^s\mathcal{F}(f)\|^q_{L^q(\mu)}}{dc(d)\|f\|^q_{L^1(\mu)}}\right)^{\frac{1}{d+qs}}$, we get the desired inequality. □

Remark 2.2. In the particular case when $p = 2$, the inequality of Theorem 2.1 is given by

$$\|f\|_{L^2(\mu)} \le K_1(s,2)\|f\|^{\frac{2s}{d+2s}}_{L^1(\mu)}\| \, |y|^s\mathcal{F}(f)\|^{\frac{d}{d+2s}}_{L^2(\mu)}.$$

Theorem 2.3 (Clarkson-type inequality). *Let $s > 0$. If $1 < p \le 2$, $q = p/(p-1)$ and $f \in L^1 \cap L^p(\mu)$, then*

$$\|f\|_{L^1(\mu)} \le D_1(s,p)\|f\|^{\frac{qs}{d+qs}}_{L^p(\mu)}\| \, |x|^s f\|^{\frac{d}{d+qs}}_{L^1(\mu)},$$

where

$$D_1(s,p) = \frac{\left(\frac{qs}{d}\right)^{\frac{d}{d+qs}} + \left(\frac{d}{qs}\right)^{\frac{qs}{d+qs}}}{\left[2^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)\right]^{\frac{s}{d+qs}}}.$$

Proof. Let $f \in L^1 \cap L^p(\mu)$, $1 < p \le 2$, $q = p/(p-1)$ and $r > 0$. Then

$$\|f\|_{L^1(\mu)} = \|\chi_{B_r}f\|_{L^1(\mu)} + \|(1-\chi_{B_r})f\|_{L^1(\mu)}. \tag{2.10}$$

Firstly,

$$\|(1-\chi_{B_r})f\|_{L^1(\mu)} \le r^{-s}\| \, |x|^s f\|_{L^1(\mu)}. \tag{2.11}$$

By (2.7) and Hölder's inequality, we get

$$\|\chi_{B_r}f\|_{L^1(\mu)} \le (\mu(B_r))^{1/q}\|f\|_{L^p(\mu)} \le (c(d)r^d)^{1/q}\|f\|_{L^p(\mu)}, \tag{2.12}$$

where $c(d)$ is the constant given by (2.8).
Combining the relations (2.10), (2.11) and (2.12), we obtain

$$\|f\|_{L^1(\mu)} \le (c(d)r^d)^{1/q}\|f\|_{L^p(\mu)} + r^{-s}\| \, |x|^s f\|_{L^1(\mu)}.$$

By setting $r = \left(\frac{qs\| \, |x|^s f\|_{L^1(\mu)}}{d(c(d))^{1/q}\|f\|_{L^p(\mu)}}\right)^{\frac{q}{d+qs}}$, we get the desired inequality. □

By combining the Nash-type inequality (Theorem 2.1) and the Clarkson-type inequality (Theorem 2.3) we obtain the following uncertainty inequality of Heisenberg-type.

Theorem 2.4. *Let $a, b > 0$. If $1 < p \leq 2$, $q = p/(p-1)$ and $f \in L^1 \cap L^p(\mu)$, then*

(i) $\|f\|_{L^1(\mu)}^{\frac{d}{d+qb}} \|f\|_{L^p(\mu)}^{-\frac{qa}{d+qa}} \|\mathcal{F}(f)\|_{L^q(\mu)} \leq C_1 \| \, |x|^a f\|_{L^1(\mu)}^{\frac{d}{d+qa}} \| \, |y|^b \mathcal{F}(f)\|_{L^q(\mu)}^{\frac{d}{d+qb}}$,

where $C_1 = D_1(a,p)K_1(b,p)$.

(ii) $\|f\|_{L^p(\mu)}^{-\frac{qa}{d+qa}} \|\mathcal{F}(f)\|_{L^q(\mu)}^{\frac{d+qb}{qb}} \leq C_2 \| \, |x|^a f\|_{L^1(\mu)}^{\frac{d}{d+qa}} \| \, |y|^b \mathcal{F}(f)\|_{L^q(\mu)}^{\frac{d}{qb}}$,

where $C_2 = D_1(a,p)(K_1(b,p))^{\frac{d+qb}{qb}}$.

Remark 2.5. The uncertainty principles given by Theorem 2.4, generalize the results obtained by Laeng-Morpurgo [9] and Morpurgo [11]. In the particular case when $p = 2$, we obtain the following Heisenberg's inequalities for the Fourier transform $\mathcal{F}$.

(i) Let $a, b > 0$ and $f \in L^1 \cap L^2(\mu)$. From Theorem 2.4 (i) we have

$$\|f\|_{L^1(\mu)}^{d+2a} \|f\|_{L^2(\mu)}^{d+2b} \leq S_1 \| \, |x|^a f\|_{L^1(\mu)}^{d+2b} \| \, |y|^b \mathcal{F}(f)\|_{L^2(\mu)}^{d+2a},$$

where $S_1 = \Big(D_1(a,2)K_1(b,2)\Big)^{\frac{(d+2a)(d+2b)}{d}}$. If $a = b = 1$ and $d = 1$,

$$\|f\|_{L^1(\mu)} \|f\|_{L^2(\mu)} \leq \frac{9\sqrt{3}}{4\pi} \| \, |x| f\|_{L^1(\mu)} \| \, |y| \mathcal{F}(f)\|_{L^2(\mu)}. \tag{2.13}$$

Let Λ be all $f \in L^1 \cap L^2(\mu)$ such that

$$\Delta_1(f) = \frac{\| \, |x| f\|_{L^1(\mu)}}{\|f\|_{L^1(\mu)}}, \quad \Delta_2(f) = \frac{\| \, |y| \mathcal{F}(f)\|_{L^2(\mu)}}{\|f\|_{L^2(\mu)}}.$$

We obtain a characterization of the region of Heisenberg's inequality (see Figure 1),

$$\Big\{(\Delta_1(f), \Delta_2(f)), f \in \Lambda\Big\} \subset \Big\{(x,y), x, y > 0, xy \geq \frac{4\pi}{9\sqrt{3}}\Big\}.$$

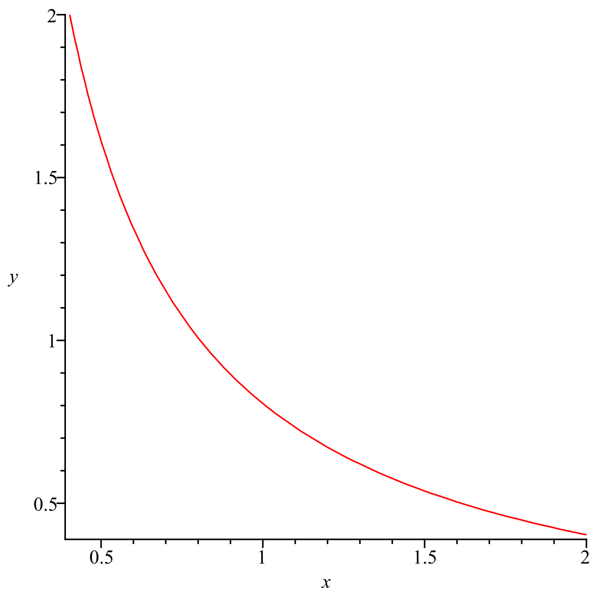

FIGURE 1. Region of the concentrated Heisenberg's inequality (2.13).

(ii) Let $a, b > 0$ and $f \in L^1 \cap L^2(\mu)$. From Theorem 2.4 (ii) we have

$$\|f\|_{L^2(\mu)}^{d+2a+2b} \leq S_2 \| \, |x|^a f\|_{L^1(\mu)}^{2b} \| \, |y|^b \mathcal{F}(f)\|_{L^2(\mu)}^{d+2a},$$

where $S_2 = (D_1(a,2))^{\frac{2b(d+2a)}{d}} (K_1(b,2))^{\frac{(d+2a)(d+2b)}{d}}$. If $a = b = 1$ and $d = 1$,

$$\|f\|_{L^2(\mu)}^5 \leq \frac{(\sqrt{3})^{21}}{(\sqrt{2})^9(\sqrt{\pi})^5} \| \, |x|f\|_{L^1(\mu)}^2 \| \, |y|\mathcal{F}(f)\|_{L^2(\mu)}^3. \tag{2.14}$$

Let Λ be all $f \in L^1 \cap L^2(\mu)$ such that

$$\Delta_1(f) = \frac{\| \, |x|f\|_{L^1(\mu)}}{\|f\|_{L^2(\mu)}}, \quad \Delta_2(f) = \frac{\| \, |y|\mathcal{F}(f)\|_{L^2(\mu)}}{\|f\|_{L^2(\mu)}}.$$

We obtain a characterization of the region of Heisenberg's inequality (see Figure 2),

$$\Big\{(\Delta_1(f), \Delta_2(f)), f \in \Lambda\Big\} \subset \Big\{(x,y), x, y > 0, x^2y^3 \geq \frac{(\sqrt{2})^9(\sqrt{\pi})^5}{(\sqrt{3})^{21}}\Big\}.$$

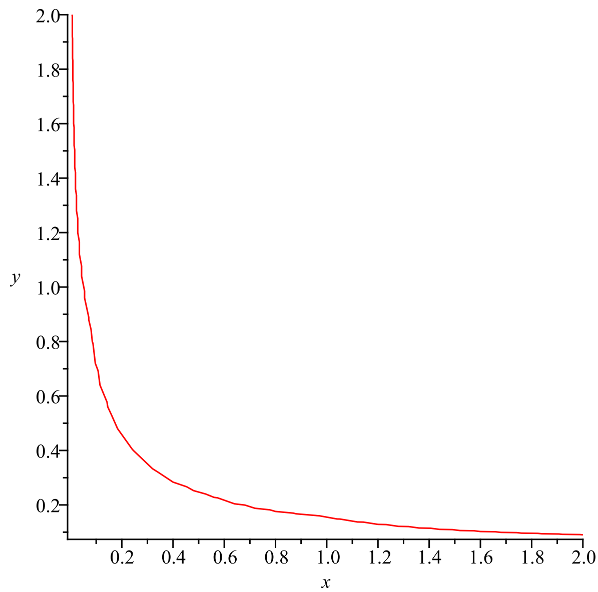

FIGURE 2. Region of the concentrated Heisenberg's inequality (2.14).

(iii) Let $a, b > 0$ and $f \in L^1 \cap L^2(\mu)$. From Remark 2.2 and Theorem 2.3 we have

$$\|f\|_{L^1(\mu)}^{d+2a+2b} \leq S_3 \| \, |x|^a f\|_{L^1(\mu)}^{d+2b} \| \, |y|^b \mathcal{F}(f)\|_{L^2(\mu)}^{2a},$$

where $S_3 = (D_1(a,2))^{\frac{(d+2a)(d+2b)}{d}} (K_1(b,2))^{\frac{2a(d+2b)}{d}}$. If $a = b = 1$ and $d = 1$,

$$\|f\|_{L^2(\mu)}^5 \leq \frac{3^{12}}{(\sqrt{2})^{11}(\sqrt{\pi})^3} \| \, |x| f\|_{L^1(\mu)}^3 \| \, |y| \mathcal{F}(f)\|_{L^2(\mu)}^2. \tag{2.15}$$

Let Λ be all $f \in L^1 \cap L^2(\mu)$ such that

$$\Delta_1(f) = \frac{\| \, |x| f\|_{L^1(\mu)}}{\|f\|_{L^2(\mu)}}, \quad \Delta_2(f) = \frac{\| \, |y| \mathcal{F}(f)\|_{L^2(\mu)}}{\|f\|_{L^2(\mu)}}.$$

We obtain a characterization of the region of Heisenberg's inequality (see Figure 3),

$$\Big\{(\Delta_1(f), \Delta_2(f)), f \in \Lambda\Big\} \subset \Big\{(x,y), x, y > 0, x^3 y^2 \geq \frac{(\sqrt{2})^{11}(\sqrt{\pi})^3}{3^{12}}\Big\}.$$

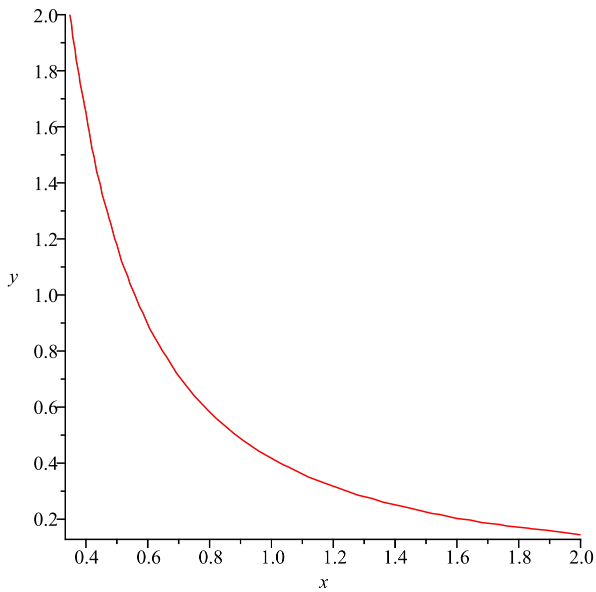

FIGURE 3. Region of the concentrated Heisenberg's inequality (2.15).

3. Heisenberg principle on $L^p \cap L^2(\mu)$

Theorem 3.1 (Nash-type inequality). *Let $s > 0$. If $1 < p < 2$, $q = p/(p-1)$ and $f \in L^2 \cap L^p(\mu)$, then*

$$\|f\|_{L^2(\mu)} \le K_2(s,p)\|f\|_{L^p(\mu)}^{\frac{2qs}{d(q-2)+2qs}} \| \, |y|^s \mathcal{F}(f)\|_{L^2(\mu)}^{\frac{d(q-2)}{d(q-2)+2qs}},$$

where

$$K_2(s,p) = \frac{\left[\left(\frac{2qs}{d(q-2)}\right)^{\frac{d(q-2)}{d(q-2)+2qs}} + \left(\frac{d(q-2)}{2qs}\right)^{\frac{2qs}{d(q-2)+2qs}}\right]^{1/2}}{\left[2^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)\right]^{\frac{(q-2)s}{d(q-2)+2qs}}}.$$

Proof. Let $f \in L^2 \cap L^p(\mu)$, $1 < p < 2$, $q = p/(p-1)$ and $r > 0$. Then

$$\|\mathcal{F}(f)\|_{L^2(\mu)}^2 = \|\chi_{B_r}\mathcal{F}(f)\|_{L^2(\mu)}^2 + \|(1-\chi_{B_r})\mathcal{F}(f)\|_{L^2(\mu)}^2. \tag{3.1}$$

Firstly,

$$\|(1-\chi_{B_r})\mathcal{F}(f)\|_{L^2(\mu)}^2 \le r^{-2s}\| \, |y|^s\mathcal{F}(f)\|_{L^2(\mu)}^2. \tag{3.2}$$

By (2.4), (2.7) and Hölder's inequality, we get

$$\|\chi_{B_r}\mathcal{F}(f)\|_{L^2(\mu)}^2 \le (\mu(B_r))^{\frac{q-2}{q}}\|\mathcal{F}(f)\|_{L^q(\mu)}^2 \le (c(d)r^d)^{\frac{q-2}{q}}\|f\|_{L^p(\mu)}^2, \tag{3.3}$$

where $c(d)$ is the constant given by (2.8). Combining the relations (3.1), (3.2) and (3.3), we obtain

$$\|f\|_{L^2(\mu)}^2 \le (c(d)r^d)^{\frac{q-2}{q}}\|f\|_{L^p(\mu)}^2 + r^{-2s}\| \, |y|^s\mathcal{F}(f)\|_{L^2(\mu)}^2.$$

By choosing $r = \left(\frac{2qs \| \, |y|^s \mathcal{F}(f) \|^2_{L^2(\mu)}}{d(q-2)(c(d))^{\frac{q-2}{q}} \|f\|^2_{L^p(\mu)}} \right)^{\frac{q}{d(q-2)+2qs}}$, we get the desired inequality.

□

Theorem 3.2 (Clarkson-type inequality). *Let $s > 0$. If $1 < p < 2$ and $f \in L^2 \cap L^p(\mu_k)$, then*

$$\|f\|_{L^p(\mu)} \leq D_2(s,p) \|f\|_{L^2(\mu)}^{\frac{2ps}{d(2-p)+2ps}} \| \, |x|^s f \|_{L^p(\mu)}^{\frac{d(2-p)}{d(2-p)+2ps}},$$

where

$$D_2(s,p) = \frac{\left[\left(\frac{2ps}{d(2-p)} \right)^{\frac{d(2-p)}{d(2-p)+2ps}} + \left(\frac{d(2-p)}{2ps} \right)^{\frac{2ps}{d(2-p)+2ps}} \right]^{1/p}}{\left[2^{\frac{d}{2}} \Gamma(\frac{d}{2}+1) \right]^{\frac{(2-p)s}{d(2-p)+2ps}}}.$$

Proof. Let $f \in L^2 \cap L^p(\mu)$, $1 < p < 2$, $q = p/(p-1)$ and $r > 0$. Then

$$\|f\|^p_{L^p(\mu)} = \|\chi_{B_r} f\|^p_{L^p(\mu)} + \|(1-\chi_{B_r}) f\|^p_{L^p(\mu)}. \tag{3.4}$$

Firstly,

$$\|(1-\chi_{B_r}) f\|^p_{L^p(\mu)} \leq r^{-ps} \| \, |x|^s f \|^p_{L^p(\mu)}. \tag{3.5}$$

By (2.7) and Hölder's inequality, we get

$$\|\chi_{B_r} f\|^p_{L^p(\mu)} \leq (\mu(B_r))^{\frac{2-p}{2}} \|f\|^p_{L^2(\mu)} \leq (c(d) r^d)^{\frac{2-p}{2}} \|f\|^p_{L^2(\mu)}, \tag{3.6}$$

where $c(d)$ is the constant given by (2.8). Combining the relations (3.4), (3.5) and (3.6), we obtain

$$\|f\|^p_{L^p(\mu)} \leq (c(d) r^d)^{\frac{2-p}{2}} \|f\|^p_{L^2(\mu)} + r^{-ps} \| \, |x|^s f \|^p_{L^p(\mu)}.$$

By setting $r = \left(\frac{2ps \| \, |x|^s f \|^p_{L^p(\mu)}}{d(2-p)(c(d))^{\frac{2-p}{p}} \|f\|^p_{L^2(\mu)}} \right)^{\frac{2}{d(2-p)+2ps}}$, we get the desired inequality.

□

By combining the Nash-type inequality (Theorem 3.1) and the Clarkson-type inequality (Theorem 3.2) we obtain the following uncertainty inequality of Heisenberg-type.

Theorem 3.3. *Let $a, b > 0$. If $1 < p < 2$, $q = p/(p-1)$ and $f \in L^2 \cap L^p(\mu)$, then*

(i) $\|f\|_{L^p(\mu)}^{\frac{d(q-2)}{d(q-2)+2qb}} \|f\|_{L^2(\mu)}^{\frac{d(2-p)}{d(2-p)+2pa}} \leq M_1 \| \, |x|^a f \|_{L^p(\mu)}^{\frac{d(2-p)}{d(2-p)+2pa}} \| \, |y|^b \mathcal{F}(f) \|_{L^2(\mu)}^{\frac{d(q-2)}{d(q-2)+2qb}}$,

where $M_1 = D_2(a,p) K_2(b,p)$.

(ii) $\|f\|_{L^2(\mu)}^{\frac{d(2-p)}{d(2-p)+2pa} + \frac{d(q-2)}{2qb}} \leq M_2 \| \, |x|^a f \|_{L^p(\mu)}^{\frac{d(2-p)}{d(2-p)+2pa}} \| \, |y|^b \mathcal{F}(f) \|_{L^2(\mu)}^{\frac{d(q-2)}{2qb}}$,

where $M_2 = D_2(a,p) (K_2(b,p))^{\frac{d(q-2)+2qb}{2qb}}$.

(iii) $\|f\|_{L^p(\mu)}^{\frac{d(q-2)}{d(q-2)+2qb} + \frac{d(2-p)}{2pa}} \leq M_3 \| \, |x|^a f \|_{L^p(\mu)}^{\frac{d(2-p)}{2pa}} \| \, |y|^b \mathcal{F}(f) \|_{L^2(\mu)}^{\frac{d(q-2)}{d(q-2)+2qb}}$,

where $M_3 = (D_2(a,p))^{\frac{d(2-p)+2pa}{2pa}} K_2(b,p)$.

4. Heisenberg principle on $L^{p_1} \cap L^{p_2}(\mu)$

Theorem 4.1. (Nash-type inequality). *Let $s > 0$. If $1 < p_1 < p_2 \leq 2$, $q_1 = p_1/(p_1-1)$, $q_2 = p_2/(p_2-1)$ and $f \in L^{p_1} \cap L^{p_2}(\mu)$, then*

$$\|\mathcal{F}(f)\|_{L^{q_2}(\mu)} \leq K_3(s,p_1,p_2)\|f\|_{L^{p_1}(\mu)}^{\frac{q_1q_2s}{d(q_1-q_2)+q_1q_2s}} \| |y|^s\mathcal{F}(f)\|_{L^{q_2}(\mu)}^{\frac{d(q_1-q_2)}{d(q_1-q_2)+q_1q_2s}},$$

where

$$K_3(s,p_1,p_2) = \frac{\left[\left(\frac{q_1q_2s}{d(q_1-q_2)}\right)^{\frac{d(q_1-q_2)}{d(q_1-q_2)+q_1q_2s}} + \left(\frac{d(q_1-q_2)}{q_1q_2s}\right)^{\frac{q_1q_2s}{d(q_1-q_2)+q_1q_2s}}\right]^{1/q_2}}{\left[2^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)\right]^{\frac{(q_1-q_2)s}{d(q_1-q_2)+q_1q_2s}}}.$$

Proof. Let $f \in L^{p_1} \cap L^{p_2}(\mu)$, $1 < p_1 < p_2 \leq 2$, $q_1 = p_1/(p_1-1)$, $q_2 = p_2/(p_2-1)$ and $r > 0$. Then

$$\|\mathcal{F}(f)\|_{L^{q_2}(\mu)}^{q_2} = \|\chi_{B_r}\mathcal{F}(f)\|_{L^{q_2}(\mu)}^{q_2} + \|(1-\chi_{B_r})\mathcal{F}(f)\|_{L^{q_2}(\mu)}^{q_2}. \tag{4.1}$$

Firstly,

$$\|(1-\chi_{B_r})\mathcal{F}(f)\|_{L^{q_2}(\mu)}^{q_2} \leq r^{-sq_2}\| |y|^s\mathcal{F}(f)\|_{L^{q_2}(\mu)}^{q_2}. \tag{4.2}$$

By (2.4), (2.7) and Hölder's inequality, we get

$$\|\chi_{B_r}\mathcal{F}(f)\|_{L^{q_2}(\mu)}^{q_2} \leq (\mu(B_r))^{\frac{q_1-q_2}{q_1}}\|\mathcal{F}(f)\|_{L^{q_1}(\mu)}^{q_2} \leq (c(d)r^d)^{\frac{q_1-q_2}{q_1}}\|f\|_{L^{p_1}(\mu)}^{q_2}, \tag{4.3}$$

where $c(d)$ is the constant given by (2.8). Combining the relations (4.1), (4.2) and (4.3), we obtain

$$\|\mathcal{F}(f)\|_{L^{q_2}(\mu)}^{q_2} \leq (c(d)r^d)^{\frac{q_1-q_2}{q_1}}\|f\|_{L^{p_1}(\mu)}^{q_2} + r^{-sq_2}\| |y|^s\mathcal{F}(f)\|_{L^{q_2}(\mu)}^{q_2}.$$

By choosing $r = \left(\frac{q_1q_2s\| |y|^s\mathcal{F}(f)\|_{L^{q_2}(\mu)}^{q_2}}{d(q_1-q_2)(c(d))^{\frac{q_1-q_2}{q_1}}\|f\|_{L^{p_1}(\mu)}^{q_2}}\right)^{\frac{q_1}{d(q_1-q_2)+q_1q_2s}}$, we get the result. □

Corollary 4.2. *Let $s > 0$. If $1 < p < 2$, $q = p/(p-1)$ and $f \in L^2 \cap L^p(\mu)$, then*

$$\|f\|_{L^2(\mu)} \leq K_3(s,p,2)\|f\|_{L^p(\mu)}^{\frac{2qs}{d(q-2)+2qs}} \| |y|^s\mathcal{F}(f)\|_{L^2(\mu)}^{\frac{d(q-2)}{d(q-2)+2qs}}.$$

Theorem 4.3 (Clarkson-type inequality). *Let $s > 0$. If $1 < p_1 < p_2 \leq 2$ and $f \in L^{p_1} \cap L^{p_2}(\mu)$, then*

$$\|f\|_{L^{p_1}(\mu)} \leq D_3(s,p_1,p_2)\|f\|_{L^{p_2}(\mu)}^{\frac{p_1p_2s}{d(p_2-p_1)+p_1p_2s}} \| |x|^s f\|_{L^{p_1}(\mu)}^{\frac{d(p_2-p_1)}{d(p_2-p_1)+p_1p_2s}},$$

where

$$D_3(s,p_1,p_2) = \frac{\left[\left(\frac{p_1p_2s}{d(p_2-p_1)}\right)^{\frac{d(p_2-p_1)}{d(p_2-p_1)+p_1p_2s}} + \left(\frac{d(p_2-p_1)}{p_1p_2s}\right)^{\frac{p_1p_2s}{d(p_2-p_1)+p_1p_2s}}\right]^{1/p_1}}{\left[2^{\frac{d}{2}}\Gamma(\frac{d}{2}+1)\right]^{\frac{(p_2-p_1)s}{d(p_2-p_1)+p_1p_2s}}}.$$

Proof. Let $f \in L^{p_1} \cap L^{p_2}(\mu)$, $1 < p_1 < p_2 \leq 2$ and $r > 0$. Then

$$\|f\|_{L^{p_1}(\mu)}^{p_1} = \|\chi_{B_r}f\|_{L^{p_1}(\mu)}^{p_1} + \|(1-\chi_{B_r})f\|_{L^{p_1}(\mu)}^{p_1}. \tag{4.4}$$

Firstly,

$$\|(1-\chi_{B_r})f\|_{L^{p_1}(\mu)}^{p_1} \leq r^{-p_1s}\| |x|^s f\|_{L^{p_1}(\mu)}^{p_1}. \tag{4.5}$$

By (2.7) and Hölder's inequality, we get

$$\|\chi_{B_r} f\|^{p_1}_{L^{p_1}(\mu)} \leq (\mu(B_r))^{\frac{p_2-p_1}{p_2}} \|f\|^{p_1}_{L^{p_2}(\mu)} \leq (c(d)r^d)^{\frac{p_2-p_1}{p_2}} \|f\|^{p_1}_{L^{p_2}(\mu)}, \tag{4.6}$$

where $c(d)$ is the constant given by (2.8). Combining the relations (4.4), (4.5) and (4.6). We obtain

$$\|f\|^{p_1}_{L^{p_1}(\mu)} \leq (c(d)r^d)^{\frac{p_2-p_1}{p_2}} \|f\|^{p_1}_{L^{p_2}(\mu)} + r^{-p_1 s}\| \, |x|^s f\|^{p_1}_{L^{p_1}(\mu)}.$$

By setting $r = \left(\dfrac{p_1 p_2 s \| \, |x|^s f\|^{p_1}_{L^{p_1}(\mu)}}{d(p_2-p_1)(c(d))^{\frac{p_2-p_1}{p_2}} \|f\|^{p_1}_{L^{p_2}(\mu)}} \right)^{\frac{p_2}{d(p_2-p_1)+p_1p_2 s}}$, we get the result. □

By combining the Nash-type inequality (Theorem 4.1) and the Clarkson-type inequality (Theorem 4.3) we obtain the following uncertainty inequality of Heisenberg-type.

Theorem 4.4. *Let $a, b > 0$. If $1 < p_1 < p_2 \leq 2$, $q_1 = p_1/(p_1-1)$, $q_2 = p_2/(p_2-1)$ and $f \in L^{p_1} \cap L^{p_2}(\mu)$, then*

(i) $$\|f\|^{\frac{d(q_1-q_2)}{d(q_1-q_2)+q_1q_2 b}}_{L^{p_1}(\mu)} \|f\|^{-\frac{p_1p_2 a}{d(p_2-p_1)+p_1p_2 a}}_{L^{p_2}(\mu)} \|\mathcal{F}(f)\|_{L^{q_2}(\mu)} \leq N_1 \| \, |x|^a f\|^{\frac{d(p_2-p_1)}{d(p_2-p_1)+p_1p_2 a}}_{L^{p_1}(\mu)} \| \, |y|^b \mathcal{F}(f)\|^{\frac{d(q_1-q_2)}{d(q_1-q_2)+q_1q_2 b}}_{L^{q_2}(\mu)},$$

where $N_1 = D_3(a, p_1, p_2) K_3(b, p_1, p_2)$.

(ii) $$\|f\|^{-\frac{p_1p_2 a}{d(p_2-p_1)+p_1p_2 a}}_{L^{p_2}(\mu)} \|\mathcal{F}(f)\|^{\frac{d(q_1-q_2)+q_1q_2 b}{q_1q_2 b}}_{L^{q_2}(\mu)} \leq N_2 \| \, |x|^a f\|^{\frac{d(p_2-p_1)}{d(p_2-p_1)+p_1p_2 a}}_{L^{p_1}(\mu)} \| \, |y|^b \mathcal{F}(f)\|^{\frac{d(q_1-q_2)}{q_1q_2 b}}_{L^{q_2}(\mu)},$$

where $N_2 = D_3(a, p_1, p_2)(K_3(b, p_1, p_2))^{\frac{d(q_1-q_2)+q_1q_2 b}{q_1q_2 b}}$.

Corollary 4.5. *Let $a, b > 0$. If $1 < p < 2$, $q = p/(p-1)$ and $f \in L^2 \cap L^p(\mu)$, then*

(i) $$\|f\|^{\frac{d(q-2)}{d(q-2)+2qb}}_{L^p(\mu)} \|f\|^{\frac{d(2-p)}{d(2-p)+2pa}}_{L^2(\mu)} \leq N_1 \| \, |x|^a f\|^{\frac{d(2-p)}{d(2-p)+2pa}}_{L^p(\mu)} \| \, |y|^b \mathcal{F}(f)\|^{\frac{d(q-2)}{d(q-2)+2qb}}_{L^2(\mu)},$$

where $N_1 = D_3(a, p, 2) K_3(b, p, 2)$.

(ii) $$\|f\|^{\frac{d(2-p)}{d(2-p)+2pa}+\frac{d(q-2)}{2qb}}_{L^2(\mu)} \leq N_2 \| \, |x|^a f\|^{\frac{d(2-p)}{d(2-p)+2pa}}_{L^p(\mu)} \| \, |y|^b \mathcal{F}(f)\|^{\frac{d(q-2)}{2qb}}_{L^2(\mu)},$$

where $N_2 = D_3(a, p, 2)(K_3(b, p, 2))^{\frac{d(q-2)+2qb}{2qb}}$.

(iii) $$\|f\|^{\frac{d(q-2)}{d(q-2)+2qb}+\frac{d(2-p)}{2pa}}_{L^p(\mu)} \leq N_3 \| \, |x|^a f\|^{\frac{d(2-p)}{2pa}}_{L^p(\mu)} \| \, |y|^b \mathcal{F}(f)\|^{\frac{d(q-2)}{d(q-2)+2qb}}_{L^2(\mu)},$$

where $N_3 = (D_3(a, p, 2))^{\frac{d(2-p)+2pa}{2pa}} K_3(b, p, 2)$.

5. Donoho-Stark principle on $L^{p_1} \cap L^{p_2}(\mu)$

Let T be a measurable subset of $\mathbb{R}^d$. We say that a function $f \in L^p(\mu)$, $1 \leq p \leq 2$, is ε-concentrated to T in $L^p(\mu)$-norm, if

$$\|f - \chi_T f\|_{L^p(\mu)} \leq \varepsilon_T \|f\|_{L^p(\mu)},$$

where χ_T is the characteristic function of the set T.

Let E be a measurable subset of $\mathbb{R}^d$, and $f \in L^p(\mu)$, $1 \le p \le 2$. We say that $\mathcal{F}(f)$ is ε_E-concentrated to E in $L^q(\mu)$-norm, $q = p/(p-1)$, if

$$\|\mathcal{F}_k(f) - \chi_E \mathcal{F}(f)\|_{L^q(\mu)} \le \varepsilon_E \|\mathcal{F}(f)\|_{L^q(\mu)}.$$

In following we state an $L^1 \cap L^p(\mu)$ uncertainty principle of concentration-type.

Theorem 5.1 (Donoho-Stark-type inequality). *Let T and E be a measurable subsets of $\mathbb{R}^d$ and $f \in L^1 \cap L^p(\mu)$, $1 < p \le 2$. If f is ε_T-concentrated to T in $L^1(\mu)$-norm and $\mathcal{F}(f)$ is ε_E-concentrated to E in $L^q(\mu)$-norm, $q = p/(p-1)$, then*

$$\|\mathcal{F}(f)\|_{L^q(\mu)} \le \frac{(\mu(T))^{1/q}(\mu(E))^{1/q}}{(1-\varepsilon_T)(1-\varepsilon_E)} \|f\|_{L^p(\mu)}.$$

Proof. Assume that $\mu(T) < \infty$ and $\mu(E) < \infty$. Let $f \in L^1 \cap L^p(\mu)$, $1 < p \le 2$. Since $\mathcal{F}(f)$ is ε_E-concentrated to E in $L^q(\mu)$-norm, $q = p/(p-1)$, then

$$\begin{aligned} \|\mathcal{F}(f)\|_{L^q(\mu)} &\le \varepsilon_E \|\mathcal{F}(f)\|_{L^q(\mu)} + \|\chi_E \mathcal{F}(f)\|_{L^q(\mu)} \\ &\le \varepsilon_E \|\mathcal{F}(f)\|_{L^q(\mu)} + (\mu(E))^{1/q} \|\mathcal{F}(f)\|_{L^\infty(\mu)}. \end{aligned}$$

Thus by (2.1),

$$\|\mathcal{F}(f)\|_{L^q(\mu)} \le \frac{(\mu(E))^{1/q}}{1-\varepsilon_E} \|f\|_{L^1(\mu)}. \tag{5.1}$$

On the other hand, since f is ε_T-concentrated to T in $L^1(\mu)$-norm,

$$\begin{aligned} \|f\|_{L^1(\mu)} &\le \varepsilon_T \|f\|_{L^1(\mu)} + \|\chi_T f\|_{L^1(\mu)} \\ &\le \varepsilon_T \|f\|_{L^1(\mu)} + (\mu(T))^{1/q} \|f\|_{L^p(\mu)}. \end{aligned}$$

Thus

$$\|f\|_{L^1(\mu)} \le \frac{(\mu(T))^{1/q}}{1-\varepsilon_T} \|f\|_{L^p(\mu)}. \tag{5.2}$$

Combining (5.1) and (5.2), we obtain the result of this theorem. □

The uncertainty principle given by Theorem 5.1, generalizes the result obtained by Donoho-Stark [3]. In the particular case when $p = 2$, we obtain the following corollary.

Corollary 5.2. *Let T and E be a measurable subsets of $\mathbb{R}^d$ and $f \in L^1 \cap L^2(\mu)$. If f is ε_T-concentrated to T in $L^1(\mu)$-norm and $\mathcal{F}(f)$ is ε_E-concentrated to E in $L^2(\mu)$-norm, then*

$$(1-\varepsilon_T)(1-\varepsilon_E) \le (\mu(T))^{1/2}(\mu(E))^{1/2}.$$

Next, we state an $L^{p_1} \cap L^{p_2}(\mu)$ uncertainty principle of concentration-type.

Theorem 5.3 (Donoho-Stark-type inequality). *Let T and E be a measurable subsets of $\mathbb{R}^d$ and $f \in L^{p_1} \cap L^{p_2}(\mu)$, $1 < p_1 < p_2 \le 2$. If f is ε_T-concentrated to T in $L^{p_1}(\mu)$-norm and $\mathcal{F}(f)$ is ε_E-concentrated to E in $L^{q_2}(\mu)$-norm, $q_2 = p_2/(p_2-1)$, then*

$$\|\mathcal{F}(f)\|_{L^{q_2}(\mu)} \le \frac{(\mu(T))^{\frac{p_2-p_1}{p_1 p_2}}(\mu(E))^{\frac{q_1-q_2}{q_1 q_2}}}{(1-\varepsilon_T)(1-\varepsilon_E)} \|f\|_{L^{p_2}(\mu)}, \quad q_1 = p_1/(p_1-1).$$

Proof. Assume that $\mu(T)<\infty$ and $\mu(E)<\infty$. Let $f\in L^{p_1}\cap L^{p_2}(\mu)$, $1<p_1<p_2\leq 2$. Since $\mathcal{F}(f)$ is ε_E-concentrated to E in $L^{q_2}(\mu)$-norm, then by Hölder's inequality we obtain

$$\begin{aligned}\|\mathcal{F}(f)\|_{L^{q_2}(\mu)} &\leq \varepsilon_E\|\mathcal{F}(f)\|_{L^{q_2}(\mu)}+\|\chi_E\mathcal{F}(f)\|_{L^{q_2}(\mu)}\\ &\leq \varepsilon_E\|\mathcal{F}(f)\|_{L^{q_2}(\mu)}+(\mu(E))^{\frac{q_1-q_2}{q_1q_2}}\|\mathcal{F}(f)\|_{L^{q_1}(\mu)}.\end{aligned}$$

Thus by (2.4),

$$\|\mathcal{F}(f)\|_{L^{q_2}(\mu)}\leq \frac{(\mu(E))^{\frac{q_1-q_2}{q_1q_2}}}{1-\varepsilon_E}\|f\|_{L^{p_1}(\mu)}. \tag{5.3}$$

On the other hand, since f is ε_T-concentrated to T in $L^{p_1}(\mu)$-norm, then by Hölder's inequality we deduce that

$$\begin{aligned}\|f\|_{L^{p_1}(\mu)} &\leq \varepsilon_T\|f\|_{L^{p_1}(\mu)}+\|\chi_T f\|_{L^{p_1}(\mu)}\\ &\leq \varepsilon_T\|f\|_{L^{p_1}(\mu)}+(\mu(T))^{\frac{p_2-p_1}{p_1p_2}}\|f\|_{L^{p_2}(\mu)}.\end{aligned}$$

Thus

$$\|f\|_{L^{p_1}(\mu)}\leq \frac{(\mu(T))^{\frac{p_2-p_1}{p_1p_2}}}{1-\varepsilon_T}\|f\|_{L^{p_2}(\mu)}. \tag{5.4}$$

Combining (5.3) and (5.4), we obtain the result of this theorem. □

Corollary 5.4. *Let T and E be a measurable subsets of $\mathbb{R}^d$ and $f\in L^2\cap L^p(\mu)$, $1<p<2$. If f is ε_T-concentrated to T in $L^p(\mu)$-norm and $\mathcal{F}(f)$ is ε_E-concentrated to E in $L^2(\mu)$-norm, then*

$$(1-\varepsilon_T)(1-\varepsilon_E)\leq (\mu(T))^{\frac{2-p}{2p}}(\mu(E))^{\frac{q-2}{2q}},\quad q=p/(p-1).$$

6. Bandlimited principle on $L^{p_1}\cap L^{p_2}(\mu)$

Let E be a measurable subset of $\mathbb{R}^d$, and $B^p(E)$, $1\leq p\leq 2$, be the set of functions $g\in L^p(\mu)$ such that $\chi_E\mathcal{F}(g)=\mathcal{F}(g)$.

We say that f is ε-bandlimited to E in $L^p(\mu)$-norm if there is a $g\in B^p(E)$ with $\|f-g\|_{L^p(\mu)}\leq\varepsilon\|f\|_{L^p(\mu)}$.

In the following, we state an $L^{p_1}\cap L^{p_2}(\mu)$ bandlimited uncertainty principle of concentration-type.

Theorem 6.1. *Let T and E be a measurable subsets of $\mathbb{R}^d$ and $f\in L^{p_1}\cap L^{p_2}(\mu)$, $1\leq p_1\leq p_2\leq 2$. If f is ε_T-concentrated to T in $L^{p_1}(\mu)$-norm and ε_E-bandlimited to E in $L^{p_2}(\mu)$-norm, then*

$$\|f\|_{L^{p_1}(\mu)}\leq\frac{(\mu(T))^{\frac{p_2-p_1}{p_1p_2}}}{1-\varepsilon_T}\Big[(1+\varepsilon_E)(\mu(T))^{1/p_2}(\mu(E))^{1/p_2}+\varepsilon_E\Big]\|f\|_{L^{p_2}(\mu)}.$$

Proof. Assume that $\mu(T)<\infty$ and $\mu(E)<\infty$. Let $f\in L^{p_1}\cap L^{p_2}(\mu)$, $1\leq p_1\leq p_2\leq 2$. Since f is ε_T-concentrated to T in $L^{p_1}(\mu)$-norm, then by Hölder's inequality we deduce that

$$\begin{aligned}\|f\|_{L^{p_1}(\mu)} &\leq \varepsilon_T\|f\|_{L^{p_1}(\mu)}+\|\chi_T f\|_{L^{p_1}(\mu)}\\ &\leq \varepsilon_T\|f\|_{L^{p_1}(\mu)}+(\mu(T))^{\frac{p_2-p_1}{p_1p_2}}\|\chi_T f\|_{L^{p_2}(\mu)}.\end{aligned}$$

Thus,

$$\|f\|_{L^{p_1}(\mu)} \leq \frac{1}{1-\varepsilon_T}(\mu(T))^{\frac{p_2-p_1}{p_1p_2}}\|\chi_T f\|_{L^{p_2}(\mu)}. \tag{6.1}$$

Since f is ε_E-bandlimited in $L^{p_2}(\mu)$-norm, by definition there is a g in $B^{p_2}(E)$ with $\|f-g\|_{L^{p_2}(\mu)} \leq \varepsilon_E\|f\|_{L^{p_2}(\mu)}$. For this g, we have

$$\begin{aligned}\|\chi_T f\|_{L^{p_2}(\mu)} &\leq \|\chi_T g\|_{L^{p_2}(\mu)} + \|\chi_T(f-g)\|_{L^{p_2}(\mu)}\\ &\leq \|\chi_T g\|_{L^{p_2}(\mu)} + \varepsilon_E\|f\|_{L^{p_2}(\mu)}.\end{aligned}$$

But for $g \in B^{p_2}(E)$, from (2.2), $g(x) = \mathcal{F}^{-1}(\chi_E\mathcal{F}(g))(x)$, and by (2.4) and Hölder's inequality, we deduce that

$$|g(x)| \leq (\mu(E))^{1/p_2}\|\mathcal{F}(g)\|_{L^{q_2}(\mu)} \leq (\mu(E))^{1/p_2}\|g\|_{L^{p_2}(\mu)}, \quad q_2 = p_2/(p_2-1).$$

Hence,

$$\|\chi_T g\|_{L^{p_2}(\mu)} = \Big(\int_T |g(x)|^p \mathrm{d}\mu(x)\Big)^{1/p_2} \leq (\mu(T))^{1/p_2}(\mu(E))^{1/p_2}\|g\|_{L^{p_2}(\mu)}.$$

Then by (6.1) and the fact that $\|g\|_{L^{p_2}(\mu)} \leq (1+\varepsilon_E)\|f\|_{L^{p_2}(\mu)}$, we get

$$\|\chi_T f\|_{L^{p_2}(\mu)} \leq \Big[(1+\varepsilon_E)(\mu(T))^{1/p_2}(\mu(E))^{1/p_2} + \varepsilon_E\Big]\|f\|_{L^{p_2}(\mu)}.$$

Corollary 6.2. *Let T and E be a measurable subsets of $\mathbb{R}^d$ and $f \in L^p(\mu)$, $1 \leq p \leq 2$. If f is ε_T-concentrated to T and ε_E-bandlimited to E in $L^p(\mu)$-norm, then*

$$\frac{1-\varepsilon_T-\varepsilon_E}{1+\varepsilon_E} \leq (\mu(T))^{1/p}(\mu(E))^{1/p}.$$

7. Matolcsi-Szücs principle on $L^{p_1} \cap L^{p_2}(\mu)$

In this section we establish uncertainty principles of Matolcsi-Szücs-type.

Theorem 7.1 (Matolcsi-Szücs-type inequality). *Let $f \in L^1 \cap L^p(\mu)$, $1 < p \leq 2$. If $A_f = \{x \in \mathbb{R}^d : f(x) \neq 0\}$ and $A_{\mathcal{F}(f)} = \{z \in \mathbb{R}^d : \mathcal{F}(f)(z) \neq 0\}$, then*

$$\|\mathcal{F}(f)\|_{L^q(\mu)} \leq (\mu(A_f))^{1/q}(\mu(A_{\mathcal{F}(f)}))^{1/q}\|f\|_{L^p(\mu)}, \quad q = p/(p-1).$$

Proof. Let $f \in L^1 \cap L^p(\mu)$, $1 < p \leq 2$ and $q = p/(p-1)$. We put $E = A_{\mathcal{F}(f)}$, then by (2.1) and Hölder's inequality we obtain

$$\begin{aligned}\|\mathcal{F}(f)\|_{L^q(\mu)} = \|\chi_E\mathcal{F}(f)\|_{L^q(\mu)} &\leq (\mu(E))^{1/q}\|\mathcal{F}(f)\|_{L^\infty(\mu)}\\ &\leq (\mu(E))^{1/q}\|f\|_{L^1(\mu)}\\ &\leq (\mu(E))^{1/q}(\mu(A_f))^{1/q}\|f\|_{L^p(\mu)},\end{aligned}$$

which gives the desired result. □

The uncertainty principle given by Theorem 7.1 generalizes the result obtained by Matolcsi-Szücs [10] and Benedicks [1]. In the particular case when $p = 2$, we obtain the following corollary.

Corollary 7.2. *Let $f \in L^1 \cap L^2(\mu)$. If $A_f = \{x \in \mathbb{R}^d : f(x) \neq 0\}$ and $A_{\mathcal{F}(f)} = \{z \in \mathbb{R}^d : \mathcal{F}(f)(z) \neq 0\}$, then*

$$\mu(A_f)\mu(A_{\mathcal{F}(f)}) \geq 1.$$

Theorem 7.3 (Matolcsi-Szücs-type inequality). *Let $f \in L^{p_1} \cap L^{p_2}(\mu)$, $1 < p_1 \leq p_2 \leq 2$. If $A_f = \{x \in \mathbb{R}^d : f(x) \neq 0\}$ and $A_{\mathcal{F}(f)} = \{z \in \mathbb{R}^d : \mathcal{F}(f)(z) \neq 0\}$, then*

$$\|\mathcal{F}(f)\|_{L^{q_2}(\mu)} \leq (\mu(A_f))^{\frac{p_2-p_1}{p_1p_2}} (\mu(A_{\mathcal{F}(f)}))^{\frac{q_1-q_2}{q_1q_2}} \|f\|_{L^{p_2}(\mu)},$$

where $q_1 = p_1/(p_1-1)$ and $q_2 = p_2/(p_2-1)$.

Proof. Let $f \in L^{p_1} \cap L^{p_2}(\mu)$, $1 < p_1 \leq p_2 \leq 2$, $q_1 = p_1/(p_1-1)$ and $q_2 = p_2/(p_2-1)$. We put $E = A_{\mathcal{F}(f)}$, then by (2.4) and Hölder's inequality we obtain

$$\begin{aligned} \|\mathcal{F}(f)\|_{L^{q_2}(\mu)} = \|\chi_E \mathcal{F}(f)\|_{L^{q_2}(\mu)} &\leq (\mu(E))^{\frac{q_1-q_2}{q_1q_2}} \|\mathcal{F}(f)\|_{L^{q_1}(\mu)} \\ &\leq (\mu(E))^{\frac{q_1-q_2}{q_1q_2}} \|f\|_{L^{p_1}(\mu)} \\ &\leq (\mu(E))^{\frac{q_1-q_2}{q_1q_2}} (\mu(A_f))^{\frac{p_2-p_1}{p_1p_2}} \|f\|_{L^{p_2}(\mu)}. \end{aligned}$$

□

Corollary 7.4. *Let $f \in L^2 \cap L^p(\mu)$, $1 < p \leq 2$. If $A_f = \{x \in \mathbb{R}^d : f(x) \neq 0\}$ and $A_{\mathcal{F}(f)} = \{z \in \mathbb{R}^d : \mathcal{F}(f)(z) \neq 0\}$, then*

$$(\mu(A_f))^{\frac{2-p}{2p}} (\mu(A_{\mathcal{F}(f)}))^{\frac{q-2}{2q}} \geq 1, \quad q = p/(p-1).$$

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